Topology

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Part I

General topology and basics of category theory

0. Preliminaries

0.1. Basic set theory.

Notation.

(1) We write $\mathbb{N}_0 := \{0, 1, 2, 3, \dots\}$ and $\mathbb{N} := \{1, 2, 3, \dots\}$.
(2) We denote by $\varnothing$ the empty set.
(3) Given a set X we write $A \subset X$ if A is a subset of X and we write $A \subsetneq X$ if A is a subset of X with $A \neq X$.
(4) Given a set X and a subset $A \subset X$ we denote by $X \setminus A := \{x \in X \,|\, x \notin A\}$ the complement of A in X.
(5) Given a set X we denote by $\mathcal{P}(X)$ its power set, i.e. $\mathcal{P}(X)$ is the set of all subsets of X.

Definition. A family of subsets of a set X consists of a set I and a map $f\colon I \to \mathcal{P}(X)$.

Convention. Let X be a set. We will view any subset $\mathcal{U} \subset \mathcal{P}(X)$ as a family of subsets of X via the index set $I = \mathcal{U}$ and the map $I = \mathcal{U} \to \mathcal{P}(X)$ given by $U \mapsto U$. This convention might sound silly, but it will be useful on many occasions.

Definition. Let X be a set.

(1) Let $\{U_i\}_{i\in I}$ be a family of subsets of X. We write

$$\begin{aligned} \bigcup_{i\in I} U_i &= \{x \in X \,|\, \text{ there exists an } i \in I \text{ with } x \in X_i\} && \text{(Union)} \\ \bigcap_{i\in I} U_i &= \{x \in X \,|\, \text{ for every } i \in I \text{ we have } x \in X_i\} && \text{(Intersection).} \end{aligned}$$

In particular, if $I = \varnothing$, then $\bigcup_{i\in I} U_i = \varnothing$ and $\bigcap_{i\in I} U_i = X$.

(2) As mentioned above we view subsets $U_1, \dots, U_k$ of X as a family of subsets. We write $U_1 \cup \dots \cup U_k := \bigcup_{i\in\{1,\dots,k\}} U_i$ and the same way for the intersection.

(3) We say that two subsets A and B of X are disjoint if $A \cap B = \varnothing$.

Lemma 0.1. Let X be a set and let $\{U_i\}_{i\in I}$ be a family of subsets of X. Given any subset A of X we have

$$A \cap \Big(\bigcup_{i\in I} U_i\Big) = \bigcup_{i\in I} (A \cap U_i).$$

Proof. The lemma follows easily from the definitions and elementary logic. ■

Notation. Let $\{U_i\}_{i\in I}$ be a family of subsets of a given set X. If the U_i are pairwise disjoint, i.e. if for any $i \neq j \in I$ we have $U_i \cap U_j = \varnothing$, then at times we write

$$\bigsqcup_{i\in I} U_i := \bigcup_{i\in I} U_i.$$

Lemma 0.2. (De Morgan's Laws) Let X be a set and let $\{A_i\}_{i\in I}$ be a family of subsets of X. If given a subset $B \subset X$ we denote its complement by $B^c := X \setminus B$, then

$$\Big(\bigcap_{i\in I} A_i\Big)^c = \bigcup_{i\in I} A_i^c \quad \text{and} \quad \Big(\bigcup_{i\in I} A_i\Big)^c = \bigcap_{i\in I} A_i^c.$$

Or equivalently, in a more cumbersome notation, we have

$$X \setminus \Big(\bigcap_{i\in I} A_i\Big) \;=\; \bigcup_{i\in I} (X \setminus A_i) \quad \text{and} \quad X \setminus \Big(\bigcup_{i\in I} A_i\Big) \;=\; \bigcap_{i\in I} (X \setminus A_i).$$

Proof. The lemma follows again easily from the definitions and elementary logic. ■

Notation. Let $f\colon X \to Y$ be a map between sets.

(1) Given a subset $A \subset X$ we write $f(A) := \{f(x) \,|\, x \in A\}$.

(2) Given a subset $A \subset Y$ we write $f^{-1}(A) := \{x \in X \,|\, f(x) \in A\}$. Furthermore, given $y \in Y$ we write $f^{-1}(y) := f^{-1}(\{y\})$.

Lemma 0.3. (Image-Preimage Lemma) Let $f\colon X \to Y$ be a map between two sets X and Y. Then the following statements hold:

(1)	(a)	for every subset $U \subset X$ we have	U	$\subset$	$f^{-1}(f(U))$
	(b)	furthermore, if f is injective , then we have	U	$=$	$f^{-1}(f(U))$
(2)	(a)	for every subset $V \subset Y$ we have	V	$\supset$	$f(f^{-1}(V))$
	(b)	furthermore, if f is surjective, then we have	V	$=$	$f(f^{-1}(V))$
(3)		for every family $\{U_i\}_{i\in I}$ of subsets of X we have	$f\Big(\bigcup_{i\in I} U_i\Big)$	$=$	$\bigcup_{i\in I} f(U_i)$
(4)		for every family $\{U_i\}_{i\in I}$ of subsets of X we have	$f\Big(\bigcap_{i\in I} U_i\Big)$	$\subset$	$\bigcap_{i\in I} f(U_i)$
(5)	(a)	for every subset $U \subset X$ we have	$f(X \setminus U)$	$\supset$	$f(X) \setminus f(U)$
	(b)	furthermore, if $f^{-1}(f(U)) = U$, then we have	$f(X \setminus U)$	$=$	$f(X) \setminus f(U)$
(6)		for every family $\{V_i\}_{i\in I}$ of subsets of Y we have	$f^{-1}\Big(\bigcup_{i\in I} V_i\Big)$	$=$	$\bigcup_{i\in I} f^{-1}(V_i)$
(7)		for every family $\{V_i\}_{i\in I}$ of subsets of Y we have	$f^{-1}\Big(\bigcap_{i\in I} V_i\Big)$	$=$	$\bigcap_{i\in I} f^{-1}(V_i)$
(8)		for every subset $V \subset Y$ we have	$f^{-1}(Y \setminus V)$	$=$	$X \setminus f^{-1}(V)$.

Proof. The lemma follows again from the definitions and elementary logic. ■

Definition. Let $\{X_i\}_{i\in I}$ be a family of sets. The product of the family $\{X_i\}_{i\in I}$ is defined as the set

$$\prod_{i\in I} X_i \;:=\; \Big\{ f\colon I \to \bigcup_{i\in I} X_i \,\Big|\, \text{for every } i \in I \text{ we have } f(i) \in X_i \Big\}.$$

Furthermore, given a set X and a set I we write

$$X^I \;:=\; \prod_{i\in I} X.$$

Axiom of Choice 0.1. For every family of sets $\{X_i\}_{i\in I}$, such that each I is non-empty and such that each X_i is non-empty, the product $\prod_{i\in I} X_i$ is non-empty.

Notation.

(1) Given finitely many sets $X_1, \ldots, X_n$ we write

$$X_1 \times \cdots \times X_n \;:=\; \{(x_1, \ldots, x_n) \,|\, \text{ for } i = 1, \ldots, n \text{ we have } x_i \in X_i\}.$$

This set has an obvious bijection to the product $\prod_{i=1}^{n} X_i$, and we will freely go back and forth between these two notations.

(2) Given a set X and given $n \in \mathbb{N}$ we write

$$X^n := \underbrace{X \times \cdots \times X}_{n \text{ times}}.$$

Notation.
(1) If $\{A_i\}_{i \in I}$ is a family of sets, then we define the disjoint union as the set

$$\bigsqcup_{i \in I} A_i := \bigcup_{i \in I} (A_i \times \{i\}).$$

(2) If $I = \{1, \ldots, k\}$, then we write $A_1 \sqcup \cdots \sqcup A_k := \bigsqcup_{i=1}^{k} A_i$.
(3) We use obvious variations on (1) and (2), for example given sets A and B we refer to

$$A \sqcup B := (A \times \{1\}) \cup (B \times \{2\})$$

as the disjoint union of A and B

0.2. Relations and equivalence relations on sets.

Definition. A relation on a set S is a subset of $S \times S$.

Definition. Let S be a set and let $R \subset S \times S$ be a relation on S.
(1) The relation is called reflexive, if for every $s \in S$ we have $(s, s) \in R$.
(2) The relation is called transitive, if for every $(s, t), (t, u) \in R$ we also have $(s, u) \in R$.
(3) The relation is called symmetric, if for every (s, t) in R we also have $(t, s) \in R$.

Notation. Let S be a set and let $R \subset S \times S$ be a relation on S. Given $x, y \in S$ we write $x \, R \, y$ if and only if $(x, y) \in R$.

Definition. An equivalence relation on a set S is a relation that is reflexive, transitive and symmetric.

Convention. Equivalence relations on a set S are almost invariably denoted by "$\sim$" or some variation thereof.

Definition. Let S be a set together with an equivalence relation "$\sim$". An equivalence class of $(S, \sim)$ is a subset C of S with the following three properties:
(a) the subset is non-empty,
(b) given any two $s, t \in C$ we have $s \sim t$, and
(c) given $s \in C$ and $t \in S$ with $s \sim t$ we have $t \in C$.

Lemma 0.4. (Equivalence Class Lemma) Let S be a set together with an equivalence relation "$\sim$".
(1) Every element of S is contained in a unique equivalence class, namely

$$[s] := \{t \in S \mid t \sim s\}.$$

(2) Two equivalence classes are either disjoint or the same.

Proof. This lemma should be familiar from an earlier course. ■

Notation. Let S be a set together with an equivalence relation "$\sim$".
(1) We denote by $S/\sim$ the set of equivalence classes.
(2) Given $s \in S$ we denote by $[s]$ the unique equivalence class that contains $[s]$.
(3) We refer to the map
$$\begin{array}{rcl} p\colon X & \to & X/\sim \\ x & \mapsto & [x] \end{array}$$
as the natural projection.

Lemma 0.5. (Quotient-Factorization Lemma) Let $\sim$ be an equivalence relation on a set X. We denote by $p\colon X \to X/\sim$ the natural projection. Let $f\colon X \to Y$ be a map to some set Y with the property that $f(x) = f(y)$ whenever $x \sim y$. Then the map
$$\begin{array}{rcl} g\colon X/\sim & \to & Y \\ [x] & \mapsto & f(x) \end{array}$$
is well-defined and it is the unique map $g\colon X/\sim \to Y$, such that $f = g \circ p$, i.e. such that the following diagram of maps commutes:
$$\begin{tikzcd} X \arrow[r, "p"] \arrow[dr, "f"'] & X/\sim \arrow[d, "\exists!\, g"] \\ & Y. \end{tikzcd}$$

Proof. Let $\sim$ be an equivalence relation on a set X and let $f\colon X \to Y$ be a map with the property that $f(x) = f(y)$ whenever $x \sim y$. Let a be an element of $X/\sim$. Since equivalences classes are by definition non-empty there exists an $x \in X$ with $a = [x]$. We set
$$g(a) \;:=\; f(x).$$
Evidently this definition does not depend on the choice of x. The map g satisfies $f = g \circ p$. Since the projection $X \to X/\sim$ is surjective we see that g is uniquely determined by this property. ■

Definition. Let S be a set and let R be a relation on S. We say $x, y \in S$ are equivalent, if there exists a sequence $x_0, \ldots, x_k$ of elements in S such that $x = x_0$, $x_k = y$ and such that for each $i \in \{0, \ldots, k-1\}$ we have $x_i \, R \, x_{i+1}$ or $x_{i+1} \, R \, x_i$. It is straightforward to verify that "$\sim$" defines an equivalence relation on S. We refer to it as the equivalence relation generated by the relation R.

Example. We consider the set $S = \mathbb{R}$ and the relation that is defined by $x \, R \, y$ if and only if $y = x + 1$. The equivalence relation generated by R is the equivalence relation that is given by $x \sim y$ if and only if $x - y \in \mathbb{Z}$.

Convention. At times we are a little sloppy in our notation and we do not distinguish in our notation between a relation and the equivalence relation it generates.

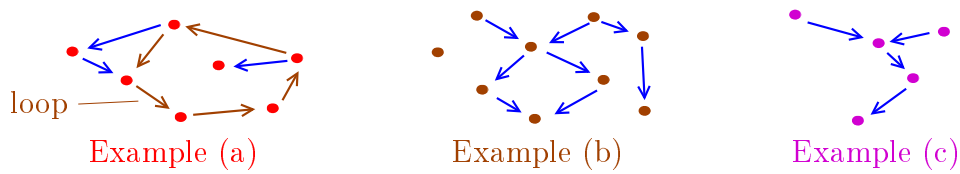
0.3. Orders on sets.

Definition.
(1) A relation "$\leq$" on a set S is called antisymmetric if $a \leq b$ and $b \leq a$ implies $a = b$.
(2) A relation that is reflexive, transitive and antisymmetric is called a partial order.

(3) A set together with a partial order is called a partially ordered set.
(4) Let "$\leq$" be a partial order on a set A. Given $a, b \in A$ we write $a < b$ if $a \leq b$ and if $a \neq b$.

Example. We consider the three reflexive transitive relations that are illustrated in the figure below. Here it is understood that $P \leq Q$ if there exists a "directed path" from Q to P.

(a) The example on the left has a "loop". Given any two points P, Q on the loop we have $P \leq Q$ and $Q \leq P$, hence this example is not a partial order.
(b) This example is a partial order.
(c) This example is also a partial order.



Definition. Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets. Given (a, b) and (a', b') in $A \times B$ we define

$$(a, b) < (a', b') \quad :\Longleftrightarrow \quad \text{either } (a < a') \text{ or } (a = a' \text{ and } b < b')$$

and we define $(a, b) \leq (a', b')$ if either $(a, b) = (a', b')$ or $(a, b) < (a', b')$. One can easily verify that "$\leq$" is a partial order on $A \times B$. We refer to this partial order on $A \times B$ as the lexicographic order on $A \times B$.

0.4. The cardinality of sets.

Definition.
(1) We say that two given sets S and T are equinumerous if there exists a bijection $\varphi \colon S \to T$.
(2) Given a set S we define the cardinality $\#S$ of S as the equinumerosity equivalence class of S in the class of all sets . Put differently we have

$$\#S \ := \ \text{class of all sets } T \text{ that are equinumerous with } S.$$

(3) We say a set is finite if there exists an $n \in \mathbb{N}_0$ such that $\#S = \#\{1, \ldots, n\}$, otherwise we say that the set is infinite
(4) If a set S is finite, then we write $\#S = n$ for the unique $n \in \mathbb{N}_0$ with $\#S = \#\{1, \ldots, n\}$. If S infinite, then sometimes we write $\#S = \infty$. At times we also write $|S| \in \mathbb{N}_0 \cup \{\infty\}$ instead of $\#S \in \mathbb{N}_0 \cup \{\infty\}$.
(5) Let S be a set. If $\#S = n$ for some $n \in \mathbb{N}_0$ or if $\#S = \#\mathbb{N}$, then we say that S is countable, otherwise we say that S is uncountable.

Lemma 0.6. (Countability Lemma)
(1) Every subset of a countable set is again countable.
(2) Let S be a countable set and let $f \colon S \to T$ be a map. The image $f(S)$ is again countable.

(3) The product of finitely many countable sets is again countable.
(4) The disjoint union of countably many countable sets is again countable.
(5) Let T be a set and let S be a subset such that $T \setminus S$ is countable. Then S is countable if and only if T is countable.
(6) If S is a finite set and if T is a countable set, then the set of maps from S to T is also countable.

Example. We consider the map
$$\begin{array}{rcl} \mathbb{Z} \times \mathbb{N} & \to & \mathbb{Q} \\ (m,n) & \mapsto & \frac{m}{n}. \end{array}$$
Evidently this map is surjective. It follows from the Countability Lemma 0.6 (2) and (3) that $\mathbb{Q}$ is countable.

Proof. This lemma should be familiar from an earlier course. ■

Definition. Let S and T be two sets.
(1) We write $\#S \leq \#T$ if there exists an injection $\varphi\colon S \to T$.
(2) We write $\#S < \#T$ if $\#S \leq \#T$ but $\#S \neq \#T$.

Theorem 0.7. (Cantor's Theorem) For every set X we have $\#X < \#\mathcal{P}(X)$.

Proof. First note that the map
$$\begin{array}{rcl} X & \to & \mathcal{P}(X) \\ x & \mapsto & \{x\} \end{array}$$
is evidently injective, hence $\#X \leq \#\mathcal{P}(X)$. Now let $f\colon X \to \mathcal{P}(X)$ be a map. We have to show that it cannot be surjective. We consider the set
$$Y \;:=\; \{x \in X \,|\, x \notin f(x)\}.$$
We claim that Y is not in the image of f. Indeed, if there did exist a $z \in X$ with $f(z) = Y$, then we would have
$$z \in Y \quad \underset{\substack{\uparrow \\ \text{by definition of } Y}}{\Longleftrightarrow} \quad z \notin f(z) \quad \underset{\substack{\uparrow \\ \text{since } f(z)=Y}}{\Longleftrightarrow} \quad z \notin Y.$$
We have thus obtained a contradiction. ■

Corollary 0.8. The set $\mathbb{R}$ of real numbers is uncountable.

Proof. We consider the maps
$$\begin{array}{rcl} f\colon \mathcal{P}(\mathbb{N}) & \to & \{\mathbb{Z}_2\}^{\mathbb{N}} \\ X & \mapsto & \left(\begin{array}{rcl} \mathbb{N} & \to & \mathbb{Z}_2 \\ k & \mapsto & \begin{cases} 1, & \text{if } k \in X \\ 0, & \text{otherwise} \end{cases} \end{array} \right) \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon \{\mathbb{Z}_2\}^{\mathbb{N}} & \to & \mathbb{R} \\ f & \mapsto & \sum\limits_{k=1}^{\infty} \frac{f(k)}{3^k}. \end{array}$$
One can easily verify that $f\colon \mathcal{P}(\mathbb{N}) \to \{\mathbb{Z}_2\}^{\mathbb{N}}$ is a bijection and that $g\colon \{\mathbb{Z}_2\}^{\mathbb{N}} \to \mathbb{R}$ is injective. Thus it follows from Cantor's Theorem 0.7 together with the Countability Lemma 0.6 (1) that $\mathbb{R}$ is uncountable. ■

Theorem 0.9. (Bernstein-Schröder Theorem) Let S and T be two sets. Then
$$\#S \leq \#T \quad \text{and} \quad \#T \leq \#S \quad \Longrightarrow \quad \#S = \#T.$$

Proof. Let $f\colon A \to B$ and $g\colon B \to A$ be two injective maps between two sets A and B. Without loss of generality we can assume that A and B are disjoint. We denote by $\sim$ the equivalence relation on $A \sqcup B$ that is generated by $a \sim f(a)$ for $a \in A$ and by $b \sim g(b)$ for $b \in B$. Note that if $a \in A$ lies in the image of g, then by the injectivity of g there exists a unique element $g^{-1}(a) \in B$ and we have $a \sim g^{-1}(a)$. The same remark works with the roles of f and g reversed.

Claim. For each equivalence class I there exists a bijection $A \cap I \to B \cap I$.

Proof. So let I be an equivalence class and let $a \in A \cap I$. The subset $I \subset A \sqcup B$ consists of all elements equivalent to a. The equivalence class I is then of one of the following three types:

$$\begin{array}{lcccccccccl} & & & A & B & A & B & A & B & & \\ (1) & a' & \dots & f^{-1}(g^{-1}(a)), & g^{-1}(a), & a, & f(a), & g(f(a)), & f(g(f(a)), & \dots & \text{with } a' \in A \setminus g(B) \\ (2) & b' & \dots & f^{-1}(g^{-1}(a)), & g^{-1}(a), & a, & f(a), & g(f(a)), & f(g(f(a)), & \dots & \text{with } b' \in B \setminus f(A) \\ (3) & \dots & \dots & f^{-1}(g^{-1}(a)), & g^{-1}(a), & a, & f(a), & g(f(a)), & f(g(f(a)), & \dots & \end{array}$$

In case (1) the map f defines a bijection $A \cap I \to B \cap I$, in case (2) the map g^{-1} defines a bijection $A \cap I \to B \cap I$ and in case (3) both f and g^{-1} define a bijection $A \cap I \to B \cap I$.⊞

Recall that by the Equivalence Class Lemma 0.4 we know that equivalence classes always give a disjoint decomposition of a set. In our case, if I_j, $j \in J$ denote the set of equivalence classes of $\sim$ on $A \sqcup B$, then we have

$$A \sqcup B = \bigsqcup_{j \in J} I_j \quad \text{and therefore also} \quad A = \bigsqcup_{j \in J} (A \cap I_j) \text{ and } B = \bigsqcup_{j \in J} (B \cap I_j).$$

Since for each $j \in J$ we have a bijection $A \cap I_j \to B \cap I_j$ we get a bijection $A \to B$. ■

0.5. The Cauchy-Schwarz Inequality.

Definition. Let R be a commutative ring and let V and W be R-modules.

(1) A pairing on $V \times W$ is a bilinear map

$$\varphi\colon V \times W \to R,$$

were by bilinear we mean that for all $v, v' \in v$ and for all $w, w' \in W$ we have the equalities $\varphi(v + v', w) = \varphi(v, w) + \varphi(v', w)$ and $\varphi(v, w + w') = \varphi(v, w) + \varphi(v, w')$.

(2) A pairing is called R-bilinear if for all $\lambda \in R$ and all $v \in V$ and $w \in W$ we have $\varphi(\lambda \cdot v, w) = \lambda \cdot \varphi(v, w)$ and $\varphi(v, \lambda \cdot w) = \varphi(v, w) \cdot \lambda$.

(3) An R-bilinear pairing $\varphi\colon V \times V \to S$ is called symmetric if $\varphi(v, w) = \varphi(w, v)$ for all $v, w \in V$.

Definition. Let V be a real vector space.

(1) We say a pairing φ is positive-definite if $\varphi(v, v) > 0$ for all $v \in V \setminus \{0\}$.

(2) We say a pairing φ is positive-semidefinite if $\varphi(v, v) \geq 0$ for all $v \in V \setminus \{0\}$.

Example. Let $n \in \mathbb{N}_0$. The standard pairing on $\mathbb{R}^n$ is given by

$$\begin{aligned} \langle -, - \rangle\colon \mathbb{R}^n \times \mathbb{R}^n &\to \mathbb{R} \\ (v, w) &\mapsto \langle v, w \rangle := \sum_{i=1}^{n} v_i \cdot w_i. \end{aligned}$$

Here we use the standard convention that for $v \in \mathbb{R}^n$ we denote by $v_1, \dots, v_n$ its coordinates. This standard pairing on $\mathbb{R}^n$ is evidently $\mathbb{R}$-bilinear, symmetric and positive-definite.

Proposition 0.10. (Cauchy-Schwarz Inequality) Let V be a real vector space and let
$$\begin{array}{rcl} \varphi\colon V \times V & \to & \mathbb{R}_{\geq 0} \\ (v,w) & \mapsto & \varphi(v,w) \end{array}$$
be an $\mathbb{R}$-bilinear form.

(1) If φ is positive semidefinite, then for any $v, w \in V$ we have
$$\varphi(v,w)^2 \;\leq\; \varphi(v,v)\cdot\varphi(w,w).$$
(2) Now suppose that φ is positive-definite. In this equality in (1) holds if and only if v and w are linearly dependent.

Proof. This statement has surely been proved in an earlier course on linear algebra. ■

Definition. Let V be a complex vector space.

(1) A pairing $\varphi\colon V \times V \to \mathbb{C}$ is called $\mathbb{C}$-sesquilinear if for all $\lambda \in \mathbb{C}$ and all $v, w \in V$ we have $\varphi(\lambda \cdot v, w) = \lambda \cdot \varphi(v,w)$ and $\varphi(v, \lambda \cdot w) = \varphi(v,w)\cdot\overline{\lambda}$.
(2) A $\mathbb{C}$-sesquilinear pairing $\varphi\colon V \times V \to \mathbb{C}$ is called hermitian if $\varphi(v,w) = \overline{\varphi(w,v)}$ for all $v, w \in V$.
(3) We say a pairing φ is positive-definite if $\varphi(v,v) \in \mathbb{R}_{>0}$ for all $v \in V \setminus \{0\}$.
(4) We say a pairing φ is positive-semidefinite if $\varphi(v,v) \in \mathbb{R}_{\geq 0}$ for all $v \in V \setminus \{0\}$.

Example. Let $n \in \mathbb{N}_0$. The standard hermitian pairing on $\mathbb{C}^n$ is given by
$$\begin{array}{rcl} \langle -,- \rangle\colon \mathbb{C}^n \times \mathbb{C}^n & \to & \mathbb{R} \\ (v,w) & \mapsto & \langle v,w\rangle := \sum_{i=1}^{n} v_i \cdot \overline{w}_i. \end{array}$$
This standard pairing on $\mathbb{C}^n$ is evidently $\mathbb{C}$-sesquilinear, hermitian and positive-definite.

Notation. Given $z = x + y\mathrm{i} \in \mathbb{C}$ with $x, y \in \mathbb{R}$ we write $|z| := \sqrt{x^2+y^2} = \sqrt{z\cdot\overline{z}}$.

Proposition 0.11. (Cauchy-Schwarz Inequality) Let V be a complex vector space and let
$$\begin{array}{rcl} \varphi\colon V \times V & \to & \mathbb{R}_{\geq 0} \\ (v,w) & \mapsto & \varphi(v,w) \end{array}$$
be a hermitian form.

(1) If φ is positive semidefinite, then for any $v, w \in V$ we have
$$|\varphi(v,w)|^2 \;\leq\; \varphi(v,v)\cdot\varphi(w,w).$$
(2) Now suppose that φ is positive-definite. In this case equality in (1) holds if and only if v and w are linearly dependent.

Proof. The proof is very similar to the proof of the above Cauchy-Schwarz Inequality 0.10. We leave it to the reader to make the necessary minor modifications. ■

0.6. The Gram-Schmidt Orthogonalization Process.

Proposition 0.12. (Gram-Schmidt Orthogonalization Process) Let V be a real vector space and let $\langle\,,\,\rangle\colon V \times V \to \mathbb{R}$ be a positive-definite symmetric $\mathbb{R}$-bilinear form. Given $v \in V$ we write $\|v\| := \sqrt{\langle v,v\rangle}$. Let $v_1, \ldots, v_n$ be a basis of V. We consider the

following sequence of n-tuples of vectors in V:

$$\begin{aligned}
&u_{0,1}, .., u_{0,n} := v_1, .., v_n \\
&u_{1,1}, .., u_{1,n} := \frac{v_1}{\|v_1\|}, \|v_1\| \cdot v_2, v_2, .., v_n \\
&\quad\vdots \qquad\qquad\qquad \vdots \\
&u_{i,1}, .., u_{i,n} := u_{i-1,1}, .., u_{i-1,i-1}, \frac{z}{\|z\|}, \|z\| \cdot v_{i+1}, v_{i+2}, .., v_n \text{ with } z := v_i - \sum_{j=1}^{i-1} \langle v_i, u_{i-1,j} \rangle \cdot u_{i-1,j} \\
&\quad\vdots \qquad\qquad\qquad \vdots \\
&u_{n,1}, .., u_{n,n} := u_{n-1,1}, .., u_{n-1,n-1}, \frac{z}{\|z\|} \qquad\qquad \text{with } z := v_n - \sum_{j=1}^{n-1} \langle v_n, u_{n-1,j} \rangle \cdot u_{n-1,j}.
\end{aligned}$$

This sequence has the following properties:

(1) For each $i \in \{1, \ldots, n\}$ the vectors $\{u_{i,1}, \ldots, u_{i,n}\}$ form a basis of V.
(2) For each $i \in \{1, \ldots, n-1\}$ the base change matrix from the basis $v_1, \ldots, v_n$ to the basis $u_{i,1}, \ldots, u_{i,n}$ has determinant 1.
(3) The base change matrix from the basis $v_1, \ldots, v_n$ to the basis $u_{n,1}, \ldots, u_{n,n}$ has positive determinant.
(4) For each $i \in \{1, \ldots, n\}$ the vectors $u_{i,1}, \ldots, u_{i,i}$ are orthonormal.
(5) If $v_1, \ldots, v_n$ are orthonormal, then the sequence of n-tuples of vectors is constant, i.e. for any $i, j \in \{1, \ldots, n\}$ we have $u_{i,j} = v_j$.

The analogous statement also holds if we are dealing with a positive-definite hermitian $\mathbb{C}$-sesquilinear form over a complex vector space.

Proof. The lemma follows from elementary arguments which we leave to the readers who suffer from linear algebra withdrawal syndrome. ■

0.7. Group actions.

Definition. Let X be a set and let G be a group with trivial element e.

(1) An action of G on X is a map

$$\begin{array}{rcl} G \times X & \to & X \\ (g, x) & \mapsto & g \cdot x \end{array}$$

with the following properties:

$$\begin{array}{rcll} e \cdot x & = & x, & \text{for all } x \in X, \\ \underset{\substack{\uparrow \\ \text{group action}}}{g} \cdot (\underset{\uparrow}{h} \cdot x) & = & (\underset{\substack{\uparrow \\ \text{multiplication in the group } G}}{g \cdot_G h}) \cdot x, & \text{for all } x \in X \text{ and } g, h \in G. \end{array}$$

(2) The action is called free, if $g \cdot x = x$ for some $x \in X$ implies that $g = e$.
(3) We say G acts transitively, if for every x and y in X there exists a $g \in G$ with $g \cdot x = y$.

Examples.

(1) Given any group G the map

$$\begin{array}{rcl} G \times G & \to & G \\ (g, h) & \mapsto & ghg^{-1} \end{array}$$

defines an action of G on itself. Let G be a non-trivial group. Considering $h = e$ we see that this action is not free and it is also straightforward to see that the action is not transitive.

(2) The group $G = \mathbb{Z}^n$ acts on $X = \mathbb{R}^n$ by addition. This action is evidently free but it is not transitive.

Lemma 0.13. (Group Action-Equivalence Relation Lemma) Let X be a set and let G be a group that acts on X. Then

$$x \sim y \quad :\Longleftrightarrow \quad \text{there exists a } g \in G \text{ such that } g \cdot x = y$$

is an equivalence relation on X.

Proof. The statement follows easily from the definitions. ■

Definition. Let X be a set and let G be a group that acts on X. By the Group Action-Equivalence Relation Lemma 0.13 we know that

$$x \sim_G y \quad :\Longleftrightarrow \quad \text{there exists a } g \in G \text{ such that } g \cdot x = y$$

is an equivalence relation on X. We denote by

$$X/G \ := \ X/\sim_G .$$

the set of equivalence classes.

0.8. Normed vector spaces.

Definition. Let V be a real vector space. A norm on V is a map $\| - \| \colon V \to \mathbb{R}_{\geq 0}$ that has the following three properties:

(1) For every $v \in V$ we have $\|v\| = 0 \iff v = 0$.
(2) For every $v \in V$ and $\lambda \in \mathbb{R}$ we have $\|\lambda \cdot v\| = |\lambda| \cdot \|v\|$.
(3) For every $v, w \in V$ we have $\|v + w\| \leq \|v\| + \|w\|$.

Lemma 0.14. Let V be a vector space. If $\langle -, - \rangle \colon V \times V \to \mathbb{R}_{\geq 0}$ is a symmetric positive definite $\mathbb{R}$-linear pairing, then the map

$$\begin{array}{rcl} V \times V & \to & \mathbb{R}_{\geq 0} \\ v & \mapsto & \|v\| := \sqrt{\langle v, v \rangle} \end{array}$$

is a norm.

Proof. It is straightforward to verify that the first two properties of a norm are satisfied. It remains to prove the third property. Given $v, w \in \mathbb{R}^n$ we see that

$$\begin{aligned} \|v + w\|^2 &= \langle v + w, v + w \rangle \overset{\substack{\text{follows from the } \mathbb{R}\text{-bilinearity of } \langle -, - \rangle \\ \downarrow}}{=} \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle \overset{\substack{\text{since } \langle w, v \rangle = \langle v, w \rangle \\ \downarrow}}{=} \langle v, v \rangle + 2 \cdot \langle v, w \rangle + \langle w, w \rangle \\ &\underset{\substack{\uparrow \\ \text{by the Cauchy-Schwarz Inequality 0.10 we have } \langle v, w \rangle^2 \leq \langle v, v \rangle \cdot \langle w, w \rangle}}{\leq} \langle v, v \rangle + 2 \cdot \sqrt{\langle v, v \rangle \cdot \langle w, w \rangle} + \langle w, w \rangle = \left(\sqrt{\langle v, v \rangle} + \sqrt{\langle w, w \rangle} \right)^2 = (\|v\| + \|w\|)^2. \end{aligned}$$

By taking square roots on both sides we obtain that $\|v + w\| \leq \|v| + \|w\|$. ■

Definition. The standard pairing on $\mathbb{R}^n$ gives rise to the Euclidean norm

$$\begin{array}{rcl} \| - \|_{\text{Eucl}} \colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ v = (v_1, \ldots, v_n) & \mapsto & \sqrt{\langle v, v \rangle} = \sqrt{v_1^2 + \cdots + v_n^2}. \end{array}$$

Usually, when there is no source of confusion, we just write $\|v\|$ instead of $\|v\|_{\text{Eucl}}$.

Example. We will now give more examples of norms $\mathbb{R}^n$. One can easily show that the following maps are norms and that for $n \geq 2$ they do *not* arise from a positive definite $\mathbb{R}$-bilinear pairing:

$$\underbrace{\begin{array}{rcl} \|-\|_{\max}\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x_1,\dots,x_n) & \mapsto & \max\{|x_1|,\dots,|x_n|\} \end{array}}_{\text{maximum norm}} \quad \text{and} \quad \underbrace{\begin{array}{rcl} \|-\|_{\text{Manh}}\colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x_1,\dots,x_n) & \mapsto & \sum\limits_{i=1}^{n} |x_i|. \end{array}}_{\text{Manhattan norm}}$$

0.9. Metric spaces.

Definition. A metric space is a pair (X,d) consisting of a set X and a metric d on X, i.e. a map

$$\begin{array}{rcl} X \times X & \to & \mathbb{R}_{\geq 0} := \{x \in \mathbb{R} \mid x \geq 0\} \\ (x,y) & \mapsto & d(x,y) \end{array}$$

with the following properties:

(1)		$d(x,y)$	$=$	$0 \iff x = y$		
(2)	for all $x,y \in X$ we have	$d(x,y)$	$=$	$d(y,x)$		(symmetry)
(3)	for all $x,y,z \in X$ we have	$d(x,z)$	$\leq$	$d(x,y)+d(y,z)$		(triangle inequality).

Sometimes we refer to $d(x,y)$ as the distance between x and y.

Lemma 0.15. Let V be a real vector space and let $\|-\|\colon V \to \mathbb{R}_{\geq 0}$ be a norm. Then

$$\begin{array}{rcl} V \times V & \to & \mathbb{R}_{\geq 0} \\ (v,w) & \mapsto & d(v,w) := \|v-w\| \end{array}$$

is a metric on V.

Proof. The three properties of a metric follow easily from the three properties of a norm. ■

Definition. Given $x,y \in \mathbb{R}^n$ we set

$$d(x,y) \;:=\; \|x-y\|_{\text{Eucl}} \quad \text{where} \quad \|v\|_{\text{Eucl}} = \|(v_1,\dots,v_n)\|_{\text{Eucl}} := \sqrt{v_1^2+\cdots+v_n^2}.$$

We refer to the map

$$\begin{array}{rcl} d\colon \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x,y) & \mapsto & d(x,y) \end{array}$$

as the Euclidean metric on $\mathbb{R}^n$. Usually we consider $\mathbb{R}^n$ as a metric space with respect to this Euclidean metric, unless we say something else.

Definition. Let X be a set. The map

$$\begin{array}{rcl} X \times X & \to & \mathbb{R}_{\geq 0} \\ (x,y) & \mapsto & \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y \end{cases} \end{array}$$

defines a metric on X, it is called the trivial metric.

1. Topological spaces: basic definitions and properties

1.1. Topological spaces.

Definition.

(1) Let X be a set. A topology on X is a subset $\mathcal{T}$ of the power set $\mathcal{P}(X)$ with the following properties:
   (a) the empty set and the entire set X lie in $\mathcal{T}$,
   (b) the intersection of *finitely many* sets in $\mathcal{T}$ is again a set in $\mathcal{T}$,
   (c) the union of *arbitrarily many* sets in $\mathcal{T}$ is again a set in $\mathcal{T}$.
   The sets in $\mathcal{T}$ are called open with respect to $\mathcal{T}$. If $\mathcal{T}$ is understood from the context, then we just say open.

(2) A topological space is a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a topology on X.

Examples.

(1) Let X be a set, then $\mathcal{T} = \{\varnothing, X\}$ is a topology on X. This topology is sometimes called the trivial topology on X.

(2) Let X be a set. Then $\mathcal{T} = \mathcal{P}(X)$ is also a topology on X. In other words, $\mathcal{T}$ is the topology such that every subset of X is open. This topology is usually referred to as the discrete topology on X. Sometimes we say that a topological space is discrete, if the topology is indeed the discrete topology.

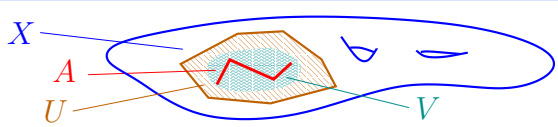
(3) Let X be the empty set. Together with $\mathcal{T} = \{\varnothing\}$ this is a topological space, called the empty topological space. It is a never ending source of counterexamples to carelessly formulated statements.

(4) Let $X = \{*\}$ be a set with a single element $*$. It follows immediately from the definition of a topology there is a unique topology on $X = \{*\}$, namely the topology given by $\mathcal{T} = \{\varnothing, \{*\}\}$. Together with the empty set this is the only case where the topology is already determined by the set. We refer to any topological space consisting of a single point as a singleton.

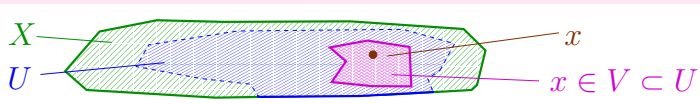
Definition. Let X be a topological space.

(1) Let $A \subset X$ be a subset. We say $U \subset X$ is a neighborhood of A if there exists an open subset V such that $A \subset V \subset U$. We say U is an open neighborhood of A, if U is furthermore open. This definition is illustrated in the figure below.

(2) An (open) neighborhood of a point $x \in X$ is an (open) neighborhood of $\{x\}$.



Lemma 1.1. (JH-Openness Criterion) Let X be a topological space and let $U \subset X$ be a subset. If for every $x \in U$ there exists a neighborhood V of x that is contained in U, then U is open.



Proof. Let X be a topological space and let $U \subset X$ be a subset. Suppose that for every $x \in U$ there exists a neighborhood V_x of x that is contained in U. By definition of a neighborhood there exists for each $x \in U$ an open subset W_x with $x \in W_x \subset V_x$. We see that

$$U = \bigcup_{x \in U} \{x\} \subset \bigcup_{x \in U} W_x \underset{\substack{\uparrow \\ \text{since each } W_x \text{ is contained in } U}}{\subset} U.$$

It follows that $U = \bigcup_{x \in U} W_x$, i.e. U is the union of open subsets of X, thus by the third property of a topology, U itself is open. ■

Definition. Let X be a topological space. We say a subset A of X is closed if the complement $X \setminus A$ is open.

Lemma 1.2. (Closed Subset-Lemma) Let X be a topological space.

(1) The empty set and the entire set X are closed.
(2) The intersection of *arbitrarily many* closed subsets is again closed.
(3) The union of *finitely many* closed subsets is again closed.

Proof. The lemma is an immediate consequence of the definition of a topology and de Morgan's Laws 0.2. ■

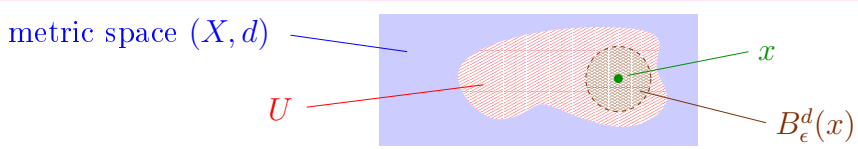
1.2. The metric topology.

Notation. Let (X, d) be a metric space. Given $x \in X$ and $r \in \mathbb{R}$ we write

$$B_r^d(x) := \{y \in X \mid d(x, y) < r\}$$

If the metric d is understood from the context, then sometimes we drop it from the notation.

Lemma 1.3. (Metric-Topology Lemma) Let (X, d) be a metric space. A subset U of X is called open if for every $x \in U$ there exists an $\epsilon > 0$ such that $B_\epsilon^d(x)$ is contained in U. These open subsets define a topology on X.



Proof. We proved this lemma in Analysis IV, alternatively see [Fri23]. ■

Definition. Let (X, d) be a metric space.

(1) We refer to the topology introduced in the Metric-Topology Lemma 1.3 as the metric topology.
(2) Unless we say something else we always equip (X, d) with the metric topology.

Convention. Let $n \in \mathbb{N}$.

(1) Unless we say something else we consider $\mathbb{R}^n$ as a metric space with the Euclidean metric given by

$$d(x, y) := \|x - y\| \quad \text{where} \quad \|z\| = \|(z_1, \ldots, z_n)\| := \sqrt{z_1^2 + \cdots + z_n^2}.$$

Furthermore, unless we say something else we equip $\mathbb{C}^n$ with the metric given by

$$d(x,y) \;:=\; \|x-y\| \quad \text{where} \quad \|z\| \;=\; \|(z_1,\dots,z_n)\| \;:=\; \sqrt{z_1\cdot\overline{z_1}+\cdots+z_n\cdot\overline{z}_n}\,.$$

(2) In particular, unless we say explicitly otherwise, we equip $\mathbb{R}^n$ with the topology arising from the Euclidean metric and we $\mathbb{C}^n$ with the topology arising from the above metric. We call these the Euclidean topologies on $\mathbb{R}^n$ and $\mathbb{C}^n$.

1.3. Basic definitions.

Proposition 1.4. Let (X,d) be a metric space, let $y\in X$ and let $r\in\mathbb{R}$.

(1) the set $B_r^d(y) := \{x\in X \,|\, d(x,y)<r\}$ is open in X,

(2) the set $\overline{B}_r^d(y) := \{x\in X \,|\, d(x,y)\le r\}$ is closed in X,

(3) the set $S_r^d(y) := \{x\in X \,|\, d(x,y)=r\}$ is closed in X.

Proof. We proved this proposition in Analysis II. ■

Notation. Let $n\in\mathbb{N}_0$. Given $y\in\mathbb{R}^n$ and $r\in\mathbb{R}$ we write

$$\begin{aligned} B_r^n(y) &:= \{x\in\mathbb{R}^n \,|\, \|x-y\|<r\} && \text{called open } n\text{-ball}\\ \overline{B}_r^n(y) &:= \{x\in\mathbb{R}^n \,|\, \|x-y\|\le r\} && \text{called closed } n\text{-ball}\\ S_r^{n-1}(y) &:= \{x\in\mathbb{R}^n \,|\, \|x-y\|=r\} && \text{called } (n-1)\text{-sphere}. \end{aligned}$$

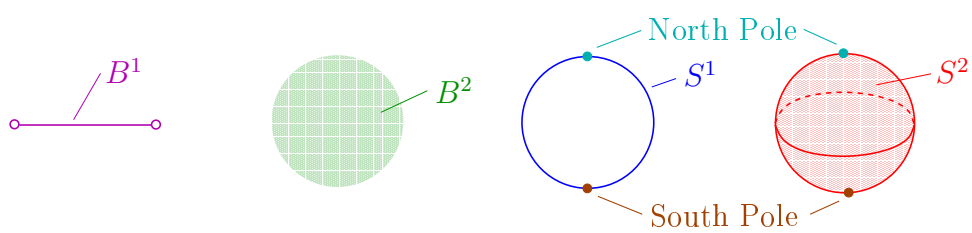
We adopt the following conventions:

(1) A 2-ball is often called a disk and a 1-sphere is often referred to as a circle.

(2) If $y=0$, then we drop y from the notation and if $r=1$, then we drop r from the notation.

(3) Sometimes we write rB^n instead of $B_r^n(0)$ and sometimes we write $r\overline{B}^n$ instead of $\overline{B}_r^n(0)$.

(4) We refer to the point $(0,\dots,0,1)\in S^n=\{x\in\mathbb{R}^{n+1} \,|\, \|x\|=1\}$ as the North Pole of S^n and we refer to the point $(0,\dots,0,-1)\in S^n=\{x\in\mathbb{R}^{n+1} \,|\, \|x\|=1\}$ as the South Pole of S^n.



1.4. The subspace topology.

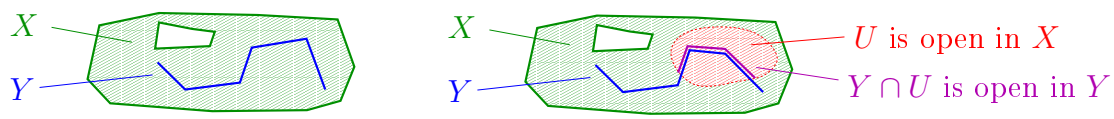
Definition. Let $(X,\mathcal{T})$ be a topological space and let $Y\subset X$ be a subset.

(1) It follows from elementary set theory, see Lemma 0.1, that

$$\mathcal{S} \;:=\; \{Y\cap U \,|\, U\in\mathcal{T}\}$$

is a topology on Y. We refer to $\mathcal{S}$ as the subspace topology on Y induced by Y.

(2) If Y is indeed equipped with the subspace topology, then sometimes we refer to Y as a subspace of X.



Convention. Unless we say something else, we consider each subset Y of $\mathbb{R}^n$ always as a topological space with respect to the subspace topology.

1.5. Covers of topological spaces.

Definition. Let X be a topological space.

(1) A cover of a subset $A \subset X$ is a family $\{U_i\}_{i \in I}$ of subsets of X with $A \subset \bigcup_{i \in I} U_i$.
(2) We say a cover $\{U_i\}_{i \in I}$ is open if each U_i is open. Similarly we define a closed cover.
(3) We say a cover $\{U_i\}_{i \in I}$ is finite if I is finite. Otherwise we say that the cover is infinite.

Lemma 1.5. (Cover-Open Subset Lemma) Let X be a topological space and let M be a subset of X.

(1) Let $\{U_i\}_{i \in I}$ be an *arbitrary* open cover of X. The following two statements hold:
  (a) M is open if and only if each intersection $M \cap U_i$ is open in U_i,
  (b) M is closed if and only if each intersection $M \cap U_i$ is closed in U_i.
(2) Let $\{A_i\}_{i \in I}$ be a *finite* closed cover of X. The following two statements hold:
  (a) M is open if and only if each intersection $M \cap A_i$ is open in A_i,
  (b) M is closed if and only if each intersection $M \cap A_i$ is closed in A_i.

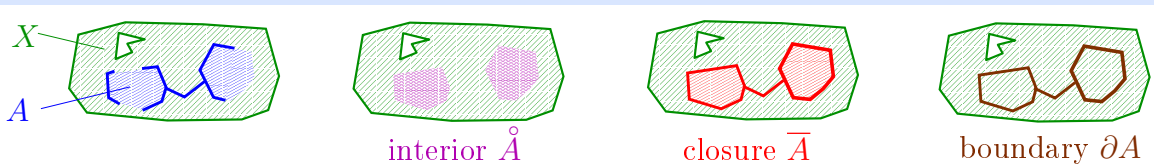
Proof. The proof is rather elementary. The curious reader is referred to [Fri23] for details. ■

1.6. The interior and the closure of a subset.

Definition. Let X be a topological space and let A be a subset of X. We make the following two definitions:

$$\text{interior } \mathring{A} := \begin{array}{c}\text{union of all open subsets of } X \\ \text{that are contained in } A\end{array} \qquad \text{closure } \overline{A} := \begin{array}{c}\text{intersection of all closed} \\ \text{subsets of } X \text{ that contain } A\end{array}$$

Sometimes, to avoid confusion, we will also write $\operatorname{int}(A)$ for the interior of a subset A. Furthermore we define the boundary of A in X as $\partial A := \overline{A} \setminus \mathring{A}$.



Lemma 1.6. (Interior-Closure Lemma) Let X be a topological space and let A be a subset of X.

(1) The interior $\mathring{A}$ is an open subset of X.
(2) The closure $\overline{A}$ of A is a closed subset of X.

Proof.

(1) The union of arbitrarily many open subsets is, by definition of a topology, again an open subset. Thus the interior $\mathring{A}$ is an open subset of X.
(2) In the Closed Subset-Lemma 1.2 we saw that the intersection of arbitrarily many closed subsets is a closed subset. It follows immediately that the closure $\overline{A}$ is a closed subset of X. ■

1.7. The Hausdorff property.

Definition. We say a topological space X is Hausdorff, if given any two distinct points $x \neq y$ there exist disjoint open neighborhoods U of x and V of y.

Proposition 1.7. (Metric Space-Hausdorff Proposition) Let (X, d) be a metric space. The corresponding topological space is Hausdorff.

Proof. Let $x, y \in X$ be two different points. We set $\epsilon := \frac{1}{2}d(x, y)$. It follows from Proposition 1.4 that $U := B_\epsilon(x)$ and $V := B_\epsilon(y)$ are neighborhoods of x respectively y. An elementary argument using the triangle inequality shows that U and V are disjoint. ■

Definition. Let $n \in \mathbb{N}$. We consider the set $X := \mathbb{R}^n \cup \{\infty\}$, i.e. X consists of $\mathbb{R}^n$ and an extra point ∞. We say $U \subset X$ is open, if both of the following two conditions are satisfied:
(1) for each point $x \in U \cap \mathbb{R}^n$ there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subset U$,
(2) if $\infty \in U$, then there exists a $C > 0$ such that $\{x \in \mathbb{R}^n \mid \|x\| > C\} \subset U$.
It is straightforward to see that this defines indeed a topology on X. We refer to $\mathbb{R}^n \cup \{\infty\}$ as $\mathbb{R}^n$ with a point at infinity. For $n = 1$ we refer to $\mathbb{R} \cup \{\infty\}$ also as the line with a point at infinity.

Definition. We consider the set $X := \mathbb{R} \cup \{*\}$, i.e. X consists of $\mathbb{R}$ and an extra point $*$. We say $U \subset X$ is open, if the following two conditions are satisfied:
(a) for each point $x \in U \cap \mathbb{R}$ there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$,
(b) if $* \in U$, then there exists an $\epsilon > 0$ such that $(-\epsilon, 0) \cup (0, \epsilon) \subset U$.
One can easily verify that this is indeed a topology on X. We refer to this topological space as the line with two zeros.

Example. Even though the "line with two zeros" looks at first glance similar to the "line with a point at infinity", they are strikingly different topological spaces. For example it is a fun exercise to show that "$\mathbb{R}^n$ with a point at infinity" is Hausdorff and that "the line with two zeros" is not Hausdorff.

Lemma 1.8. (Points-in-Hausdorff Lemma) Let X be a topological space. If X is Hausdorff, then for every $x \in X$ the corresponding subset $\{x\}$ is closed.

Proof. Let X be a topological space which is Hausdorff. Let $x \in X$. Since X is Hausdorff we know in particular that for every $y \neq x$ there exists an open neighborhoods V_y of y that

is disjoint from x. Note that

$$X \setminus \{x\} \;=\; \bigcup_{y \in X \setminus \{x\}} \{y\} \;\underset{\substack{\uparrow \\ \text{since } y \in V_y}}{\subset}\; \bigcup_{y \in X \setminus \{x\}} V_y \;\underset{\substack{\uparrow \\ \text{since } x \notin V_y}}{\subset}\; X \setminus \{x\}.$$

Thus we see that the inclusions are equalities. Since each V_y is open we see that $X \setminus \{x\}$ is the union of open subsets, in particular $X \setminus \{x\}$ is open. In other words, $\{x\}$ is a closed subset of X. ■

1.8. Compact topological spaces.

Definition. We say a topological space X is compact if for each open cover $\{U_i\}_{i \in I}$ of X there exist finitely many indices $i_1, \ldots, i_k \in I$ such that

$$X \;=\; U_{i_1} \cup \cdots \cup U_{i_k}.$$

Example. Every finite topological space (regardless of the topology) is compact, since there exist only finitely many distinct open subsets. This simple example can sometimes be quite useful.

Convention. Let P be a property of a topological space (e.g. being discrete, Hausdorff, compact).

(1) If X is a topological space that satisfies P we might just say that "X is a P-space". For example, a Hausdorff space is a topological space that is Hausdorff.
(2) We say that a subset Y of a topological space X has the property P if Y, equipped with the subspace topology, has the property P.

Lemma 1.9. (Compact-Closed Lemma) Let X be a topological space.

(1) If X is compact, then any closed subset of X is also compact.
(2) Let $A \subset X$ be a compact subset. If X is Hausdorff, then A is a closed subset of X.

Example. Let X be a topological space, let A be a closed subset and let K be a compact subset of X. It follows from the Compact-Closed Lemma 1.9 (1) that $A \cap K$ is also compact. This simple observation will get used frequently.

Proof. We proved this lemma in Analysis II and Analysis IV. Alternatively see [Fri23]. ■

1.9. The Heine-Borel Theorem.

Definition. Let (X, d) be a metric space. A subset A of X is called bounded if there exists a $C \geq 0$ such that for all $x_0, x_1 \in A$ we have $d(x_0, x_1) \leq C$.

Lemma 1.10. Let (X, d) be a metric space and let $A \subset X$ be a subset. If A is compact, then A is closed and it is bounded.

Proof. Let (X, d) be a metric space and let $A \subset X$ be a subset.

(1) We want to show that $X \setminus A$ is open. Let $x \in X \setminus A$. We need to show that there exists an $\epsilon > 0$ with $B^d_\epsilon(x) \subset X \setminus A$. Given $j \in \mathbb{N}$ we consider the subset $U_j = X \setminus \overline{B}^d_{\frac{1}{j}}(x)$. By Proposition 1.4 we know that the U_j are open. Evidently these open subsets cover A. Since A is compact there exist $j_1, \ldots, j_k \in \mathbb{N}$ with $A \subset U_{j_1} \cup \cdots \cup U_{j_k}$. We set $\epsilon := \min\{\frac{1}{j_1}, \ldots, \frac{1}{j_k}\}$. We obtain that $A \subset X \setminus \overline{B}^d_\epsilon(x)$. Thus we have found the desired ϵ.

(2) We need to show that A is bounded. If A is empty, then there is nothing to show. Now suppose that A is non-empty. We pick $x_0 \in A$. It follows from the triangle inequality that it suffices to show that there exists an $R \geq 0$ with $A \subset \overline{B}^d_R(x_0)$. We consider the open subsets $U_j = B^d_j(x_0)$ with $j \in \mathbb{N}$. Evidently these sets cover A. Since A is compact we obtain that there exist $j_1, \ldots, j_k \in \mathbb{N}$ such that $A \subset U_{j_1} \cup \cdots \cup U_{j_k}$. We set $R := \max\{j_1, \ldots, j_k\}$. We obtain that $A \subset B^d_R(x_0) \subset \overline{B}^d_R(x_0)$. We have thus found the desired R. ■

Theorem 1.11. (Heine-Borel Theorem) A subset of $\mathbb{R}^n$ is compact if and only if it is bounded and closed.

Proof. We proved this theorem in Analysis II. Alternatively see [Fri23]. ■

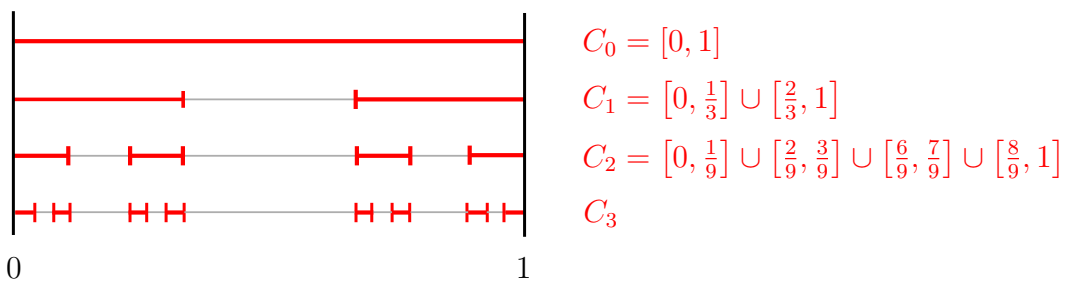
Example. Let $y \in \mathbb{R}^n$ and $r \geq 0$. It follows from the Heine-Borel Theorem 1.11 and Proposition 1.4 that the closed ball $\overline{B}^n_r(y)$ and the sphere $S^{n-1}_r(y)$ are compact.

Definition. We set $C_0 := [0,1]$. Iteratively we now define sets $C_2, C_3, \ldots$ as follows:

$$C_i \;:=\; \underbrace{\{\tfrac{1}{3} \cdot x \,|\, x \in C_{i-1}\}}_{=\frac{1}{3}\cdot C_{i-1}} \;\cup\; \underbrace{\{\tfrac{2}{3} + \tfrac{1}{3} \cdot x \,|\, x \in C_{i-1}\}}_{=\frac{2}{3}+\frac{1}{3}\cdot C_{i-1}}.$$

Note that each C_i consists of 2^i closed intervals of length $\frac{1}{3^i}$ and that we obtain C_i from C_{i-1} by removing the "open middle third" of each interval. We define

$$\text{Cantor set } C \;:=\; \bigcap_{i \in \mathbb{N}_0} C_i.$$



Lemma 1.12. The Cantor set is compact.

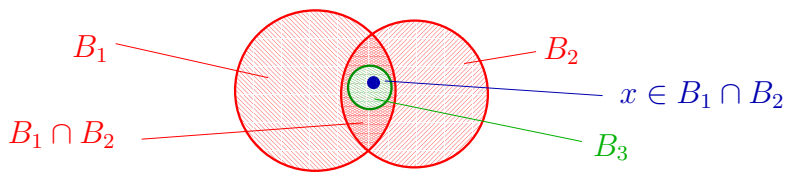
Proof. Each C_i is the union of 2^i closed intervals, hence each C_i is closed. It follows from the Closed Subset-Lemma 1.2 that C itself is a closed subset of $\mathbb{R}$. It is evidently bounded, hence the Heine-Borel Theorem 1.11 implies that the Cantor set is compact. ■

1.10. Bases of a topology.

Definition. Let X be a set and let $\mathcal{B} \subset \mathcal{P}(X)$ be a subset of the power set of X. We say $\mathcal{B}$ has the basis property if the following two conditions are satisfied:

(B1) For every $x \in X$ there exists a $B \in \mathcal{B}$ with $x \in B$.

(B2) Given any $B_1, B_2 \in \mathcal{B}$ and given any $x \in B_1 \cap B_2$ there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.



Example. Let (X, d) be a metric space. It follows from an elementary argument, see e.g. the proof of Proposition 1.4 (1), that the set of all open balls

$$\mathcal{B} \;=\; \{\text{all balls } B_r(x) \text{ with } r > 0 \text{ and } x \in X\} \;\subset\; \mathcal{P}(X)$$

satisfies (B1) and (B2).

Lemma 1.13. (Topology-from-Basis Lemma) Let X be a set and let $\mathcal{B} \subset \mathcal{P}(X)$. We set

$$\mathcal{T}(\mathcal{B}) \;:=\; \{V \subset X \,|\, \text{for every } x \in V \text{ there exists a } B \in \mathcal{B} \text{ with } x \in B \subset V\}.$$

If $\mathcal{B}$ has the basis property, then $\mathcal{T}(\mathcal{B})$ is a topology on X.

Proof. We proved the lemma in Analysis IV. Alternatively see [Fri23]. ■

Definition. Let X be a set.

(1) If $\mathcal{B} \subset \mathcal{P}(X)$ has the basis property, then we refer to $\mathcal{T}(\mathcal{B})$ as the topology on X generated by $\mathcal{B}$.

(2) Conversely, if $\mathcal{T}$ is a given topology on X, if $\mathcal{B} \subset \mathcal{P}(X)$ has the basis property and if $\mathcal{T} = \mathcal{T}(\mathcal{B})$, then we say that $\mathcal{B}$ is a basis of the topology $\mathcal{T}$ of X.

Examples.

(1) If (X, d) is a metric space, then the topology generated by the set of all open balls, is by definition precisely the metric topology we already introduced on page 31.

(2) Let $n \in \mathbb{N}$ and let $\mathbb{R}^n \cup \{\infty\}$ be the topological space defined on page 33. It follows immediately from the definition of the topology that a basis for the topological space $\mathbb{R}^n \cup \{\infty\}$ is given by

$$\mathcal{B} \;=\; \big\{B_r^n(x) \,\big|\, r > 0 \text{ and } x \in \mathbb{R}^n\big\} \;\cup\; \big\{\{\infty\} \cup (\mathbb{R}^n \setminus \overline{B}_C^n(0)) \,\big|\, C \in \mathbb{R}\big\}.$$

2. Continuity

2.1. Continuous maps.

Definition. We say a map $f\colon X \to Y$ between two topological spaces X and Y is continuous, if for each open subset U in Y the preimage $f^{-1}(U)$ is open in X. Otherwise the map is called discontinuous.

Examples.

(1) Let X and Y be two topological spaces. If Y is equipped with the trivial topology or if X is equipped with the discrete topology, then every map $f\colon X \to Y$ is continuous.
(2) We say a map $X \to Y$ between two sets is *constant* if for any $x_1, x_2 \in X$ we have $f(x_1) = f(x_2)$. Any constant map $f\colon X \to Y$ between two topological spaces is continuous since the preimage of an open subset of Y is either the empty set or X.

Lemma 2.1. (Basics-of-Continuity Lemma)

(1) If $f\colon X \to Y$ and $g\colon Y \to Z$ are two continuous maps between topological spaces, then the composition $g \circ f\colon X \to Z$ is also continuous.
(2) Let X be a topological space and let A be a subset of X. If we equip A with the subspace topology, then the inclusion map $i\colon A \to X$ is continuous.
(3) Let $f\colon X \to Y$ be a continuous map between topological spaces and let A be a subset of X. If we equip A with the subspace topology, then the restriction of f to the map $f|_A\colon A \to Y$ is also continuous.
(4) Let $f\colon X \to Y$ be a continuous map. If $Z \subset Y$ is a subset with $f(X) \subset Z$, then the map $f\colon X \to Z$ is also continuous.

Proof.

(1) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two continuous maps between topological spaces. We need to show that the composition $g \circ f\colon X \to Z$ is also continuous. Let $U \subset Z$ be an open subset. Then

$$(g \circ f)^{-1}(U) \;=\; f^{-1}(g^{-1}(U)) \underset{\substack{\uparrow \\ \text{since } g \text{ is continuous and } U \text{ is open}}}{=} f^{-1}(\text{the open subset } g^{-1}(U)) \underset{\substack{\uparrow \\ \text{since } f \text{ is continuous}}}{=} \text{open.}$$

(2) This statement follows immediately from the definitions.
(3) This statement is an immediate consequence of (1) and (2) since the restriction of f to A is precisely the same as $f \circ i$ where $i\colon A \to X$ denotes the inclusion.
(4) For clarity we denote by $g\colon X \to Z$ the map given by $x \mapsto f(x)$. Let $U \subset Z$ be an open subset. We need to show that $g^{-1}(U)$ is an open subset of X. By definition of the subspace topology there exists an open subset $V \subset X$ with $U = Z \cap V$. Thus we see that

$$g^{-1}(U) \;=\; g^{-1}(Z \cap V) \underset{\substack{\uparrow \\ \text{definition of } g}}{=} f^{-1}(Z \cap V) \underset{\substack{\uparrow \\ \text{since } f(X) \subset Z}}{=} f^{-1}(V) \underset{\substack{\uparrow \\ \text{since } f \text{ is continuous}}}{=} \text{open subset of } X.$$

■

Proposition 2.2. (Basis-Continuity Proposition) Let $f\colon X \to Y$ be a map between topological spaces. Let $\mathcal{C}$ be a basis for the topology of Y. Then the following holds

$$f \text{ is continuous} \quad \Longleftrightarrow \quad \text{for each } C \in \mathcal{C} \text{ the preimage } f^{-1}(C) \text{ is open in } X.$$

Proof. We proved the proposition in Analysis IV. Alternatively see [Fri23]. ■

Lemma 2.3. (Continuity-via-Closed Subsets Lemma) A map $f\colon X \to Y$ between two topological spaces X and Y is continuous if and only if for each closed subset A in Y the preimage $f^{-1}(A)$ is closed in X.

Proof. This lemma is a straightforward consequence of the Image-Preimage Lemma 0.3 (8) and the definitions. ■

Proposition 2.4. (Pasting Proposition) Let $f\colon X \to Y$ be a map between topological spaces.

(1) If there exists an open cover $\{W_i\}_{i\in I}$ of X such that each restriction $f|_{W_i}$ is continuous, then f itself is continuous.

(2) If there exists a finite closed cover $\{A_i\}_{i\in I}$ of X such that each restriction $f|_{A_i}$ is continuous, then f itself is continuous.

In practice often the following variation on (2) gets used:

(2′) If X is Hausdorff and if there exists a locally finite cover $\{A_i\}_{i\in I}$ of X such that each A_i is compact and such that each restriction $f|_{A_i}$ is continuous, then f itself is continuous.

Proof. Statements (1) and (2) of the Pasting Proposition follow fairly easily from the Cover-Open Subset Lemma 1.5. Statement (2′) follows from Statement (2) and the Compact-Closed Lemma 1.9 (2). We refer to [Fri23] for details. ■

Proposition 2.5. (Metric Continuity Proposition) Let $f\colon X \to Y$ be a map between metric spaces. Then the following holds:

$$f \text{ is continuous} \quad \Longleftrightarrow \quad \underset{x\in X}{\forall}\ \underset{\epsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x'\in B_\delta(x)}{\forall}\ d(f(x), f(x')) < \epsilon.$$

Proof. We proved the Metric Continuity Proposition 2.5 in Analysis II. Alternatively see [Fri23]. ■

Example. Let $n \in \mathbb{N}$. For each $i \in \{1, \dots, n\}$ the projection

$$\begin{array}{rcl} \mathbb{R}^n & \to & \mathbb{R} \\ (x_1, \dots, x_n) & \mapsto & x_i \end{array}$$

is continuous since the "$\epsilon - \delta$"-condition is satisfied with $\delta = \epsilon$.

Lemma 2.6. (Sum-Product-Continuity Lemma) Let X be a topological space. If $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ are continuous maps, then the maps

$$\begin{array}{rcl} X & \to & \mathbb{R} \\ x & \mapsto & f(x)+g(x) \end{array} \qquad \begin{array}{rcl} X & \to & \mathbb{R} \\ x & \mapsto & f(x)\cdot g(x) \end{array} \qquad \begin{array}{rcl} X & \to & \mathbb{R} \\ x & \mapsto & \max\{f(x), g(x)\} \end{array} \qquad \begin{array}{rcl} X & \to & \mathbb{R} \\ x & \mapsto & \min\{f(x), g(x)\} \end{array}$$

are also continuous. The obvious analogues of the first two statements also hold if we replace $\mathbb{R}$ by $\mathbb{C}$.

Proof. For brevity's sake we outsource the proof to [Fri23]. ■

2.2. Open and closed maps.

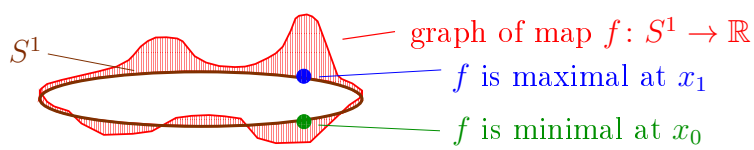
Definition. Let $f\colon X \to Y$ be a continuous map between topological spaces.
(1) We say f is open if the image of every open subset of X is open in Y.
(2) We say f is closed if the image of every closed subset of X is closed in Y.

Lemma 2.7. (Open-Closed Inclusion Lemma) Let X be a topological space.
(1) If $U \subset X$ is an open subset, then the inclusion map $i\colon U \to X$ is open.
(2) If $A \subset X$ is a closed subset, then the inclusion map $i\colon A \to X$ is closed.

Proof.
(1) Suppose that V is an open subset of U. By definition of the subspace topology there exists an open subset $W \subset X$ with $V = U \cap W$. But then $i(V) = V = U \cap W$ is the intersection of two open subsets of X, thus it is an open subset of X.
(2) The proof of this statement is analogous to the proof of Statement (1). ■

2.3. The Compact Image Lemma.

Lemma 2.8. (Compact Image Lemma)
(1) Let $f\colon X \to Y$ be a continuous map. If X is compact, then $f(X)$ is also compact.
(2) Let $f\colon X \to \mathbb{R}$ be a continuous map. If X is compact and non-empty, then f assumes its maximum and its minimum, i.e. there exist $x_0, x_1 \in X$ such that
$$f(x_0) \;\le\; f(x) \;\le\; f(x_1) \qquad \text{for all } x \in X.$$
(3) Let $f\colon X \to Y$ be a continuous map. If X is compact and if Y is Hausdorff, then $f(X)$ is a closed subset of Y.



Proof.
(1) Let $\{U_i\}_{i\in I}$ be an open cover of the topological space $f(X)$. We need to show that $f(X)$ is contained in the union of finitely many U_i's. We have
$$X \;\underset{\substack{\uparrow \\ \text{Image-Preimage Lemma 0.3 (1)}}}{=}\; f^{-1}(f(X)) \;\subset\; f^{-1}\Big(\bigcup_{i\in I} U_i\Big) \;\underset{\substack{\uparrow \\ \text{Image-Preimage Lemma 0.3 (6)}}}{=}\; \bigcup_{i\in I} f^{-1}(U_i).$$
By hypothesis the map $f\colon X \to Y$ is continuous. It follows from the elementary Basics-of-Continuity Lemma 2.1 that the map $f\colon X \to f(X)$ is also continuous, in particular each $f^{-1}(U_i)$ is an open subset of X. This observation together with the above discussion shows that $\{f^{-1}(U_i)\}_{i\in I}$ is an open cover of X. Since X is compact there exist $i_1, \dots, i_k \in I$ such that
$$X \;=\; f^{-1}(U_{i_1}) \cup \cdots \cup f^{-1}(U_{i_k}).$$
Therefore
$$\begin{aligned} f(X) &= f\left(f^{-1}(U_{i_1}) \cup \cdots \cup f^{-1}(U_{i_k})\right) \\ &\underset{\substack{\uparrow \\ \text{Image-Preimage Lemma 0.3 (3)}}}{=} f\left(f^{-1}(U_{i_1})\right) \cup \cdots \cup f\left(f^{-1}(U_{i_k})\right) \;\underset{\substack{\uparrow \\ \text{Image-Preimage Lemma 0.3 (2)}}}{\subset}\; U_{i_1} \cup \cdots \cup U_{i_k}. \end{aligned}$$

(2) It follows from (1) and the Heine-Borel Theorem 1.11 that $f(X) \subset \mathbb{R}$ is bounded and closed. Since X is non-empty we thus see that $y_0 := \inf(f(X))$ and $y_1 := \sup(f(X))$ exist and they lie in $f(X)$. Any choice of points $x_0 \in f^{-1}(y_0)$ and $x_1 \in f^{-1}(y_1)$ have the desired properties.

(3) This statement is an immediate consequence of the first statement and of the Compact-Closed Lemma 1.9 (1). ■

2.4. Homeomorphisms.

Definition.

(1) We say a map $f\colon X \to Y$ between two topological spaces X and Y is a homeomorphism if the following three properties are satisfied:
   (a) f is continuous,
   (b) f is bijective,
   (c) the inverse map $f^{-1}\colon Y \to X$ is also continuous.
   If there exists a homeomorphism between X and Y we say that X and Y are homeomorphic and sometimes we write $X \cong Y$.

(2) We say a map $f\colon X \to Y$ between two topological spaces X and Y is an embedding if $f\colon X \to f(X)$, where $f(X)$ is equipped with the subspace topology, is a homeomorphism. We say the embedding is closed if $f(X)$ is a closed subset of Y and we say the embedding is open if $f(X)$ is an open subset of Y.

Notation. Given $n_0 \in \mathbb{N}$ we write

$$\begin{aligned} S^n_{\geq 0} &:= \{(x_1, \dots, x_{n+1}) \in S^n \mid x_{n+1} \geq 0\} && \text{the upper hemisphere} \\ S^n_{\leq 0} &:= \{(x_1, \dots, x_{n+1}) \in S^n \mid x_{n+1} \leq 0\} && \text{the lower hemisphere} \\ S^n_{=0} &:= \{(x_1, \dots, x_{n+1}) \in S^n \mid x_{n+1} = 0\} && \text{the equator.} \end{aligned}$$

Similarly we define

$$\begin{aligned} B^n_{\geq 0} &:= \{(x_1, \dots, x_n) \in B^n \mid x_n \geq 0\} \\ B^n_{\leq 0} &:= \{(x_1, \dots, x_n) \in B^n \mid x_n \leq 0\} \end{aligned}$$

Finally, the same way we also define $\overline{B}^n_{\geq 0}$ and $\overline{B}^n_{\leq 0}$.

Examples.

(1) The maps

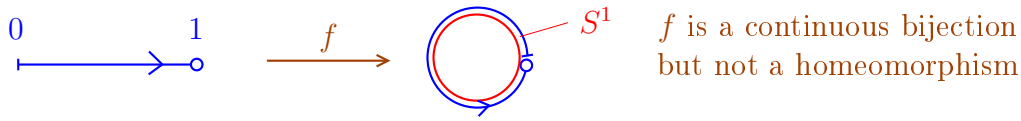
$$\begin{aligned} \varphi\colon \overline{B}^n &\to S^n_{\geq 0} \\ x &\mapsto \left(x, \sqrt{1 - \|x\|^2}\right) \end{aligned} \qquad \text{and} \qquad \begin{aligned} p\colon S^n_{\geq 0} &\to \overline{B}^n \\ (x_1, \dots, x_n, x_{n+1}) &\mapsto (x_1, \dots, x_n) \end{aligned}$$

are easily seen to be continuous and inverses to one another. Thus both maps are homeomorphisms. The same way we see that the lower hemisphere is also homeomorphic to the closed ball $\overline{B}^n$.

(2) We consider the map

$$\begin{aligned} f\colon [0,1) &\to S^1 \\ t &\mapsto \exp(2\pi i t). \end{aligned}$$

The map f is a continuous bijection but it is not a homeomorphism. One can see this by either directly showing that f^{-1} is not continuous or by noting that $[0,1)$ is not compact whereas S^1 is compact. The importance of this example cannot be overestimated. It is an unfortunate fact of life that all too often one writes down a continuous map $f\colon X \to Y$ between topological spaces, checks that it is a bijection, and walks away under the warm, but mistaken, impression that one now has found a homeomorphism.



Lemma 2.9. (Open-Injective Lemma) Let $f\colon X \to Y$ be a continuous map between topological spaces. We assume that f is open or that f is closed.

(1) If f is an injection, then f is an open (respectively closed) embedding.
(2) If f is a bijection, then f is a homeomorphism.

Proof. Let $f\colon X \to Y$ be a continuous map between topological spaces.

(2) We assume that f is a bijection. Furthermore we now assume that f is open. We want to show that f is a homeomorphism. It remains to show that $g := f^{-1}\colon Y \to X$ is continuous. Thus let $U \subset X$ be an open subset. We need to show that $g^{-1}(U)$ is an open subset of Y. But this is obvious, since $g^{-1}(U) = f(U)$ is an open subset since we assume that f is an open map.

Now suppose that the map f is closed. The proof is almost identical to the above argument, except that now we have to use the continuity criterion provided by the Continuity-via-Closed Subsets Lemma 2.3.

(1) We assume that f is an injection and we assume that f is open. We write $Z := f(X)$. Since $f\colon X \to Y$ is open it follows basically immediately from the definition of the subspace topology that $f\colon X \to Z$ is also open. Thus we obtain from (2) that f is an embedding. Furthermore, since f is open we see that f is an open embedding. The proof in the case that f is closed is basically identical. ■

Proposition 2.10. (Compact-Hausdorff Proposition) Let $f\colon X \to Y$ be a continuous map between topological spaces. If X is compact and if Y is Hausdorff, then the following statements hold:

(1) The map $f\colon X \to Y$ is closed.
(2) If f is an injection, then f is a closed embedding.
(3) If f is a bijection, then f is a homeomorphism.

Proof. Let $f\colon X \to Y$ be a continuous map between topological spaces. We assume that X is compact and that Y is Hausdorff.

(1) Let A be a closed subset of X. We need to show that $f(A)$ is a closed subset of Y. In fact we see that

$$A \subset X \text{ closed} \underset{\substack{\uparrow \\ \text{by the Compact-Closed} \\ \text{Lemma 1.9 since } X \text{ is compact}}}{\Longrightarrow} A \text{ is compact} \underset{\substack{\uparrow \\ \text{by the Compact Image} \\ \text{Lemma 2.8 (1)}}}{\Longrightarrow} f(A) \text{ is compact} \underset{\substack{\uparrow \\ \text{by the Compact-Closed Lemma 1.9 (2)} \\ \text{since } Y \text{ is Hausdorff}}}{\Longrightarrow} f(A) \subset Y \text{ is closed.}$$

(2) If f is an injection, then it follows immediately from (1) together with the Open-Injective Lemma 2.9 (1) that f is a closed embedding.
(3) This follows immediately from (2). ■

Lemma 2.11. (Stereographic Projection Lemma) Let $n \in \mathbb{N}$. We consider the map

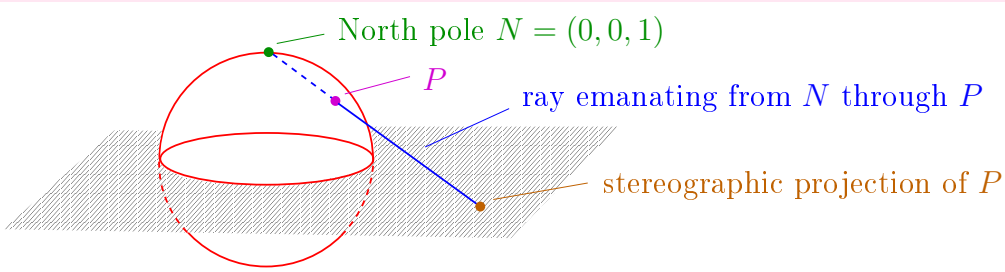
$$\begin{array}{rcl} \Phi\colon S^n & \to & \mathbb{R}^n \cup \{\infty\} \\ (x_1, \ldots, x_{n+1}) & \mapsto & \begin{cases} \left(\frac{x_1}{1-x_{n+1}}, \ldots, \frac{x_n}{1-x_{n+1}}\right), & \text{if } x_{n+1} < 1, \\ \infty, & \text{if } x_{n+1} = 1. \end{cases} \end{array}$$

where we equip $\mathbb{R}^n \cup \{\infty\}$ with the topology that we introduced on page 33. We refer to Φ as the stereographic projection. This map has the following properties:

(1) The map Φ sends the North Pole $N = (0, \ldots, 0, 1) \in S^n$ to ∞.
(2) For any $P \in S^n$ that does not equal the North Pole N the point $\Phi(P) \in \mathbb{R}^n$ is the unique point such that the ray emanating from N and that goes through P intersects the plane $\mathbb{R}^n \times \{0\}$ in $(\Phi(P), 0)$.
(3) For any $(v_1, \ldots, v_n, 0) \in S^n \cap (\mathbb{R}^n \times \{0\})$ we have $\Phi(v_1, \ldots, v_n, 0) = (v_1, \ldots, v_n)$.
(4) The map Φ is a homeomorphism.
(5) (a) The restriction of Φ to a map $S^n \setminus \{N\} \to \mathbb{R}^n$ is also a homeomorphism, where the inverse is given by

$$\begin{array}{rcl} \mathbb{R}^n & \to & S^n \setminus \{N\} \\ x & \mapsto & \left(x \cdot \frac{2}{\|x\|^2+1}, \underbrace{\frac{\|x\|^2-1}{\|x\|^2+1}}_{=1-\frac{2}{\|x\|^2+1}}\right). \end{array}$$

(b) The restriction of Φ to a map $S^n_{\leq 0} \to \overline{B}^n$ is a homeomorphism.



Identification. Let $n \in \mathbb{N}$. On many occasions we will use the homeomorphism from the Stereographic Projection Lemma 2.11 to make the identification $S^n =_i \mathbb{R}^n \cup \{\infty\}$. In particular we have the identification $S^2 =_i \mathbb{C} \cup \{\infty\}$.

Proof.

(1) This statement follows immediately from the definition of Φ.
(2) This statement follows from an elementary calculation which we leave to the reader.
(3) This statement follows immediately from the definition of Φ.
(4) Using the Subbasis-Continuity Proposition 2.2 one can easily show that Φ is continuous. We leave it to the reader to verify that Φ is a bijection. Note that S^n is compact by the Heine-Borel Theorem 1.11. Furthermore note that on page 33 we pointed out that $\mathbb{R}^n \cup \{\infty\}$ is Hausdorff. Hence it follows from the Compact-Hausdorff Proposition 2.10 (3) that Φ is a homeomorphism.

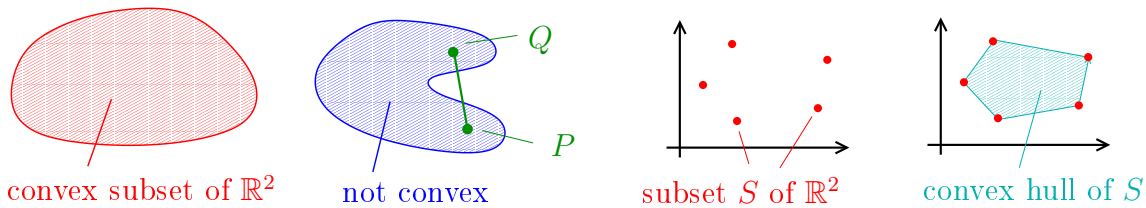
(5) It follows almost immediately from (4) that the two maps are homeomorphism. One can easily verify that the inverse of Φ is indeed given by the map we provided. ■

2.5. Convex subsets.

Definition.

(1) We say that a subset $A \subset \mathbb{R}^n$ is convex, if for every two points P and Q in A the segment $\overline{PQ} := \{t \cdot P + (1-t) \cdot Q \,|\, t \in [0,1]\}$ also lies in A.

(2) Given a subset S of $\mathbb{R}^k$ the convex hull of S is defined as the intersection of all convex subsets of $\mathbb{R}^k$ that contain S. Since the intersection of convex sets is again convex we see that the convex hull of S is a convex subset of $\mathbb{R}^n$.



Examples.

(1) For any point $x \in \mathbb{R}^n$ and any $\epsilon > 0$ the open ball $B^n_\epsilon(x)$ is convex. Indeed, let $P, Q \in B^n_\epsilon(x)$. Then for any $t \in [0,1]$ we have

$$\begin{aligned} \|t\cdot P + (1-t)\cdot Q - x\| &= \|t\cdot(P-x) + (1-t)\cdot(Q-x)\| \\ &\underset{\substack{\uparrow \\ \text{triangle inequality and homogeneity of } \|-\|}}{\leq} t\cdot\|P-x\| + (1-t)\cdot\|Q-x\| \underset{\substack{\uparrow \\ \text{since } P,Q \in B^n_\epsilon(x)}}{<} t\cdot\epsilon + (1-t)\cdot\epsilon = \epsilon. \end{aligned}$$

We have thus shown that $t \cdot P + (1-t) \cdot Q \in B^n_\epsilon(x)$. Basically the same way one shows that the closed ball $\overline{B}^n_\epsilon(x)$ is also convex.

(2) It is straightforward to verify that any hyperrectangle $[a_1, b_1] \times \cdots \times [a_n, b_n]$ is convex.

Proposition 2.12. (Convex-to-Ball Proposition)

(1) Let A be a bounded *open* convex subset of $\mathbb{R}^n$ and let $Q \in A$. We consider the map

$$\begin{aligned} \rho_A \colon A \setminus \{Q\} &\to \mathbb{R}_{\geq 0} \\ x &\mapsto \sup\left\{\|r \cdot x\| \,\middle|\, r \in \mathbb{R}_{>0} \text{ and } r \cdot (x - Q) + Q \in A\right\}. \end{aligned}$$

This map is well-defined in the sense that the supremum exists for each $x \in A \setminus \{Q\}$. Furthermore the map ρ_A is continuous and the map

$$\begin{aligned} f_Q \colon A &\to B^n \\ x &\mapsto \begin{cases} (x-Q) \cdot \frac{1}{\rho_A(x)}, & \text{if } x \neq Q, \\ 0, & \text{if } x = Q. \end{cases} \end{aligned}$$

is a homeomorphism with $f_Q(Q) = 0$.

(2) Let A be a bounded *closed* convex subset of $\mathbb{R}^n$. If Q is a point in the interior of A, then the same expression as in (1) defines a homeomorphism

$$f_Q \colon A \to \overline{B}^n = \{x \in \mathbb{R}^n \,|\, \|x\| \leq 1\}$$

with the following properties:
(a) the homeomorphism f_Q sends Q to the origin,
(b) the homeomorphism f_Q restricts to a homeomorphism $\partial A \to S^{n-1}$.

Lemma 2.13. Let A be a bounded closed convex subset of $\mathbb{R}^n$ such that the origin 0 lies in the interior of A.

(1) The map $\rho_A \colon A \setminus \{0\} \to \mathbb{R}_{\geq 0}$ defined in the Convex-to-Ball Proposition 2.12 is continuous.
(2) The map
$$\begin{array}{rcl} f_0 \colon A & \to & \overline{B}^n \\ x & \mapsto & \begin{cases} x \cdot \frac{1}{\rho_A(x)}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases} \end{array}$$
is continuous and it satisfies $f_0^{-1}(S^{n-1}) = \partial A$.

Proof of Lemma 2.13. We start out with the following claim.

Claim. The map $\rho_A \colon A \setminus \{0\} \to \mathbb{R}$ is defined, it is continuous, it is bounded from above and it is bounded from below by 1.

Proof. Since A is bounded it follows that ρ_A is defined and that it is bounded from above. It follows from the definition of ρ_A that ρ_A is bounded from below by 1. Therefore it remains to show that ρ_A is continuous. Now let $x \in A \setminus \{0\}$ and let $\epsilon > 0$. It suffices to show the following two statements:

(1) there exists a neighborhood U of $x \in A \setminus \{0\}$ with $\rho_A(y) > \rho_A(x) - \epsilon$ for all $y \in U$,
(2) there exists a neighborhood V of $x \in A \setminus \{0\}$ with $\rho_A(y) < \rho_A(x) + \epsilon$ for all $y \in V$.

We first show the existence of U. After possibly replacing ϵ by $\min\{\frac{1}{2}\rho_A(x), \epsilon\}$ we can suppose that $\epsilon \in (0, \rho_A(x))$.

Since A is closed we have $f_0(x) \in A$. Since 0 lies in the interior of A there exists an $\eta > 0$ such that $B^n_\eta(0) \subset A$. Since A is convex, the convex hull C of $B^n_\eta(0) \cup \{f_0(x)\}$ is still contained in A. Furthermore, since $B^n_\eta(0)$ is open it is straightforward to see that $C' := C \setminus \{f_0(x)\}$ is an open subset of $\mathbb{R}^n$. We denote by $S^{n-1}_{\rho_A(x)-\epsilon}$ the sphere of radius $\rho_A(x) - \epsilon$ around 0. The point $x \cdot \frac{\rho_A(x)-\epsilon}{\|x\|}$ lies on $S^{n-1}_{\rho_A(x)-\epsilon}$ and it lies in C'. Since C' is open there exists an open neighborhood U' on $S^{n-1}_{\rho_A(x)-\epsilon}$ that is contained in C'. We set
$$U \ := \ \{r \cdot z \,|\, z \in U' \text{ and } r \in (0,1)\}.$$
This is an open neighborhood of x and for any $y \in U$ we have $(\rho_A(x) - \epsilon) \cdot y \in A$, i.e. for any $y \in A$ we have $\rho_A(y) \geq \rho_A(x) - \epsilon$. Thus we have shown that the desired neighborhood U exists. The existence of V is proved in a very similar way. We refer to [Ber09, Chapter 11.3] for full details. ⊞

From the continuity of ρ_A it follows that f is continuous on $A \setminus \{0\}$. From the fact that ρ_A is bounded it follows easily that f_Q is continuous in 0. Finally one can easily verify, using the convexity of A, that $f_0^{-1}(S^{n-1}) = \partial A$. ■

Proof of the Convex-to-Ball Proposition 2.12.

(2) Let A be a bounded closed convex subset of $\mathbb{R}^n$ and let Q be a point in the interior of A. After a translation we can assume that $Q = 0$ is the origin, in particular we can

assume that the origin 0 lies in the interior of A. By Lemma 2.13 we know that the map $\rho_A\colon A \setminus \{0\} \to \mathbb{R}_{\geq 0}$ is defined and continuous. Next we consider the map

$$\begin{array}{rcl} f_0\colon A & \to & \overline{B}^n \\ x & \mapsto & \begin{cases} x \cdot \frac{1}{\rho_A(x)}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \end{array}$$

It is straightforward to verify that this map is bijective and by Lemma 2.13 we know that f_0 is continuous and that $f_0^{-1}(S^{n-1}) = \partial A$. Note that our hypothesis on A and the Heine-Borel Theorem 1.11 imply that A is compact. Thus we obtain from the Compact-Hausdorff Proposition 2.10 (3) that f_0 is in fact a homeomorphism. Finally note that it follows from $f_0^{-1}(S^{n-1}) = \partial A$ that f_0 restricts to a homeomorphism $\partial A \to S^{n-1}$.

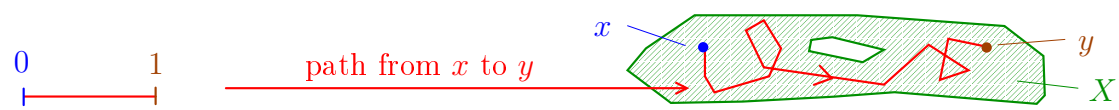
(1) We denote by $C = \overline{A}$ the closure of A. Since A is open and convex it follows fairly easily that $A = \mathring{C}$. One verifies fairly easily that C is convex. It is clear that C is bounded and it follows from the Interior-Closure Lemma 1.6 that C is closed. One verifies easily that the map f_0 defined for $C = \overline{A}$ restricts to the map f_0 defined for A. Thus it follows from (2) that the map $f_0\colon C = \overline{A} \to \overline{B}^n$ is a homeomorphism and it restricts to a homeomorphism $f_0\colon \mathring{C} \to B^n$. Since $\mathring{C} = A$ we get the desired homeomorphism. ■

2.6. The two notions of connected spaces.

Definition.

(1) Let X be a topological space. A path is a continuous map $p\colon [0,1] \to X$. Often we say that p is a path from $p(0)$ to $p(1)$.

(2) We say that a topological space X is path-connected if given any two points x and y there exists a path from x to y.



Definition. We say that a topological space X is connected if the following holds

$$\text{if } X = U \sqcup V \text{ is the disjoint union of open subsets } U, V \quad \Longrightarrow \quad X = U \text{ or } X = V.$$

A topological space which is not connected is called disconnected.

Example.

(1) It follows basically from the definitions that any convex subset of some $\mathbb{R}^n$ is path-connected. But it is much less clear whether say $\mathbb{R}^n$ is connected in the above sense.

(2) It follows from the Intermediate Value Theorem that the rational numbers $\mathbb{Q}$ are not path-connected. But $\mathbb{Q}$ is also not connected, for example we can write the topological space $\mathbb{Q}$ as the disjoint union of the two open non-empty subsets $\{x \in \mathbb{Q} \mid x^2 < 2\}$ and $\{x \in \mathbb{Q} \mid x^2 > 2\}$.

So we see that both notions of connectedness have their advantages and disadvantages. It is often easy to show that a given topological space is path-connected but it can be a pain to show that a given topological space is not path-connected. With connectedness it is precisely the other way round.

Proposition 2.14. (Interval-Connected Proposition) The interval $[0,1]$ is connected.

Proof. We proved the proposition in Analysis II, alternatively see [Fri23]. ■

Corollary 2.15. (Path-Connected Implies Connected Corollary) Every path-connected topological space is also connected.

Proof. Let X be a topological space that is path-connected. Suppose that X is not connected. This means that there exist disjoint open non-empty subsets U and V with $X = U \cup V$. Since U and V are non-empty we can find points $x \in U$ and $y \in V$. Since X is path-connected there exists a path $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Then $\gamma^{-1}(U) \cup \gamma^{-1}(V)$ is a decomposition of $[0,1]$ into two disjoint open non-empty subsets. But that is a contradiction to the Interval-Connected Proposition 2.14. ■

Lemma 2.16. (Image-Connected Lemma) Let $f\colon X \to Y$ be a continuous map between topological spaces.

(1) If X is path-connected, then $f(X)$ is also path-connected.
(2) If X is connected, then $f(X)$ is also connected.

Proof.

(1) We suppose that X is path-connected. Let $y_0 = f(x_0)$ and $y_1 = f(x_1)$ be two points in $f(X)$. Since X is path-connected there exists a path $\gamma\colon [0,1] \to X$ from x_0 to x_1. But then $f \circ \gamma\colon [0,1] \to f(X)$ is a path from $f(x_0)$ to $f(x_1)$.
(2) We suppose that $f(Y)$ is disconnected. This means that there exist disjoint open non-empty subsets U and V of $f(Y)$ with $f(Y) = U \cup V$. By the Image-Preimage Lemma 0.3 we know that $f^{-1}(U)$ and $f^{-1}(V)$ are also disjoint and we know that $X = f^{-1}(U) \cup f^{-1}(V)$. By definition of a continuous map $f^{-1}(U)$ and $f^{-1}(V)$ are open. Finally, since $U, V \subset f(X)$ we see that $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty. Thus we have shown that X is also disconnected. ■

Lemma 2.17. (Ball-Sphere-Connected Lemma)

(1) Let $n \geq 1$. Every open ball $B_r^n(y)$, every closed ball $\overline{B}_r^n(y)$ and $\mathbb{R}^n$ are path-connected and connected.
(2) Every sphere $S_r^n(y)$ with $n \geq 1$ is path-connected and connected.

Proof. By the Path-Connected Implies Connected Corollary 2.15 it suffices to show that all these topological spaces are path-connected.

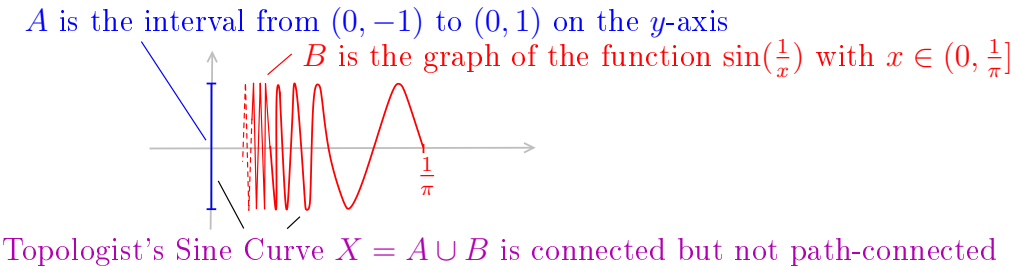
(1) It follows from the example on page 43 that all balls (open or closed) and $\mathbb{R}^n$ are convex. But as we already pointed out on page 45, convex subsets are basically by definition path-connected.
(2) We leave it to the reader to show that spheres of dimension ≥ 1 are path-connected. ■

Example. The *Topologist's Sine Curve* is defined as the following topological space

$$X \;:=\; \big\{(0,y)\,\big|\,y \in [-1,1]\big\} \cup \big\{(x, \sin(\tfrac{1}{x}))\,\big|\,x \in (0, \tfrac{1}{\pi}]\big\} \;\subset\; \mathbb{R}^2,$$

equipped with the subspace topology. In other words, the Topologist's Sine Curve is the union of the graph of $x \mapsto \sin(\frac{1}{x})$ for $x \in (0,\pi]$ and an interval on the y-axis. It is a very

instructive exercise to show that X is connected and it is an equally instructive exercise to show that X is *not* path-connected.

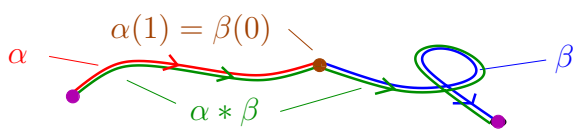


Lemma 2.18. (Connected-to-Discrete Lemma) Every continuous map $f\colon X \to Y$ from a connected space X to a discrete space is constant.

Proof. We need to show that for any $x \in X$ we have $f(\{x\}) = f(X)$. Since Y is a discrete space we know that $\{f(x)\}$ and $Y \setminus \{f(x)\}$ are open subsets of Y. The preimages $U := f^{-1}(f(x))$ and $V := f^{-1}(Y \setminus \{f(x)\})$ are also open and they are evidently disjoint. Since U is non-empty and since X is connected we see that $V = \varnothing$. But this means that f is constant. ■

Definition. Let X be a topological space and let $\alpha\colon [0,1] \to X$ and $\beta\colon [0,1] \to X$ be two paths with $\alpha(1) = \beta(0)$. We define the concatenation of α and β as the path $\alpha * \beta$ which is given by

$$\begin{array}{rcl} \alpha * \beta\colon [0,1] & \to & X \\ t & \mapsto & \begin{cases} \alpha(2t), & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta((2t-1)), & \text{if } t \in \left(\frac{1}{2}, 1\right]. \end{cases} \end{array}$$



Lemma 2.19. (Union Connected Lemma) Let X be a topological space and let $\{B_i\}_{i \in I}$ be a family of subsets such that the intersection $\bigcap_{i \in I} B_i$ is non-empty.

(1) If each B_i, $i \in I$ is path-connected, then $\bigcup_{i \in I} B_i$ is path-connected.

(2) If each B_i, $i \in I$ is connected, then $\bigcup_{i \in I} B_i$ is connected.

Proof. Let X be a topological space and let $\{B_i\}_{i \in I}$ be a family of subsets such that the intersection $\bigcap_{i \in I} B_i$ contains a point z. We write $Z := \bigcup_{i \in I} B_i$.

(1) We suppose that each B_i, $i \in I$ is path-connected. So let $x, y \in Z$. There exists an $i \in I$ with $x \in B_i$ and there exists a $j \in I$ with $y \in B_j$. Since B_i and B_j are path-connected we can pick a path $\gamma\colon [0,1] \to B_i$ from x to z and we can pick a path $\delta\colon [0,1] \to B_j$ from z to y. Note that the concatenation of the paths γ and δ is a path from x to y.

(2) We suppose that each B_i, $i \in I$ is connected. We write $Z = U \sqcup V$ where U and V are open subsets. Without loss of generality we can assume that $z \in U$. For each $i \in I$ we have a decomposition $B_i = (U \cap B_i) \sqcup (V \cap B_i)$ into disjoint open subsets (here we use

the definition of the subspace topology). Since B_i is connected and since $U \cap B_i \neq \varnothing$ we see that $B_i = U \cap B_i$. Thus we have shown that

$$U \;=\; \bigcup_{i\in I} (B_i \cap U) \;=\; \bigcup_{i\in I} B_i \;=\; Z. \qquad \blacksquare$$

2.7. The (path-) components of a topological space.

Definition. Let x and y be two points in a topological space X. We say that the points x and y are path-equivalent, if there exists a continuous map $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Lemma 2.20. (Path-Equivalence Lemma) For every topological space X the notion of path-equivalence is indeed an equivalence relation.

Proof. All properties of an equivalence relation are trivial except possibly for transitivity. The fact that path-equivalence is transitive can be shown by concatenating paths. ■

Definition. Let X be a topological space.

(1) We call the path-equivalence classes of points the path-components of X.
(2) We denote by $\pi_0(X)$ the set of path-components of X. Furthermore we write $\pi_0(X) = 0$ if X is non-empty and if it consists of a single path-component.

Example. It is a straightforward consequence of the Intermediate Value Theorem that each point in $\mathbb{Q}$ defines a path-component of $\mathbb{Q}$.

Lemma 2.21. (Path-Component Lemma) Let X be a topological space.

(1) As a set, X is the disjoint union of its path-components.
(2) Each path-component of X is path-connected.
(3) Each path-component C of X is a maximal path-connected subset, i.e. if $C \subsetneq D$, then D is not path-connected.
(4) Let A be a subset of X.
    (a) If no point in A is path-equivalent to a point in $X \setminus A$, then A is the union of path-components of X.
    (b) If A is path-connected, then A is contained in a path-component of X.
    (c) If A is path-connected and if no point in A is path-equivalent to a point in $X \setminus A$, then A is a path-component of X.

Proof. The lemma follows immediately from the definitions and basic properties of equivalence relations, see the Equivalence Class Lemma 0.4. ■

Example. We consider the topological space $\mathbb{R} \setminus \{0\}$. The subsets $\mathbb{R}_{<0}$ and $\mathbb{R}_{>0}$ are evidently path-connected. It follows from the Intermediate Value Theorem that no point in $\mathbb{R}_{<0}$ is path-equivalent to a point in $\mathbb{R}_{>0}$. Thus it follows from the Path-Component Lemma 2.21 (4c) that $\mathbb{R}_{<0}$ and $\mathbb{R}_{>0}$ are the path-components of $\mathbb{R} \setminus \{0\}$.

2.8. The path-components of matrix groups.

Notation. Given a commutative ring R we write

$$\begin{aligned} \mathrm{GL}(n,R) &:= \{A \in \mathrm{M}(n\times n, R) \mid \det(A) \text{ is a multiplicative unit in } R\} \\ \mathrm{SL}(n,R) &:= \{A \in \mathrm{M}(n\times n, R) \mid \det(A) = 1\}. \end{aligned}$$

For $\mathbb{K} = \mathbb{R}$ we have the following extra definitions:

$$\begin{aligned}
\mathrm{GL}_+(n,\mathbb{R}) &:= \{A \in \mathrm{M}(n\times n,\mathbb{R}) \mid \det(A) > 0\} \\
\mathrm{GL}_-(n,\mathbb{R}) &:= \{A \in \mathrm{M}(n\times n,\mathbb{R}) \mid \det(A) < 0\} \\
\mathrm{O}(n) &:= \{A \in \mathrm{M}(n\times n,\mathbb{R}) \mid A^T\cdot A = \mathrm{id}\} \\
\mathrm{SO}(n) &:= \mathrm{O}(n)\cap \mathrm{SL}(n,\mathbb{R}) \;=\; \{A \in \mathrm{M}(n\times n,\mathbb{R}) \mid A^T\cdot A = \mathrm{id} \text{ and } \det(A) = 1\}.
\end{aligned}$$

Furthermore for $\mathbb{K} = \mathbb{C}$ we have the following extra definitions:

$$\begin{aligned}
\mathrm{U}(n) &:= \{A \in \mathrm{M}(n\times n,\mathbb{C}) \mid \overline{A}^T\cdot A = \mathrm{id}\} \\
\mathrm{SU}(n) &:= \mathrm{U}(n)\cap \mathrm{SL}(n,\mathbb{C}) \;=\; \{A \in \mathrm{M}(n\times n,\mathbb{C}) \mid \overline{A}^T\cdot A = \mathrm{id} \text{ and } \det(A) = 1\}.
\end{aligned}$$

Identification. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Given $m, n \in \mathbb{N}$ we make the identification

$$\mathrm{M}(m\times n,\mathbb{K}) \;=_i\; \mathbb{K}^{m\cdot n}$$

by "juxtaposing" the rows of the matrix. We equip $\mathbb{K}^{m\cdot n}$ and thus also $\mathrm{M}(m\times n,\mathbb{K})$ with the standard topology that we introduced on page 31. We equip each subset of $\mathrm{M}(m\times n,\mathbb{K})$ with the corresponding subspace topology that we introduced on page 31.

Lemma 2.22. (Matrix Group Lemma) Let $n \in \mathbb{N}$.

(1) $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}_+(n,\mathbb{R})$ and $\mathrm{GL}_-(n,\mathbb{R})$, are open subsets of $\mathrm{M}(n\times n,\mathbb{R})$.
(2) $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{O}(n)$ and $\mathrm{SO}(n)$, are closed subsets of $\mathrm{M}(n\times n,\mathbb{R})$.
(3) $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are compact.
(4) $\mathrm{GL}(n,\mathbb{C})$ is an open subset of $\mathrm{M}(n\times n,\mathbb{C})$.
(5) $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$, are closed subsets of $\mathrm{M}(n\times n,\mathbb{C})$.
(6) $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are compact.

Proof. We consider the following two maps

$$\begin{aligned}
\det\colon \mathrm{M}(n\times n,\mathbb{R}) &\to \mathbb{R} \\
A &\mapsto \det(A)
\end{aligned}
\quad\text{and}\quad
\begin{aligned}
\varphi\colon \mathrm{M}(n\times n,\mathbb{R}) &\to \mathrm{M}(n\times n,\mathbb{R}) \\
A &\mapsto A^T\cdot A.
\end{aligned}$$

It follows from the Leibniz formula and the rather elementary the Sum-Product-Continuity Lemma 2.6 that both maps are continuous.

(1) It follows from the above that $\mathrm{GL}(n,\mathbb{R}) = \det^{-1}(\mathbb{R}\setminus\{0\})$, $\mathrm{GL}_+(n,\mathbb{R}) = \det^{-1}((0,\infty))$ and $\mathrm{GL}_-(n,\mathbb{R}) = \det^{-1}((-\infty,0))$ are open subsets of $\mathrm{M}(n\times n,\mathbb{R})$.
(2) Note that it follows from the above and the Continuity-via-Closed Subsets Lemma 2.3 that $\mathrm{SL}(n,\mathbb{R}) = \det^{-1}(\{1\})$, $\mathrm{O}(n) = \varphi^{-1}(\{\mathrm{id}\})$ and $\mathrm{SO}(n) = \mathrm{SL}(n,\mathbb{R})\cap \mathrm{O}(n,\mathbb{R})$ are closed subsets of $\mathrm{M}(n\times n,\mathbb{R})$.
(3) Note that each column of an orthogonal matrix has norm one. This implies that $\mathrm{O}(n) \subset \mathbb{R}^{n^2}$ is a bounded subset of $\mathrm{M}(n\times n,\mathbb{R}) = \mathbb{R}^{n^2}$. It follows from this observation together with (2) and the Heine-Borel Theorem 1.11 that $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are compact.

The proofs of Statements (4), (5) and (6) are almost identical to the above proofs. ■

Proposition 2.23. (Matrix Group-π_0-Proposition) Let $n \in \mathbb{N}$.

	topological space	path-components
(1)	$\mathrm{GL}(n,\mathbb{R})$	path-components are $\mathrm{GL}_+(n,\mathbb{R})$ and $\mathrm{GL}_-(n,\mathbb{R})$
(2)	$\mathrm{SL}(n,\mathbb{R})$	path-connected
(3)	$\mathrm{O}(n)$	path-components are $\mathrm{SO}(n)$ and $\mathrm{O}(n) \cap \mathrm{GL}_-(n,\mathbb{R})$
(4)	$\mathrm{GL}(n,\mathbb{C})$	path-connected
(5)	$\mathrm{SL}(n,\mathbb{C})$	path-connected
(6)	$\mathrm{U}(n)$	path-connected
(7)	$\mathrm{SU}(n)$	path-connected.

Proof. The proof is provided in [Fri23]. ■

Proof of Proposition 2.23 (4), (5), (6) and (7).

Claim 4.

(1) Given any $A \in \mathrm{GL}(n,\mathbb{C})$ there exists a path $\delta\colon [0,1] \to \mathrm{GL}(n,\mathbb{C})$ with $\gamma(0) = A$ and $\gamma(1) \in \mathrm{SL}(n,\mathbb{C})$.

(2) If $A \in \mathrm{U}(n)$, then we can find such a path that takes values in $\mathrm{U}(n)$.

Proof. Let $A \in \mathrm{GL}(n,\mathbb{C})$. We pick $r \in \mathbb{R}_{>0}$ and $\varphi \in \mathbb{R}$ with $\det(A) = r \cdot \exp(\mathrm{i}\varphi)$. The map

$$\begin{array}{rcl} \delta\colon [0,1] & \to & \mathrm{GL}(n,\mathbb{C}) \\ t & \mapsto & r^{-t/n} \cdot \exp(-\mathrm{i}\varphi \cdot t/n) \cdot A \end{array}$$

is a path from A to a matrix in $\mathrm{SL}(n,\mathbb{C})$. If $A \in \mathrm{U}(n)$, then $r = 1$ and we see that the path takes values in $\mathrm{U}(n)$. ⊞

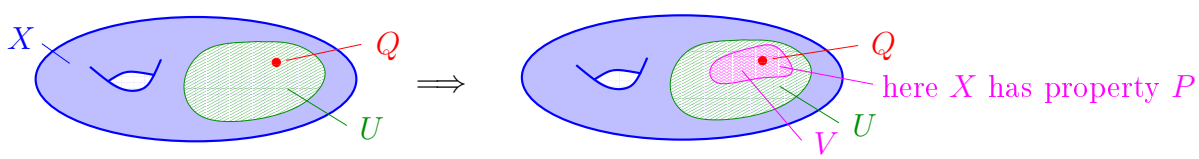
Using Claim 4 one sees that it suffices to show that $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{SU}(n)$ are connected. This is achieved by making straightforward modifications to the Claims 2 and 3 that we considered in the proof of the real case. ■

2.9. Local and regional properties.

Definition. Let P be a property of topological spaces and let X be a topological space.

(1) We say X is regionally P if given any $Q \in X$ and any neighborhood U of Q there exists a neighborhood V of Q that is contained in U that has the property P.

(2) We say X is locally P if given any $Q \in X$ and any neighborhood U of Q there exists an *open* neighborhood V of Q that is contained in U that has the property P.



Examples.

(1) Every open subset of $\mathbb{R}^n$ is locally path-connected. Indeed, suppose A is an open subset of $\mathbb{R}^n$. Let $Q \in A$ and let U be a neighborhood of Q in A. Then there exists an $\epsilon > 0$ such that $B^n_\epsilon(x) \subset U$. The ϵ-ball $B^n_\epsilon(x)$ is of course path-connected.

(2) The topological space $X = (-2,1) \cup [3,6] \subset \mathbb{R}$ with the usual topology is not path-connected but it is locally path-connected. On the other hand $X = \mathbb{Q}$ is not even locally path-connected.

Lemma 2.24. (Locally Path-Connected Lemma) Let X be a topological space. If X is locally path-connected, then X is connected if and only if X is path-connected.

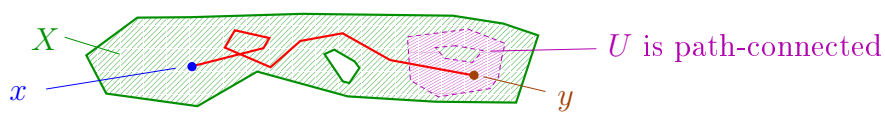
Proof. First note that if X is path-connected, then we know by the Path-Connected Implies Connected Corollary 2.15 that X is connected. This gives us the cheap "if"-direction.

We now assume that X is connected. We start out with the following claim.

Claim 1. Each path-component of X is open.

Proof. Let P be a path-component. We pick $x \in X$. To show that P is open it suffices by the JH-Openness Criterion 1.1 to show that each $y \in P$ admits a neighborhood that is contained in P.

Thus let $y \in P$. We pick a path $p\colon [0,1] \to X$ with $p(0) = x$ and $p(1) = y$. Since X is locally path-connected there exists an open neighborhood U of y that is path-connected. It follows from the Image-Connected Lemma 2.16 (1) together with the Union Connected Lemma 2.19 that $p([0,1]) \cup U$ is path-connected. It follows from the Path-Component Lemma 2.21 (4b) that $p([0,1]) \cup U \subset P$. Finally note that since U is open we know that $p([0,1]) \cup U$ is a neighborhood of x. ⊞



Claim 2. Each path-component of X is closed.

Proof. By Claim 1 we now know that each path-component is open. By the Path-Component Lemma 2.21 (1) we know that the complement of a path-component is the union of path-components. Since the union of open subsets is open we see that each path-component is also closed. ⊞

Since X is connected we obtain from Claim 1 and Claim 2 that each path-component equals X, i.e. X is path-connected. ■

Examples.

(1) Every open subset of $\mathbb{R}^n$ is regionally compact. Indeed, suppose A is an open subset of $\mathbb{R}^n$. Let $Q \in A$ and let U be a neighborhood of Q in A. Then there exists an $\epsilon > 0$ such that $B^n_\epsilon(x) \subset U$. The closed ϵ-ball $\overline{B}^n_{\frac{\epsilon}{2}}(x)$ is a neighborhood of x and it is compact by the Heine-Borel Theorem 1.11.

(2) Similar to (1) one can easily show that every interval is regionally compact. In fact later on we will mostly need the fact that the interval $[0,1]$ is regionally compact.

Definition. Let P be a property of maps between topological spaces and let $f\colon X \to Y$ be a map between topological spaces.

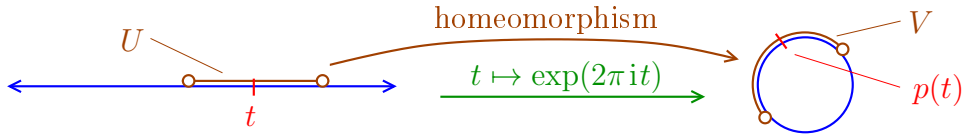
(1) We say that $f\colon X \to Y$ is locally P at $x \in X$ if there exists an open neighborhood U of x and an open neighborhood V of $f(x)$ such that $f\colon U \to V$ has the property P.

(2) We say that f is locally P if it is locally P at any $x \in X$.

Example. The map

$$\begin{array}{rcl} p\colon \mathbb{R} & \to & S^1 \\ t & \mapsto & \exp(2\pi \mathrm{i} t) \end{array}$$

is a local homeomorphism. Indeed, let $t \in \mathbb{R}$. We set $U = (t - \frac{1}{4}, t + \frac{1}{4})$ and $V = p(U)$. The restriction of p to $p\colon U \to V$ is continuous and a bijection. Furthermore it is not difficult to show that p is an open map. It follows from the Open-Injective Lemma 2.9 (2) that $p\colon U \to V$ is a homeomorphism.



3. Categories, functors and natural transformations

3.1. Definition and examples of categories.

Definition. A category $\mathcal{C}$ consists of the following data:

(1) A class $\mathrm{Ob}(\mathcal{C})$, the elements of which are called the objects of the category.

(2) For each pair (X, Y) of objects there exists a set $\mathrm{Mor}_{\mathcal{C}}(X, Y)$, the elements of which are called the morphisms from X to Y.

(3) For any three objects X, Y and Z we are provided a map

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{C}}(X,Y) \times \mathrm{Mor}_{\mathcal{C}}(Y,Z) & \to & \mathrm{Mor}_{\mathcal{C}}(X,Z) \\ (f,g) & \mapsto & g \circ f, \end{array}$$

called composition map, that satisfies the following axioms:

(A) (associativity): For every $f \in \mathrm{Mor}_{\mathcal{C}}(W, X), g \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ and $h \in \mathrm{Mor}_{\mathcal{C}}(Y, Z)$ we have

$$(h \circ g) \circ f \;=\; h \circ (g \circ f) \;\in\; \mathrm{Mor}_{\mathcal{C}}(W, Z).$$

(B) (identity): For every object X there exists a morphism $\mathrm{id}_X \in \mathrm{Mor}_{\mathcal{C}}(X, X)$ with the property that

$$\begin{array}{lrcll} \text{(i)} & \mathrm{id}_X \circ f & = & f & \text{for all } W \in \mathrm{Ob}(\mathcal{C}) \text{ and all } f \in \mathrm{Mor}_{\mathcal{C}}(W,X), \quad \text{and} \\ \text{(ii)} & f \circ \mathrm{id}_X & = & f & \text{for all } Y \in \mathrm{Ob}(\mathcal{C}) \text{ and all } f \in \mathrm{Mor}_{\mathcal{C}}(X,Y). \end{array}$$

Lemma 3.1. (Identity-Uniqueness Lemma) Let $\mathcal{C}$ be a category. For every $X \in \mathrm{Ob}(\mathcal{C})$ there exists a *unique* morphism $\mathrm{id}_X \in \mathrm{Mor}_{\mathcal{C}}(X, X)$ with the property that

$$\begin{array}{lrcll} \text{(i)} & \mathrm{id}_X \circ f & = & f & \text{for all } W \in \mathrm{Ob}(\mathcal{C}) \text{ and all } f \in \mathrm{Mor}_{\mathcal{C}}(W,X), \\ \text{(ii)} & f \circ \mathrm{id}_X & = & f & \text{for all } X \in \mathrm{Ob}(\mathcal{C}) \text{ and all } f \in \mathrm{Mor}_{\mathcal{C}}(X,Y). \end{array}$$

Proof. The existence of id_X is given by property (3B) of a category. Now suppose that $I \in \mathrm{Mor}_{\mathcal{C}}(X, X)$ has the same property. Then we have

$$\mathrm{id}_X \underset{\substack{\uparrow \\ \text{property (i) of } I \text{ applied to } f = \mathrm{id}_X}}{=} I \circ \mathrm{id}_X \underset{\substack{\uparrow \\ \text{property (ii) of } \mathrm{id}_X \text{ applied to } f = I}}{=} I.$$

■

Definition. Let $\mathcal{C}$ be a category and let $X \in \mathrm{Ob}(\mathcal{C})$. We refer to the morphism that is uniquely defined in the Identity-Uniqueness Lemma 3.1 as the identity morphism id_X.

Definition.

(a) We refer to the category $\mathcal{S}et$ with

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{S}et) & := & \text{all sets}, \\ \mathrm{Mor}(X,Y) & := & \text{all maps from } X \text{ to } Y, \end{array}$$

and together with the usual composition of maps as the category of sets.

(b) We refer to the category $\mathcal{T}op$ with

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{T}op) & := & \text{all topological spaces}, \\ \mathrm{Mor}(X,Y) & := & \text{all continuous maps from } X \text{ to } Y, \end{array}$$

with the usual composition of maps as the category of topological spaces.

(c) Let $\mathbb{K}$ be a field. We refer to the category $\mathcal{Vec}_{\mathbb{K}}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{Vec}_{\mathbb{K}}) &:= \text{all } \mathbb{K}\text{-vector spaces}, \\ \mathrm{Mor}(X,Y) &:= \mathrm{Hom}_{\mathbb{K}}(X,Y) = \text{all } \mathbb{K}\text{-homomorphisms from } X \text{ to } Y, \end{aligned}$$

with the usual composition of homomorphisms as the category of $\mathbb{K}$-vector spaces.

(d) We refer to the category $\mathcal{Group}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{Group}) &:= \text{all groups}, \\ \mathrm{Mor}(X,Y) &:= \mathrm{Hom}(X,Y) = \text{all group homomorphisms from } X \text{ to } Y, \end{aligned}$$

with the usual composition of group homomorphisms as the category of groups. Similarly we define the category $\mathcal{AbGroup}$ of abelian groups, the category $\mathcal{Ring}$ of rings and the category $\mathcal{CRing}$ of commutative rings.

(e) A pair of sets is a pair (X, A), where X is a set and A is a subset of X. We refer to the category $\mathcal{PairSet}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{PairSet}) &:= \text{all pairs of sets}, \\ \mathrm{Mor}((X,A),(Y,B)) &:= \text{all maps } f \text{ from } X \text{ to } Y \text{ with } f(A) \subset B \end{aligned}$$

with the usual composition of maps as the category of pairs of sets. With the same approach we could also introduce the category of pairs of topological spaces and so on.

Example.

(f) Let G be an arbitrary group. We consider the category $\mathcal{C}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}) &:= \text{the set with the unique element } *, \\ \mathrm{Mor}(*,*) &:= G. \end{aligned}$$

We define

$$\begin{aligned} \mathrm{Mor}(*,*) \times \mathrm{Mor}(*,*) &\to \mathrm{Mor}(*,*) \\ (f,g) &\mapsto g \circ f := \underbrace{g \cdot f}_{\text{multiplication in } G} \end{aligned}$$

via the group structure on $\mathrm{Mor}(*,*) = G$. The axioms of a category can be deduced easily from the group axioms of G.

Our last example deserves being a proper definition again.

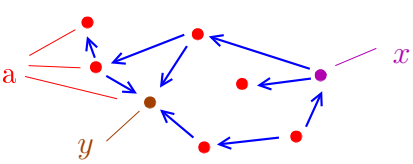
Definition.

(g) Let $(P, \leq)$ be a partially ordered set, as defined on page 22. We fix a set $\{*\}$ consisting of a single element. We define the order category $\mathcal{Order}_{(P,\leq)}$ to be the category that is given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{Order}_{(P,\leq)}) &= P \\ \mathrm{Mor}(x,y) &= \begin{cases} \{*\}, & \text{if } x \leq y, \\ \varnothing, & \text{otherwise.} \end{cases} \end{aligned}$$

together with the only possible way to compose morphisms.

partially ordered set where $a \leq b$ if there exists a directed path from a to b



$\mathrm{Mor}(x,y) = \{*\}$
$\mathrm{Mor}(y,x) = \varnothing$
$\mathrm{Mor}(x,x) = \{*\}$
$\mathrm{Mor}(y,y) = \{*\}$

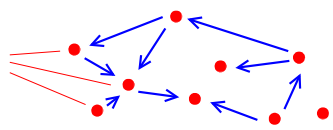
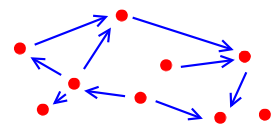
Definition. Let $\mathcal{C}$ be a category. The opposite category $\mathcal{C}^{\mathrm{op}}$ of $\mathcal{C}$ is defined as the category where

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) &:= \mathrm{Ob}(\mathcal{C}) \\ \mathrm{Mor}_{\mathcal{C}^{\mathrm{op}}}(X,Y) &:= \mathrm{Mor}_{\mathcal{C}}(Y,X) \end{aligned}$$

with the composition given by $f \circ_{\mathcal{C}^{\mathrm{op}}} g := g \circ_{\mathcal{C}} f$.

Example.

(1) Let $(P, \leq)$ be a partially ordered set. We consider the corresponding order category $\mathit{Order}_{(P,\leq)}$. The opposite of the order category $\mathit{Order}_{(P,\leq)}$ is the order category corresponding to the partially ordered set $(P, \leq^{\mathrm{op}})$ where $x \leq^{\mathrm{op}} y$ if and only if $y \leq x$.

partially ordered set $(P, \leq)$  $(P, \leq^{\mathrm{op}})$ 

(2) For any category $\mathcal{C}$ we evidently have $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$.

3.2. Isomorphisms.

Definition. Let $\mathcal{C}$ be a category and let X, Y be two objects in $\mathcal{C}$.

(1) Let $f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ be a morphism and let $g \in \mathrm{Mor}_{\mathcal{C}}(Y,X)$.
  (a) We say g is left inverse of f if $g \circ f = \mathrm{id}_X$.
  (b) We say g is right inverse of f if $f \circ g = \mathrm{id}_Y$.
  (c) We say g is an inverse of f if g is a left and a right inverse of f.

(2) A morphism $f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ is called invertible, or an isomorphism, if it admits an inverse.

(3) Two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ are called isomorphic, if there exists an isomorphism in $\mathrm{Mor}_{\mathcal{C}}(X,Y)$.

(4) Given an object $X \in \mathrm{Ob}(\mathcal{C})$ we refer to the class of all objects that are isomorphic to X as the isomorphism type of X.

Example. In the category of topological spaces the isomorphisms are precisely the homeomorphisms.

Lemma 3.2. (Inverse Uniqueness Lemma) Let $\mathcal{C}$ be a category.

(1) Let $X, Y \in \mathrm{Ob}(\mathcal{C})$ and let $f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$. If f admits a left inverse $\widetilde{g}$ and a right inverse g, then they agree, i.e. $g = \widetilde{g}$. In particular f is invertible.

(2) Let $X, Y \in \mathrm{Ob}(\mathcal{C})$ and let $f \in \mathrm{Ob}(\mathcal{C})$. If f admits an inverse, then the inverse is unique.

Definition. Let $\mathcal{C}$ be a category, let $X, Y \in \mathrm{Ob}(\mathcal{C})$ and let $f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)_{\mathrm{inv}}$ be an invertible morphism. By the Inverse Uniqueness Lemma 3.2 there exists a *unique* morphism $g \in \mathrm{Mor}_{\mathcal{C}}(Y,X)$ with $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. We refer to this morphism as the inverse of f and usually we denote it by f^{-1}.

Proof of Lemma 3.2.

(1) Let $g \in \mathrm{Mor}_{\mathcal{C}}(Y, X)$ be a morphism with $f \circ g = \mathrm{id}_Y$ and let $\widetilde{g} \in \mathrm{Mor}_{\mathcal{C}}(Y, X)$ be a morphism with $\widetilde{g} \circ f = \mathrm{id}_X$. We have the following equalities of morphisms:

$$g \underset{\substack{\uparrow \\ \text{defining} \\ \text{property of } \mathrm{id}_X}}{=} \mathrm{id}_X \circ g \underset{\substack{\uparrow \\ \text{defining} \\ \text{property of } \widetilde{g}}}{=} (\widetilde{g} \circ f) \circ g \underset{\substack{\uparrow \\ \text{associativity of “}\circ\text{”}}}{=} \widetilde{g} \circ (f \circ g) \underset{\substack{\uparrow \\ \text{defining} \\ \text{property of } g}}{=} \widetilde{g} \circ \mathrm{id}_Y \underset{\substack{\uparrow \\ \text{defining} \\ \text{property of } \mathrm{id}_Y}}{=} \widetilde{g}.$$

(2) Suppose we have $g, \widetilde{g} \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ with $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$ and with $f \circ \widetilde{g} = \mathrm{id}_Y$ and $\widetilde{g} \circ f = \mathrm{id}_X$. It follows immediately from (1) that $g = \widetilde{g}$. ■

Lemma 3.3. (Isomorphism Lemma) Let $\mathcal{C}$ be a category.

(1) Let $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(Y, Z)$ be morphisms. If two morphisms out of $f, g, g \circ f$ are invertible, then so is the third and we have $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

(2) “Being isomorphic” is an equivalence relation in the class of objects of $\mathcal{C}$.

Proof. The two statements are basically trivial. ■

3.3. Monomorphisms and epimorphisms.

Definition. Let $\mathcal{C}$ be a category.

(1) A morphism $g \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ is called a monomorphism if for every two morphisms f_1, f_2 in $\mathrm{Mor}_{\mathcal{C}}(W, X)$ with $g \circ f_1 = g \circ f_2$ we also have $f_1 = f_2$.

(2) A morphism $g \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ is called an epimorphism if for every two morphisms h_1, h_2 in $\mathrm{Mor}_{\mathcal{C}}(Y, Z)$ with $h_1 \circ g = h_2 \circ g$ we also have $h_1 = h_2$.

Proposition 3.4. (Monomorphism Proposition) Suppose we are in one of the following categories:

(1) The category $\mathcal{Group}$ of groups.

(2) The category $\mathcal{AbGroup}$ of abelian groups.

(3) The category $\mathcal{Vec}_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces, where $\mathbb{K}$ is a fixed field.

(4) The category $\mathcal{Ring}$ of rings.

If $\varphi\colon G \to H$ is a monomorphism in any of the above categories, then φ, viewed as a map between sets, is injective.

Proof. Let $\varphi\colon G \to H$ be a monomorphism in one of the first three categories. We set $K := \ker(\varphi) := \{g \in G \mid \varphi(g) = e\}$, where e is the trivial element of the group or the zero in the vector space. Note that in the first three categories we listed K is again an object of the category and φ is injective if and only if K is trivial.

We consider the trivial homomorphisms $\alpha\colon K \to G$ and we consider the inclusion homomorphism $\beta\colon K \to G$. Note that $\varphi \circ \alpha = \varphi \circ \beta$. Since φ is a monomorphism we see that $\alpha = \beta$. But this implies that K is trivial.

Finally we turn to the category $\mathcal{Ring}$ of rings. Note that the argument above does not apply since the multiplicatively neutral element never lies in the kernel, which implies that the kernel of a ring homomorphism is not a ring. Therefore we have to work a little harder. Let $\varphi\colon A \to B$ be a monomorphism in the category $\mathcal{Ring}$ of rings and let $a_1, a_2 \in A$ with $\varphi(a_1) = \varphi(a_2)$. We consider the polynomial ring $\mathbb{Z}[x]$ and we consider the two homomorphisms $f_1, f_2\colon \mathbb{Z}[x] \to A$ that are defined by $f_1(x) = a_1$ and $f_2(x) = a_2$. Then $(\varphi \circ f_1)(x) = (\varphi \circ f_2)(x)$ and hence $\varphi \circ f_1 = \varphi \circ f_2\colon \mathbb{Z}[x] \to B$. Since φ is a monomorphism we see that $f_1 = f_2$ and thus $x = y$. ■

Proposition 3.5. (Epimorphism Proposition) Suppose we are in one of the following categories:

(1) The category $\mathcal{Group}$ of groups.
(2) The category $\mathcal{AbGroup}$ of abelian groups.
(3) The category $\mathcal{Vec}_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces, where $\mathbb{K}$ is a fixed field.

If $\varphi\colon G \to H$ is an epimorphism in any of the above categories, then φ, viewed as a map between sets, is surjective.

Example. The perspicacious reader will have noticed that in the Epimorphism Proposition 3.5 we do not say anything about the category of rings. In fact one can easily show that in the category $\mathcal{Ring}$ of rings the inclusion $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism.

3.4. Initial and terminal objects.

Definition. Let $\mathcal{C}$ be a category.

(1) An object $X \in \mathrm{Ob}(\mathcal{C})$ is called initial if for each object $Y \in \mathrm{Ob}(\mathcal{C})$ there exists a unique morphism from X to Y.
(2) An object $X \in \mathrm{Ob}(\mathcal{C})$ is called terminal if for each object $Y \in \mathrm{Ob}(\mathcal{C})$ there exists a unique morphism from Y to X.

Example.

(1) In the category $\mathcal{Set}$ of sets an initial object is given by the empty set and a terminal object is given by any set consisting of a single element. Similarly in the category $\mathcal{Top}$ of topological spaces an initial object is given by the empty topological space and a terminal object is given by a topological space consisting of a single element.
(2) In the category $\mathcal{Group}$ of groups any group consisting of a single element is an initial and also a terminal object.
(3) Let X be an initial object of a category $\mathcal{C}$. It follows easily from the definitions that it is also a terminal object of the opposite category $\mathcal{C}^{\mathrm{op}}$.

Lemma 3.6. (Initial-Terminal Object Uniqueness Lemma) Let $\mathcal{C}$ be a category.

(1) If an initial object exists, then it is unique up to isomorphism.
(2) If a terminal object exists, then it is unique up to isomorphism.

Proof. The statements follow easily from the definition. ■

3.5. Covariant functors.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A covariant functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of a map

$$F\colon\ \mathrm{Ob}(\mathcal{C}) \ \to\ \mathrm{Ob}(\mathcal{D})$$

and for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ of a map

$$F\colon\ \mathrm{Mor}_{\mathcal{C}}(X,Y) \ \to\ \mathrm{Mor}_{\mathcal{D}}(F(X),F(Y)),$$

such that the following axioms are satisfied:

(F1) For all $X \in \mathrm{Ob}(\mathcal{C})$ we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.
(F2) For all $\phi \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ and all $\psi \in \mathrm{Mor}_{\mathcal{C}}(Y,Z)$ the following equality holds

$$F(\psi \circ \phi) \ = \ F(\psi) \circ F(\phi) \qquad \in \mathrm{Mor}_{\mathcal{D}}(F(X),F(Z)).$$

Convention. If the covariant functor F is understood from the context, then given a morphism $\phi \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ we often write ϕ_* instead of $F(\phi)$. The first axiom (F1) then becomes $(\mathrm{id}_X)_* = \mathrm{id}_{F(X)}$ and the second axiom (F2) becomes $(\psi \circ \phi)_* = \psi_* \circ \phi_*$.

Examples.

(1) The map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{V}ec_{\mathbb{R}}) & \to & \mathrm{Ob}(\mathcal{S}et) \\ (V,+,\cdot) & \mapsto & V, \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{V}ec_{\mathbb{R}}}(V,W) & \to & \mathrm{Mor}_{\mathcal{S}et}(V,W) \\ (\varphi\colon V \to W) & \mapsto & (\varphi\colon V \to W) \end{array}$$

define a covariant functor from the category $\mathcal{V}ec_{\mathbb{R}}$ of real vector spaces to the category $\mathcal{S}et$ of sets. Functors of this type are colloquially called *forgetful functor*.

(2) Let $\mathcal{T}op$ be the category of topological spaces and let $\mathcal{S}et$ be the category of sets. One can easily show that the map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{T}op) & \to & \mathrm{Ob}(\mathcal{S}et) \\ X & \mapsto & \pi_0(X) := \text{set of path-components of } X, \end{array}$$

and the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{T}op}(X,Y) & \to & \mathrm{Mor}_{\mathcal{S}et}(\pi_0(X),\pi_0(Y)) \\ f & \mapsto & \left(\begin{array}{rcl} \pi_0(X) & \to & \pi_0(Y) \\ C & \mapsto & \begin{array}{l}\text{unique path-component} \\ \text{of } Y \text{ that contains } f(C)\end{array} \end{array}\right) \end{array}$$

define a covariant functor from the category $\mathcal{T}op$ of topological spaces to the category $\mathcal{S}et$ of sets.

Lemma 3.7. (Fixed Domain-Covariant Functor Lemma) Let $\mathcal{D}$ be a category and let $\mathcal{S}et$ be the category of sets. Furthermore let $W \in \mathrm{Ob}(\mathcal{D})$. The map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{D}) & \to & \mathrm{Ob}(\mathcal{S}et) \\ X & \mapsto & \mathrm{Mor}_{\mathcal{D}}(W,X) \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{D}}(X,Y) & \to & \mathrm{Mor}_{\mathcal{S}et}(\mathrm{Mor}_{\mathcal{D}}(W,X),\mathrm{Mor}_{\mathcal{D}}(W,Y)) \\ \phi & \mapsto & \left(\begin{array}{rcl} \mathrm{Mor}_{\mathcal{D}}(W,X) & \to & \mathrm{Mor}_{\mathcal{D}}(W,Y) \\ g & \mapsto & \phi \circ g \end{array}\right) \end{array}$$

is a covariant functor from the category $\mathcal{D}$ to the category $\mathcal{S}et$ of sets.

Proof. The statement follows easily from the axioms of a category. ■

Example. The example of a covariant functor in the Fixed Domain-Covariant Functor Lemma 3.7 comes with many variations, in particular if $\mathrm{Mor}_{\mathcal{D}}(W,X)$ has more structure than that of a set. For instance let W be a real vector space. Then the above map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{V}ec_{\mathbb{R}}) & \to & \mathrm{Ob}(\mathcal{V}ec_{\mathbb{R}}) \\ V & \mapsto & \underbrace{\mathrm{Hom}_{\mathbb{R}}(W,V)}_{\text{real vector space}}, \end{array}$$

and the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{V}ec_{\mathbb{R}}}(U,V) & \to & \mathrm{Mor}_{\mathcal{V}ec_{\mathbb{R}}}(F(U),F(V)) \\ (\phi\colon U \to V) & \mapsto & \left(\begin{array}{rcl} \mathrm{Hom}_{\mathbb{R}}(W,U) & \to & \mathrm{Hom}_{\mathbb{R}}(W,V) \\ g & \mapsto & \phi \circ g \end{array}\right) \end{array}$$

define a covariant functor from the category $\mathcal{V}ec_{\mathbb{R}}$ of real vector spaces to itself.

3.6. Contravariant functors.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A contravariant functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of a map

$$F\colon \operatorname{Ob}(\mathcal{C}) \to \operatorname{Ob}(\mathcal{D})$$

and for all $X, Y \in \operatorname{Ob}(\mathcal{C})$ of a map

$$\operatorname{Mor}_{\mathcal{C}}(X, Y) \to \operatorname{Mor}_{\mathcal{D}}(F(Y), F(X))$$

such that the following axioms are satisfied:

(F1) For all $X \in \operatorname{Ob}(\mathcal{C})$ we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

(F2) For all $\phi \in \operatorname{Mor}_{\mathcal{C}}(X, Y)$ and all $\psi \in \operatorname{Mor}_{\mathcal{C}}(Y, Z)$ we have

$$F(\psi \circ \phi) = F(\phi) \circ F(\psi) \qquad \in \operatorname{Mor}_{\mathcal{D}}(F(Z), F(X)).$$

Convention. If the contravariant functor F is understood from the context, then given a morphism $\phi \in \operatorname{Mor}_{\mathcal{C}}(X, Y)$ we often write ϕ^* instead of $F(\phi)$. The first axiom then becomes $(\mathrm{id}_X)^* = \mathrm{id}_{F(X)}$ and the second axiom becomes $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

Lemma 3.8. (Fixed Target-Contravariant Functor Lemma) Let $\mathcal{D}$ be a category and let $\mathcal{S}et$ be the category of sets. Furthermore let $W \in \operatorname{Ob}(\mathcal{D})$. The map

$$\begin{array}{rcl} F\colon \operatorname{Ob}(\mathcal{D}) & \to & \operatorname{Ob}(\mathcal{S}et) \\ X & \mapsto & \operatorname{Mor}_{\mathcal{D}}(X, W) \end{array}$$

together with the maps

$$\begin{array}{rcl} \operatorname{Mor}_{\mathcal{D}}(X, Y) & \to & \operatorname{Mor}_{\mathcal{S}et}(\operatorname{Mor}_{\mathcal{D}}(Y, W), \operatorname{Mor}_{\mathcal{D}}(X, W)) \\ f & \mapsto & \left(\begin{array}{rcl} \operatorname{Mor}_{\mathcal{D}}(Y, W) & \to & \operatorname{Mor}_{\mathcal{D}}(X, W) \\ g & \mapsto & g \circ f \end{array} \right) \end{array}$$

is a contravariant functor from the category $\mathcal{D}$ to the category $\mathcal{S}et$ of sets.

Proof. Again the statement follows easily from the axioms of a category. ■

Example. The Fixed Target-Contravariant Functor Lemma 3.8 can be viewed as the mother of all contravariant functors. As in the case of the Fixed Domain-Covariant Functor Lemma 3.7 it spawns many variations if $\operatorname{Mor}_{\mathcal{D}}(X, W)$ has more structure than just being a set. The map

$$\begin{array}{rcl} F\colon \operatorname{Ob}(\mathcal{V}ec_{\mathbb{R}}) & \to & \operatorname{Ob}(\mathcal{V}ec_{\mathbb{R}}) \\ V & \mapsto & V^* := \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R}) =: \text{dual vector space of } V, \end{array}$$

together with the maps

$$\begin{array}{rcl} \operatorname{Mor}_{\mathcal{V}ec_{\mathbb{R}}}(U, V) & \to & \operatorname{Mor}_{\mathcal{V}ec_{\mathbb{R}}}(V^*, U^*) \\ (\phi\colon U \to V) & \mapsto & \left(\begin{array}{rcl} \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R}) & \to & \operatorname{Hom}_{\mathbb{R}}(U, \mathbb{R}) \\ f & \mapsto & f \circ \phi \end{array} \right) \end{array}$$

is a contravariant functor from the category $\mathcal{V}ec_{\mathbb{R}}$ of real vector spaces to itself.

3.7. Natural transformations.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two covariant functors.

(1) A natural transformation between the functors F and G assigns to each $X \in \operatorname{Ob}(\mathcal{C})$ a morphism $\Phi_X\colon F(X) \to G(X)$ in $\mathcal{D}$ such that for each morphism $\mu\colon X \to Y$ in $\mathcal{C}$ the

following diagram of morphisms in $\mathcal{D}$ commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(\mu)"] \arrow[d, "\Phi_X"'] & F(Y) \arrow[d, "\Phi_Y"] \\ G(X) \arrow[r, "G(\mu)"] & G(Y). \end{tikzcd}$$

(2) If for every $X \in \mathrm{Ob}(\mathcal{C})$ the morphism $\Phi_X \colon F(X) \to G(X)$ is an isomorphism, then we refer to the natural transformation as a natural isomorphism. In this case, given $X \in \mathrm{Ob}(\mathcal{C})$ we often say that $F(X)$ and $G(X)$ are naturally isomorphic.

The same way we can of course also define the notion of a natural transformation and a natural isomorphism between two contravariant functors.

Example. Let $\mathbb{K}$ be a field. We consider the category $\mathcal{V}ec_{\mathbb{K}}$ of vector spaces over $\mathbb{K}$. Given a $\mathbb{K}$-vector space V we write $V^* := \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ and given $f \colon V \to W$ we denote by $f^* \colon W^* \to V^*$ the induced map. We consider the following two covariant functors from $\mathcal{V}ec_{\mathbb{K}}$ to $\mathcal{V}ec_{\mathbb{K}}$:

$$\begin{array}{rcl} \mathrm{id} \colon V & \mapsto & V \\ (f \colon V \to W) & \mapsto & (f \colon V \to W) \end{array} \quad \text{and} \quad \underbrace{\begin{array}{rcl} F \colon V & \mapsto & (V^*)^* \\ (f \colon V \to W) & \mapsto & ((f^*)^* \colon (V^*)^* \to (W^*)^*). \end{array}}_{\text{this is a covariant functor}}$$

For each $\mathbb{K}$-vector space V we denote by Φ_V the homomorphism

$$\begin{array}{rcl} V & \to & (V^*)^* \\ v & \mapsto & \left(\begin{array}{rcl} V^* & \to & \mathbb{K} \\ (\varphi \colon V \to \mathbb{K}) & \mapsto & \varphi(v) \end{array} \right). \end{array}$$

It is straightforward to verify that these maps define a natural transformation from the identity functor $\mathrm{id} \colon \mathcal{V}ec_{\mathbb{K}} \to \mathcal{V}ec_{\mathbb{K}}$ to the functor $F \colon \mathcal{V}ec_{\mathbb{K}} \to \mathcal{V}ec_{\mathbb{K}}$.

4. Products and Coproducts – Limits and colimits

4.1. Products.

Definition. Let $\mathcal{C}$ be a category and let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$. A product of $\{X_i\}_{i\in I}$ in the category $\mathcal{C}$ is an object $P \in \mathrm{Ob}(\mathcal{C})$ together with a family of morphisms $\{p_i\colon P \to X_i\}_{i\in I}$ such that the following universal property is satisfied: Whenever we are given a $Z \in \mathrm{Ob}(\mathcal{C})$ and a family of morphisms $\{f_i\colon Z \to X_i\}_{i\in I}$ there exists a unique morphism $\Phi\colon Z \to P$ such that for each $i \in I$ we have $f_i = p_i \circ \Phi$, i.e. such that the following diagram commutes:

$$\begin{tikzcd} Z \arrow[r, dashed, "\exists!\,\Phi"] \arrow[dr, "f_i"'] & \text{product } P \arrow[d, "p_i"] \\ & X_i. \end{tikzcd}$$

Example. If we take the empty family of objects in a category $\mathcal{C}$, then a corresponding product is the same as a terminal object in the category $\mathcal{C}$.

Lemma 4.1. (Product Uniqueness Lemma) Let $\mathcal{C}$ be a category and let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$. If we are given products $(P, \{p_i\colon P \to X_i\}_{i\in I})$ and $(\widetilde{P}, \{\widetilde{p}_i\colon \widetilde{P} \to X_i\}_{i\in I})$ then there exist uniquely determined isomorphisms $\Phi\colon \widetilde{P} \to P$ and $\widetilde{\Phi}\colon P \to \widetilde{P}$ which are inverses of one another and such that for each $i \in I$ the following diagram commutes:

$$\begin{tikzcd} \widetilde{P} \arrow[rr, shift left, "\Phi"] \arrow[dr, "p_i"'] & & P \arrow[ll, shift left, "\widetilde{\Phi}"] \arrow[dl, "\widetilde{p}_i"] \\ & X_i. & \end{tikzcd}$$

Sketch of proof. The two morphisms $\Phi\colon \widetilde{P} \to P$ and $\widetilde{\Phi}\colon P \to \widetilde{P}$ come from the universal properties of P and $\widetilde{P}$. By comparing $\mathrm{id}_P\colon P \to P$ and $\Phi \circ \widetilde{\Phi}\colon P \to P$ one sees, by the uniqueness property, that $\mathrm{id}_P = \Phi \circ \widetilde{\Phi}$ and similarly one sees that $\mathrm{id}_{\widetilde{P}} = \widetilde{\Phi} \circ \Phi$, i.e. Φ and $\widetilde{\Phi}$ are isomorphisms. ■

Notation. Let $\mathcal{C}$ be a category. Let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$ such that a product exists.

(1) We usually denote the product by $\prod_{i\in I} X_i$ and we refer to the morphisms $\prod_{i\in I} X_i \to X_i$ as natural projections. Furthermore, given morphisms $f_i\colon Z \to X_i$ we denote the corresponding morphism $Z \to \prod_{i\in I} X_i$ by $\prod_{i\in I} f_i$.

(2) If $I = \{1, \dots, k\}$, then often we denote the product by $X_1 \times \cdots \times X_k$. Furthermore given morphisms $f_i\colon Z \to X_i$, $i = 1, \dots, k$ we denote the corresponding morphism $Z \to X_1 \times \cdots \times X_k$ by $(f_1, \dots, f_k)$.

Lemma 4.2. (Product-of-Sets-and-Groups Lemma)

(1) Let $\mathcal{Set}$ be the category of sets and let $\{X_i\}_{i\in I}$ be a family of sets. The product in $\mathcal{Set}$ is precisely the product

$$\prod_{i\in I} X_i \;:=\; \left\{ f\colon I \to \bigcup_{i\in I} X_i \;\middle|\; \text{for every } i \in I \text{ we have } f(i) \in X_i \right\}$$

together with the obvious maps to the X_i that are given by $f \mapsto f(i)$.

(2) Let $\mathcal{Group}$ be the category of groups and let $\{X_i\}_{i\in I}$ be a family of groups. The product in $\mathcal{Group}$ is the set

$$\prod_{i\in I} X_i \;=\; \left\{ f\colon I \to \bigcup_{i\in I} X_i \;\middle|\; \text{for every } i \in I \text{ we have } f(i) \in X_i \right\}$$

as in (1), which we now view as a group by the obvious "entry wise" multiplication, together with the obvious homomorphisms to X_i.

(3) The same approach as in (2) also applies to the category $\mathcal{AbGroup}$ of abelian groups and the category $\mathcal{Vec}_{\mathbb{K}}$ of vector spaces over a fixed field $\mathbb{K}$.

Proof.

(1) So let $Z \in \mathrm{Ob}(\mathcal{Set})$ and let $\{f_i\colon Z \to X_i\}_{i\in I}$ be a family of maps between sets. We define

$$\begin{array}{rcl} \Phi\colon Z & \to & \prod\limits_{i\in I} X_i \\ z & \mapsto & \begin{pmatrix} I & \to & \bigcup\limits_{i\in I} X_i \\ i & \mapsto & f_i(z) \end{pmatrix}. \end{array}$$

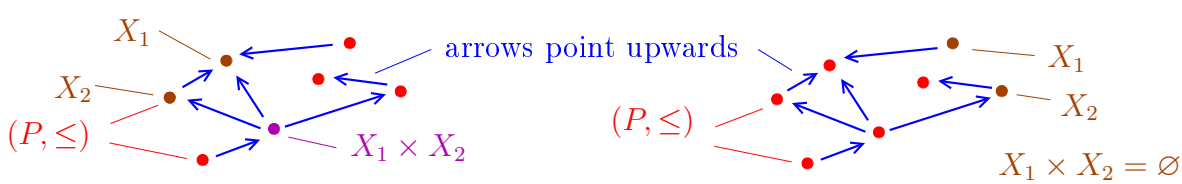
It follows immediately from the definition that for each $i \in I$ we have $f_i = p_i \circ \Phi$. It remains to show that Φ is unique. So let $\Psi\colon Z \to \prod\limits_{i\in I} X_i$ be a continuous map such that for each $i \in I$ we have $f_i = p_i \circ \Psi$. We need to show that $\Phi = \Psi$, i.e. we need to show that for each $z \in Z$ and $i \in I$ we have $\Phi(z)(i) = \Psi(z)(i) \in X_i$. But this follows from the observation that $\Phi(z)(i) = p_i(\Phi(z)) = f_i(z) = p_i(\Psi(z)) = \Psi(z)(i) \in X_i$.

(2) The same argument as in (1) also works for the category $\mathcal{Group}$ of groups.

(3) The argument of (2) also works in (3). ■

Example. Let $(P, \leq)$ be a partially ordered set. We consider the corresponding order category $\mathcal{Order}_{(P,\leq)}$ as defined on page 54. Given a family $\{X_i\}_{i\in I}$ of objects in $\mathcal{Order}_{(P,\leq)}$, i.e. given a family $\{X_i\}_{i\in I}$ of elements of P, one can easily show that

$$\prod_{i\in I} X_i \;:=\; \begin{cases} \inf(\{X_i\}_{i\in I}), & \text{i.e. a maximal element among all lower bounds on } \{X_i\}_{i\in I} \\ \varnothing, & \text{if no such infimum exists.} \end{cases}$$



4.2. Coproducts.

Definition. Let $\mathcal{C}$ be a category and let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$. A coproduct of $\{X_i\}_{i\in I}$ in the category $\mathcal{C}$ is an object $C \in \mathrm{Ob}(\mathcal{C})$ together with a family of morphisms

$\{\iota_i\colon X_i \to C\}_{i\in I}$ such that the following property is satisfied: If we are given a $Z \in \mathrm{Ob}(\mathcal{C})$ and a family of morphisms $\{f_i\colon X_i \to Z\}_{i\in I}$ there exists a unique morphism $\Phi\colon C \to Z$ such that for each $i \in I$ we have $f_i = \Phi \circ \iota_i$, i.e. such that the following diagram commutes:

$$\begin{tikzcd} X_i \arrow[d, "\iota_i"'] \arrow[dr, "f_i"] & \\ \text{coproduct } C \arrow[r, dashed, "\exists!\,\Phi"'] & Z. \end{tikzcd}$$

Example. Let $\mathcal{C}$ be a category.

(1) If we take the empty family of objects in $\mathcal{C}$, then it follows easily from the definitions that the corresponding coproduct is an initial object in the category $\mathcal{C}$.

(2) Basically by definition coproducts in a category $\mathcal{C}$ correspond to products in the opposite category $\mathcal{C}^{\mathrm{op}}$.

Lemma 4.3. (Coproduct Uniqueness Lemma) Let $\mathcal{C}$ be a category and let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$. If we are given two coproducts $(C, \{\iota_i\colon X_i \to C\}_{i\in I})$ and $(\widetilde{C}, \{\widetilde{\iota}_i\colon X_i \to \widetilde{C}\}_{i\in I})$, then there exist unique isomorphisms $\Phi\colon C \to \widetilde{C}$ and $\widetilde{\Phi}\colon \widetilde{C} \to C$ which are inverses of one another and such that for each $i \in I$ the following diagram commutes:

$$\begin{tikzcd} & X_i \arrow[dl, "\iota_i"'] \arrow[dr, "\widetilde{\iota}_i"] & \\ C \arrow[rr, shift left, "\Phi"] & & \widetilde{C}. \arrow[ll, shift left, "\widetilde{\Phi}"] \end{tikzcd}$$

Proof. The lemma is just a reformulation of the Product Uniqueness Lemma 4.1 since coproducts in $\mathcal{C}$ are products in the opposite category $\mathcal{C}^{\mathrm{op}}$. ■

Notation. Let $\mathcal{C}$ be a category. Let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$ such that a coproduct exists. Often we denote the coproduct by $\coprod_{i\in I} X_i$ and we refer to the morphisms $X_i \to \coprod_{i\in I} X_i$ as natural injections.

Lemma 4.4. (Coproduct-of-Sets-and-Abelian Groups Lemma)

(1) Let $\mathcal{S}et$ be the category of sets and let $\{X_i\}_{i\in I}$ be a family of sets. The coproduct in $\mathcal{S}et$ is the disjoint union
$$\bigsqcup_{i\in I} X_i \;:=\; \bigcup_{i\in I} (X_i \times \{i\}),$$
as defined on page 20, together with the natural injections of the X_i.

(2) Let $\mathcal{A}b\mathcal{G}roup$ be the category of *abelian* groups and let $\{X_i\}_{i\in I}$ be a family of groups. The coproduct in $\mathcal{A}b\mathcal{G}roup$ is given by
$$\bigoplus_{i\in I} X_i \;:=\; \left\{ (g_i)_{i\in I} \in \prod_{i\in I} X_i \,\middle|\, \text{all but finitely many } g_i \text{ are trivial} \right\}$$
and the homomorphisms

$$\begin{array}{rcl} X_i & \to & \bigoplus_{i\in I} X_i \\ x & \mapsto & \left(\begin{array}{rcl} I & \to & \bigcup_{i\in I} X_i \\ j & \mapsto & \begin{cases} x_i, & \text{if } i = j, \\ 0_{X_j}, & \text{if } i \neq j. \end{cases} \end{array} \right) \end{array}$$

(3) The same approach as in (2) also applies to the category $\mathcal{Vec}_{\mathbb{K}}$ of vector spaces over a fixed field $\mathbb{K}$.

Notation. Let $\{X_i\}_{i\in I}$ be objects in the category $\mathcal{AbGroup}$ of abelian groups. We write

$$\bigoplus_{i\in I} X_i \;:=\; \left\{ (g_i)_{i\in I} \in \prod_{i\in I} X_i \,\middle|\, \text{all but finitely many } g_i \text{ are trivial} \right\}$$

and we refer to it as the direct sum of the X_i. We proceed similarly with objects in the category $\mathcal{Vec}_{\mathbb{K}}$ of vector spaces over a fixed field $\mathbb{K}$.

Proof.

(1) So let $Z \in \mathrm{Ob}(\mathcal{Set})$ and let $\{f_i \colon X_i \to Z\}_{i\in I}$ be a family of maps between sets. We define

$$\begin{array}{rcl} \Phi \colon \bigsqcup_{i\in I} X_i & \to & Z \\ (z_i, i) & \mapsto & f_i(z_i). \end{array}$$

It follows immediately from the definition that for each $i \in I$ we have $f_i = \Phi \circ \iota_i$. It remains to show that Φ is unique. So let $\Psi \colon \bigsqcup_{i\in I} X_i \to Z$ be a continuous map such that for each $i \in I$ we have $f_i = \Psi \circ \iota_i$. We need to show that $\Phi = \Psi$, i.e. we need to show that for each $i \in I$ and each $x_i \in X_i$ we have $\Phi((x_i, i)) = \Psi((x_i, i)) \in Z$. But this follows from the observation that $\Phi((x_i, i)) = \Phi(\iota_i(x_i)) = f_i(z) = \Psi(\iota_i(x_i)) = \Psi((x_i, i)) \in Z$.

(2) Let a $Z \in \mathrm{Ob}(\mathcal{AbGroup})$ and let $\{f_i \colon X_i \to Z\}_{i\in I}$ be a family of homomorphisms. We set

$$\begin{array}{rcl} \Phi \colon \left\{ (g_i)_{i\in I} \in \prod_{i\in I} X_i \,\middle|\, \text{all but finitely many } g_i \text{ are trivial} \right\} & \to & Z \\ (g_i)_{i\in I} & \mapsto & \sum\limits_{\substack{i\in I \text{ with} \\ g_i \neq e}} f_i(g_i). \end{array}$$

Note that this map makes sense since we take the product only of finitely elements of Z. Furthermore note that this map is a homomorphism since we work with the category of abelian groups.

It follows immediately from the definition that for each $i \in I$ we have $f_i = \Phi \circ \iota_i$. It remains to show that Φ is unique. So let Ψ be another homomorphism such that for each $i \in I$ we have $f_i = \Psi \circ \iota_i$. We need to show that $\Phi = \Psi$. So suppose we are given $(g_i)_{i\in I} \in \prod_{i\in I} X_i$ such that there exist only finitely many $i_1, \ldots, i_k \in I$ such that the corresponding group element is non-trivial. Then

$$\Phi((g_i)_{i\in I}) \;=\; \Phi\Big(\sum_{r=1}^{k} \iota_{i_r}(g_{i_r}) \Big) \;=\; \sum_{r=1}^{k} \Phi(\iota_{i_r}(g_{i_r})) \;=\; \sum_{r=1}^{k} f_{i_r}(g_{i_r}) \;\underset{\substack{\uparrow \\ \text{same argument backwards}}}{=}\; \Psi((g_i)_{i\in I}).$$

(3) This statement is proved the same way as we proved (2). ■

4.3. Limits.

Definition. Let $(P, \leq)$ be a partially ordered set and let $\mathcal{C}$ be a category. A diagram in $\mathcal{C}$ over $(P, \leq)$ is a covariant functor from the order category $\mathit{Order}_{(P,\leq)}$ to $\mathcal{C}$.

Example.

(1) We consider the partially ordered set $(P, \leq_{\mathrm{tr}})$ that is given by a set P with the trivial partial order $\leq_{\mathrm{tr}}$, i.e. $x \leq_{\mathrm{tr}} y$ if and only if $x = y$. In this setting a diagram over $(P, \leq_{\mathrm{tr}})$ is the same data as a family of objects $\{X_i\}_{i \in P}$.

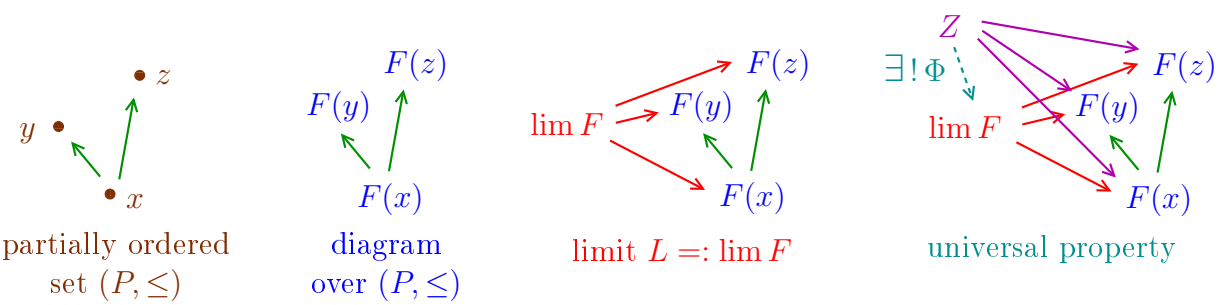
(2) We consider the partially ordered set $(\mathbb{N}, \leq)$. In this setting a diagram over $(\mathbb{N}, \leq)$ is the same data as a sequence
$$X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} X_3 \xrightarrow{\varphi_3} \dots$$
of morphisms $\varphi_i \colon X_i \to X_{i+1}$. For $i < j$ the morphisms $X_i \to X_j$ are understood to be given by $\varphi_{j-1} \circ \cdots \circ \varphi_{i+1} \circ \varphi_i$.

Definition. Let $(P, \leq)$ be a partially ordered set, let $\mathcal{C}$ be a category and let F be a diagram in $\mathcal{C}$ over $(P, \leq)$. A limit of F consists of an object $L \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $f_x \colon L \to F(x)$ for each $x \in P$ such that the following two conditions are satisfied:

(1) For any $x \leq y$ we have $f_y = F(x \to y) \circ f_x \colon L \to F(y)$.

(2) The following universal property holds: If Z is an object in $\mathcal{C}$ and if we are given morphisms $g_x \colon Z \to F(x)$ for each $x \in P$ such that (1) is satisfied, then there exists a unique morphism $\Phi \colon Z \to L$ such that $g_x = f_x \circ \Phi$ for all $x \in P$.

If a limit exists, then often we denote it by $\lim F$.



Lemma 4.5. (Limit Uniqueness Lemma) Let $(P, \leq)$ be a partially ordered set, let $\mathcal{C}$ be a category and let F be a diagram in $\mathcal{C}$ over $(P, \leq)$. Suppose that we are given two limits $(L, \{f_x \colon L \to F(x)\}_{x \in P})$ and $(\widetilde{L}, \{\widetilde{f}_x \colon \widetilde{L} \to F(x)\}_{x \in P})$. Then there exist uniquely determined isomorphisms $\Phi \colon \widetilde{L} \to L$ and $\widetilde{\Phi} \colon L \to \widetilde{L}$ which are inverses of one another and such that for each $x \in P$ the following diagram commutes:

$$\begin{tikzcd}
\widetilde{L} \arrow[rr, shift left, "\Phi"] \arrow[dr, "f_x"'] & & L. \arrow[ll, shift left, "\widetilde{\Phi}"] \arrow[dl, "\widetilde{f}_x"] \\
& F(x) &
\end{tikzcd}$$

Proof. As in the case of the Product Uniqueness Lemma 4.1 the statement follows easily from the definitions. ■

Example. If we consider a trivial partially ordered set $(P, \leq_{\mathrm{tr}})$, then the limit is by definition precisely the product in the sense of the definition on page 61.

Definition. We consider the set $P = \{0, x, y\}$ with the partial order that is given by the trivial relations $0 \leq 0, x \leq x, y \leq y$ and the non-trivial relations $x \leq 0$ and $y \leq 0$:

the partially ordered set
$$\begin{tikzcd} & x \arrow[d] \\ y \arrow[r] & 0 \end{tikzcd}$$
gives rise to the diagram
$$\begin{tikzcd} & x \arrow[d, "\varphi_x"] \\ Y \arrow[r, "\varphi_y"] & A \end{tikzcd}$$
the corresponding limit is called a pullback
$$\begin{tikzcd} \text{pullback } L \arrow[r, "f_X"] \arrow[d, "f_Y"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "\varphi_X"] \\ Y \arrow[r, "\varphi_Y"'] & A. \end{tikzcd}$$

Here the symbol "$\lrcorner$" indicates that L is the pullback of the diagram.

Example. We consider the following pullback diagram in the category of groups:

$$\begin{tikzcd} & G \arrow[d, "\varphi"] \\ \{e\} \arrow[r] & H \end{tikzcd}$$
one can easily show that the corresponding pullback is given by
$$\begin{tikzcd} \ker(\varphi) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & G \arrow[d, "\varphi"] \\ \{e\} \arrow[r] & H \end{tikzcd}$$

Lemma 4.6. (Pullback Lemma) Let $\mathcal{C}$ be a category.

(1) Suppose that $\mathcal{C}$ admits a terminal object $*$. Then the pullback of the diagram

$$\begin{tikzcd} X \times Y \arrow[r, "p_X"] \arrow[d, "p_Y"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "\varphi_X"] \\ Y \arrow[r, "\varphi_Y"'] & * \end{tikzcd}$$

is given by the product $X \times Y$.

(2) Let

$$\begin{tikzcd} L \arrow[r, "f_X"] \arrow[d, "f_Y"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "\varphi_X"] \\ Y \arrow[r, "\varphi_Y"'] & A. \end{tikzcd}$$

be a pullback.

(a) If φ_X is an isomorphism, then so is the "opposite" morphism f_Y.

(b) If φ_X is a monomorphism, then so is the "opposite" morphism f_Y.

The analogous statements of course also hold with the roles of X and Y swapped.

(3) If we are given pullbacks

$$\begin{tikzcd} L \arrow[r, "f_X"] \arrow[d, "f_Y"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "\varphi_X"] \\ Y \arrow[r, "\varphi_Y"] & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "\theta"] \arrow[d, "\varphi_X"'] \arrow[dr, phantom, "\lrcorner"] & Z \arrow[d, "\psi_Z"] \\ A \arrow[r, "\psi_A"] & B \end{tikzcd}$$
then the "composition"
$$\begin{tikzcd} L \arrow[r, "\theta \circ f_X"] \arrow[d, "f_y"'] \arrow[dr, phantom, "\lrcorner"] & Z \arrow[d, "\psi_Z"] \\ Y \arrow[r, "\psi_A \circ \varphi_Y"] & B \end{tikzcd}$$

is also a pullback.

Proof.

(1) This statement follows easily from the definitions.

(2) (a) This statement also follows easily from the definitions.

(b) We assume that φ_X is a monomorphism. We need to show that f_Y is also a monomorphism. Thus let $\alpha\colon Z \to L$ and $\beta\colon Z \to L$ be two morphisms such that $f_Y \circ \alpha = f_Y \circ \beta$.

$$\begin{tikzcd}
Z \arrow[dr, bend left, "\alpha"] \arrow[drr, dashed, bend left] \arrow[ddr, bend right, "f_Y\circ\alpha=f_Y\circ\beta"'] \arrow[dr, bend right, "\beta"'] & & \\
& L \arrow[r, "f_X"] \arrow[d, "f_Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\varphi_X"] \\
& Y \arrow[r, "\varphi_Y"'] & A.
\end{tikzcd}$$

We need to show that $\alpha = \beta$. We proceed as follows:

(i) We assume that $f_Y \circ \alpha = f_Y \circ \beta$.

(ii) By (i) we also have $\varphi_Y \circ f_Y \circ \alpha = \varphi_Y \circ f_Y \circ \beta$.

(iii) Since the diagram commutes we get $\varphi_X \circ f_X \circ \alpha = \varphi_X \circ f_X \circ \beta\colon Z \to A$.

(iv) Since φ_X is a monomorphism we see that $f_X \circ \alpha = f_X \circ \beta\colon Z \to X$.

(v) Since we dealing with a pullback there exists a *unique* morphism $\Phi\colon Z \to L$ such that $f_Y \circ \Phi = f_Y \circ \alpha = f_Y \circ \beta$ and such that $f_X \circ \Phi = f_X \circ \alpha = f_X \circ \beta$.

(vi) From (i), (iv) and (v) we obtain that $\alpha = \Phi = \beta$.

(3) This statement follows from purely formal arguments which do not become clearer by writing them down. Thus we leave it to the reader to go through the argument. ■

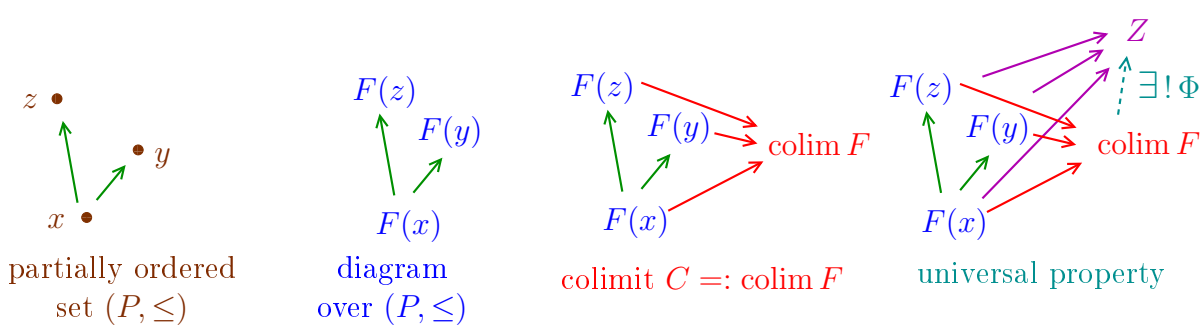
4.4. Colimits.

Definition. Let $(P, \leq)$ be a partially ordered set, let $\mathcal{C}$ be a category and let F be a diagram in $\mathcal{C}$ over $(P, \leq)$. A colimit of F consists of an object $C \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $f_x\colon F(x) \to C$ for each $x \in P$ such that the following two conditions are satisfied:

(1) For any $x \leq y$ we have $f_x = f_y \circ F(x \to y)\colon F(x) \to C$.

(2) The following universal property holds: If Z is an object in $\mathcal{C}$ and if we are given morphisms $g_x\colon F(x) \to Z$ for each $x \in P$ such that (1) is satisfied, then there exists a unique morphism $\Phi\colon C \to Z$ such that $g_x = \Phi \circ f_x$ for all $x \in P$.

If a colimit exists, then often we denote it by $\operatorname{colim} F$.



Lemma 4.7. (Colimit Uniqueness Lemma) Let $(P, \leq)$ be a partially ordered set, let $\mathcal{C}$ be a category and let F be a diagram in $\mathcal{C}$ over $(P, \leq)$. Suppose that we are given two colimits $(C, \{f_x\colon F(x) \to C\}_{x \in P})$ and $(\widetilde{C}, \{\widetilde{f}_x\colon F(X) \to \widetilde{C}\}_{x \in P})$. Then there exist

uniquely determined isomorphisms $\Phi\colon C \to \widetilde{C}$ and $\widetilde{\Phi}\colon \widetilde{C} \to C$ which are inverses of one another and such that for each $x \in P$ the following diagram commutes:

$$\begin{tikzcd} & F(X) \arrow[dl, "f_X"'] \arrow[dr, "\widetilde{f}_X"] & \\ C \arrow[rr, shift left, "\Phi"] & & \widetilde{C}. \arrow[ll, shift left, "\widetilde{\Phi}"] \end{tikzcd}$$

Proof. As always the proof follows easily from the definitions. ■

Example. If we consider a trivial partially ordered set $(P, \leq_{\mathrm{tr}})$, then the colimit is by definition precisely the coproduct, as defined on page 63.

Definition. We consider the set $P = \{0, x, y\}$ with the partial order that is given by the trivial relations $0 \leq 0, x \leq x, y \leq y$ and the non-trivial relations $0 \leq x$ and $0 \leq y$:

the partially ordered set $\begin{tikzcd} 0 \arrow[r] \arrow[d] & x \\ y \end{tikzcd}$ gives rise to the diagram $\begin{tikzcd} A \arrow[r, "\varphi_x"] \arrow[d, "\varphi_y"] & X \\ Y \end{tikzcd}$ the corresponding colimit is called a pushout $\begin{tikzcd} A \arrow[r, "\varphi_X"] \arrow[d, "\varphi_Y"] & X \arrow[d, "f_X"] \\ Y \arrow[r, "f_Y"'] \arrow[ur, phantom, "\lrcorner"] & \text{pushout } C. \end{tikzcd}$

Here the symbol "$\lrcorner$" indicates that C is the pushout of the diagram.

Example. Let G be a group and let A be a *normal* subgroup of G. We consider the following pushout diagram in the category of groups:

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & \{e\} \\ G \end{tikzcd}$$

one can easily show that the corresponding pushout is given by

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & \{e\} \arrow[d] \\ G \arrow[r] \arrow[ur, phantom, "\lrcorner"] & G/A. \end{tikzcd}$$

Lemma 4.8. (Pushout Lemma) Let $\mathcal{C}$ be a category.

(1) If $\mathcal{C}$ has an initial object $*$, then the pushout of

$$\begin{tikzcd} * \arrow[r, "\varphi_X"] \arrow[d, "\varphi_Y"] & X \arrow[d, "\iota_X"] \\ Y \arrow[r, "\iota_Y"] \arrow[ur, phantom, "\lrcorner"] & X \coprod Y \end{tikzcd}$$

is given by the coproduct $X \coprod Y$.

(2) Let

$$\begin{tikzcd} A \arrow[r, "\varphi_X"] \arrow[d, "\varphi_Y"] & X \arrow[d, "f_X"] \\ Y \arrow[r, "f_Y"] \arrow[ur, phantom, "\lrcorner"] & C. \end{tikzcd}$$

be a pushout.

(a) If φ_X is an isomorphism, then so is the "opposite" morphism f_Y.

(b) If φ_X is an epimorphism, then so is the "opposite" morphism f_Y.

The analogous statements of course also hold with the roles of X and Y swapped.

(3) If we are given pushouts

$$\begin{tikzcd} A \arrow[r, "\varphi_X"] \arrow[d, "\varphi_Y"'] & X \arrow[d, "f_X"] \\ Y \arrow[r, "f_Y"] \arrow[ur, phantom, "\lrcorner"] & L \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "\theta"] \arrow[d, "f_X"'] & Z \arrow[d, "g_Z"] \\ L \arrow[r, "g_L"] \arrow[ur, phantom, "\lrcorner"] & W \end{tikzcd} \quad \text{then the "composition"} \quad \begin{tikzcd} A \arrow[r, "\theta\circ\varphi_X"] \arrow[d, "\varphi_Y"'] & Z \arrow[d, "g_Z"] \\ Y \arrow[r, "g_L\circ f_Y"] \arrow[ur, phantom, "\lrcorner"] & W \end{tikzcd}$$

is also a pushout.

Proof.

(1) This statement follows easily from the definitions.

(2) (a) This statement also follows easily from the definitions.

(b) See exercise sheet 11.

(3) This statement follows from purely formal arguments which do not become clearer by writing them down. Thus we leave it to the reader to go through the argument. ■

5. Products and disjoint unions of topological spaces

5.1. The product of topological spaces.

Proposition 5.1. (Topological-Product Proposition) Let $\{X_i\}_{i\in I}$ be a family of topological spaces. The following statements hold:

(0) The set

$$\mathcal{B} := \left\{ \prod_{i\in I} U_i \subset \prod_{i\in I} X_i \;\middle|\; \begin{array}{ll} (1) & \text{each } U_i \text{ is open in } X_i \text{ and} \\ (2) & \text{there exist only finitely many } i\text{'s with } U_i \neq X_i \end{array} \right\}$$

has the basis property that we introduced on page 36. Thus it follows from the Topology-from-Basis Lemma 1.13 that $\mathcal{B}$ defines a topology on $\prod_{i\in I} X_i$.

(1) (a) For any $i \in I$ the projection

$$\begin{array}{rcl} p_i \colon \prod_{i\in I} X_i & \to & X_i \\ f & \mapsto & f(i) \end{array}$$

is continuous with respect to the topology from (0).

(b) Given a topological space Z and given a family $\{f_i \colon Z \to X_i\}_{i\in I}$ of continuous maps the map

$$\begin{array}{rcl} F \colon Z & \to & \prod_{i\in I} X_i \\ z & \mapsto & \begin{pmatrix} I & \to & \bigcup_{i\in I} X_i \\ i & \mapsto & f_i(z) \end{pmatrix}. \end{array}$$

is continuous with respect to the topology from (0).

(∗) A map $F \colon Z \to \prod_{i\in I} X_i$ from some topological space Z is continuous if and only if each map $p_i \circ F \colon Z \to X_i$ is continuous.

(2) The topological space $\prod_{i\in I} X_i$ together with the natural projections $p_j \colon \prod_{i\in I} X_i \to X_j$ is a product (in the sense of the definition on page 61) of $\{X_i\}_{i\in I}$ in the category $\mathcal{T}op$ of topological spaces.

Proof of Proposition 5.1. Let $\{X_i\}_{i\in I}$ be a family of topological spaces.

(0) We need to verify that

$$\mathcal{B} := \left\{ \prod_{i\in I} U_i \subset \prod_{i\in I} X_i \;\middle|\; \begin{array}{ll} (1) & \text{each } U_i \text{ is open in } X_i \text{ and} \\ (2) & \text{there exist only finitely many } i\text{'s with } U_i \neq X_i \end{array} \right\}$$

satisfies the conditions (B1) and (B2) formulated on page 36.

(B1) Evidently we have $\prod_{i\in I} X_i \in \mathcal{B}$. It follows immediately that $\mathcal{B}$ satisfies (B1).

(B2) Let $\prod_{i\in I} U_i$ and $\prod_{i\in I} V_i$ be two sets in $\mathcal{B}$. Then

$$\prod_{i\in I} \underbrace{U_i}_{\substack{\text{only finitely} \\ \text{many are } \neq X_i}} \cap \prod_{i\in I} \underbrace{V_i}_{\substack{\text{only finitely} \\ \text{many are } \neq X_i}} = \prod_{i\in I} \underbrace{(U_i \cap V_i)}_{\substack{\text{only finitely} \\ \text{many are } \neq X_i}}$$

is again a set in $\mathcal{B}$. Thus $\mathcal{B}$ also satisfies (B2).

(1) (a) It follows immediately from the definitions that the projection maps p_i are continuous with respect to the given topology.

(b) Suppose we are given a topological space Z and continuous maps $f_i\colon Z \to X_i$. We denote by $F\colon Z \to \prod_{i\in I} X_i$ the map that is given by $z \mapsto (i \mapsto f_i(z))$. We claim that F is continuous. By the Subbasis-Continuity Proposition 2.2 it suffices to verify that the preimage of any set in the basis $\mathcal{B}$ is open. So let $\prod_{i\in I} U_i$ be a set in $\mathcal{B}$. By definition there exists a finite set $J \subset I$ such that $U_i = X_i$ for any $i \notin J$. Then

$$F^{-1}\Big(\prod_{i\in I} U_i\Big) = \bigcap_{i\in I} f_i^{-1}(U_i) = \bigcap_{j\in j} \underbrace{f_j^{-1}(U_j)}_{\text{open in } Z} = \begin{array}{c}\text{intersection of } \textit{finitely} \\ \text{many open subsets of } Z.\end{array}$$

This concludes the proof that F is continuous.

(∗) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) Let Z be a topological space and let $\{f_i\colon Z \to X_i\}_{i\in I}$ be a family of continuous maps. By the definition of a product on page 61 we need to show that there exists a unique continuous map $\Phi\colon Z \to \prod_{i\in I} X_i$ such that $f_i = p_i \circ \Phi$ for all $i \in I$. To do so, first note that it follows from the Product-of-Sets-and-Groups Lemma 4.2 that the map

$$\begin{array}{rcl} \Phi\colon Z & \to & \prod_{i\in I} X_i \\ z & \mapsto & (f_i(z))_{i\in I} \end{array}$$

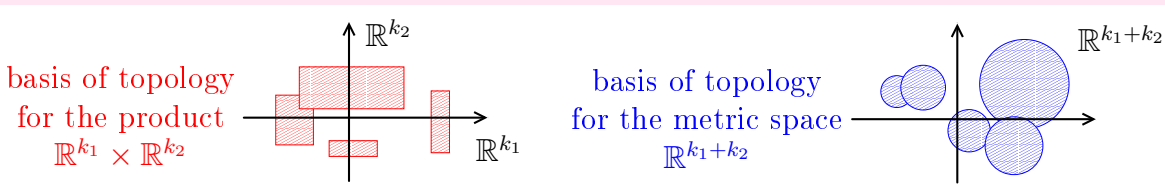
is the unique map (of sets) $\Phi\colon Z \to \prod_{i\in I} X_i$ such that for each $i \in I$ we have the equality $f_i = p_i \circ \Phi$. In (1b) we just showed that this map is fortunately continuous. These two observations give us the desired result. ■

Definition. Let $\{X_i\}_{i\in I}$ be a family of topological spaces. We refer to the topology on the set $\prod_{i\in I} X_i$ that we defined in the Topological-Product Proposition 5.1 (0) as the product topology on $\prod_{i\in I} X_i$.

Lemma 5.2. (Product Metric Lemma) Let $k_1, \dots, k_m \in \mathbb{N}_0$. The map

$$\begin{array}{ccc} \overbrace{\mathbb{R}^{k_1+\cdots+k_m}}^{\text{Euclidean topology}} & \to & \overbrace{\mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_m}}^{\text{product of Euclidean topologies}} \\ (x_1, \dots, x_{k_1}, \dots, x_1, \dots, x_{k_m}) & \mapsto & ((x_1, \dots, x_{k_1}), \dots, (x_1, \dots, x_{k_m})) \end{array}$$

is a homeomorphism.



Proof. The proof, which is not very difficult, can be found in [Fri23]. ■

Lemma 5.3. (Product Topology-Basics Lemma)

(1) Let $\{X_i\}_{i\in I}$ be a family of topological spaces.

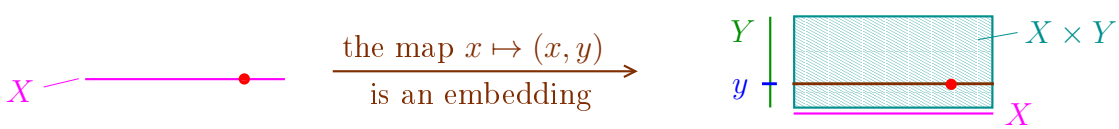
(a) For each $i \in I$ let A_i be a subspace of X_i.

(i) The subspace topology of $\prod_{i\in I} A_i \subset \prod_{i\in I} X_i$ agrees with product topology on $\prod_{i\in I} A_i$.

(ii) The natural map
$$\prod_{i\in I} A_i \;\to\; \prod_{i\in I} X_i$$
given by the inclusions $A_i \to X_i$ is an embedding.

(b) If $J \subset I$ is a subset such that for each $j \in J$ the set X_j consists of a single element x_j, then the map
$$\prod_{i\in I\setminus J} X_i \;\to\; \prod_{i\in I} X_i$$
$$(f\colon I\setminus J \to \bigcup_{i\in I\setminus J} X_i) \;\mapsto\; \left(\begin{array}{ccl} I & \to & \bigcup_{i\in I} X_i \\ i & \mapsto & \begin{cases} x_i, & \text{if } i \in J, \\ f(i), & \text{if } i \notin J \end{cases} \end{array}\right).$$
is a homeomorphism.

(2) If $\{X_i\}_{i\in I}$ is a family of topological spaces and if for each $i \in I$ we are given a basis $\mathcal{B}_i$ of the topological space X_i, then
$$\mathcal{C} \;=\; \left\{ \prod_{i\in I} U_i \;\subset\; \prod_{i\in I} X_i \;\middle|\; \begin{array}{ll} (1) & \text{each } U_i \neq X_i \text{ is contained in } \mathcal{B}_i \text{ and} \\ (2) & \text{there exist only finitely many } i\text{'s with } U_i \neq X_i \end{array} \right\}$$
is a basis for $\prod_{i\in I} X_i$.



Proof of Lemma 5.3.

(1) (a) (i) This statement follows easily from the observation that if we take the basis of the topology of the product $\prod_{i\in I} X_i$ from the Topological-Product Proposition 5.1 (0) and if we intersect each basis element with $\prod_{i\in I} A_i$ we obtain precisely the basis of the product $\prod_{i\in I} A_i$ from the Topological-Product Proposition 5.1 (0).

(ii) The given map is clearly an injection. It follows from the Topological-Product Proposition 5.1 (0) that this map is continuous and it follows from (i) that this map is a homeomorphism onto its image.

(b) It follows from the Topological-Product Proposition 5.1 that the given map is continuous. It has an obvious inverse, which is also continuous by the Topological-Product Proposition 5.1.

(2) First note that by definition the sets in $\mathcal{C}$ are open. Next let $W \subset \prod_{i\in I} X_i$ be an open subset and let $(x_i)_{i\in I} \in W$. By definition of the product topology there exist open subsets $V_i \subset X_i$, $i \in I$ such that $X_i = V_i$ for all but finitely many $i \in I$ and such that $(x_i)_{i\in I} \subset \prod_{i\in I} V_i \subset W$. Since $\mathcal{B}_i$ is a basis for X_i we see that for each i with $V_i \neq X_i$ there

exists a $U_i \in \mathcal{B}_i$ with $x \in U_i \subset V_i$. For all other i we set $U_i := V_i = X_i$. It follows that

$$(x_i)_{i\in I} \in \prod_{i\in I} U_i, \subset \prod_{i\in I} V_i \subset W.$$

We deduce that the given set $\mathcal{C}$ is indeed a basis for $\prod_{i\in I} X_i$. ■

5.2. Properties of products of topological spaces I.

Proposition 5.4. (Product Topology-Properties Proposition) Let $X_1, \dots, X_k$ be non-empty topological spaces.

(1) The product $X_1 \times \cdots \times X_k$ is compact if and only if each X_i is compact.
(2) The product $X_1 \times \cdots \times X_k$ is (path)-connected if and only if each X_i is (path)-connected.
(3) The product $X_1 \times \cdots \times X_k$ is Hausdorff if and only if each X_i is Hausdorff.

Proof. We proved the proposition in Analysis IV. Alternatively see [Fri23]. ■

Definition.

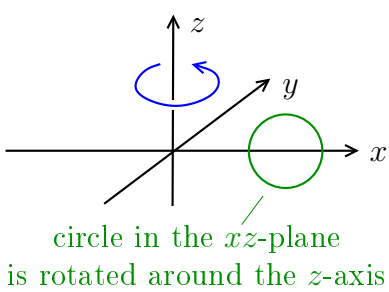
(1) We refer to $(S^1)^n$ as the n-dimensional torus. For $n = 2$ we just say the torus.
(2) We refer to $S^1 \times \overline{B}^2$ as the solid torus and we refer to $S^1 \times B^2$ as the open solid torus.

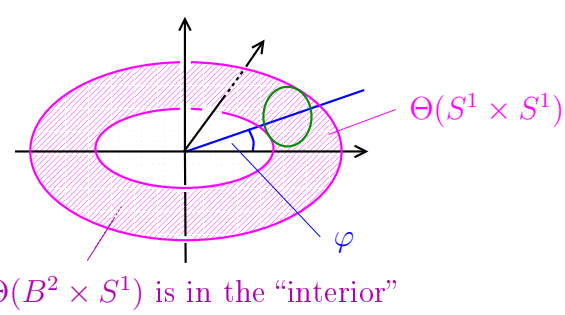
We deduce from the Product Topology-Properties Proposition 5.4 that the n-dimensional torus $(S^1)^n$ and the solid torus $S^1 \times \overline{B}^2$ are compact.

Lemma 5.5. (Torus Embedding Lemma) The map

$$\begin{aligned} \Theta\colon \overline{B}^2 \times S^1 &\to \mathbb{R}^3 \\ ((x,y), \exp(\mathrm{i}\varphi)) &\mapsto \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation around } z\text{-axis}} \cdot \underbrace{\begin{pmatrix} 1+\frac{1}{3}x \\ 0 \\ -\frac{1}{3}y \end{pmatrix}}_{\substack{\text{describes circle} \\ \text{in } xz\text{-plane}}} \end{aligned}$$

is an embedding. The restrictions of Θ to $S^1 \times S^1$ and $B^2 \times S^1$ are also embeddings.





Proof. The map $\Theta\colon \overline{B}^2 \times S^1 \to \mathbb{R}^3$ can be written as the composition of the maps

$$\begin{aligned} \overline{B}^2 \times S^1 \to \mathbb{R}^2 \times (\mathbb{R}^2 \setminus \{0\}) \xleftarrow{\cong} \{(x,y,a,b) \in \mathbb{R}^4 \mid a^2+b^2 > 0\} &\to \mathbb{R}^3 \\ (x,y,a,b) &\mapsto \begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1+\frac{1}{3}x \\ 0 \\ -\frac{1}{3}y. \end{pmatrix} \end{aligned}$$

The first map is continuous by the Product Topology-Properties Proposition 5.4 (1), the second map is a homeomorphism by the Product Metric Lemma 5.2 and the last map is continuous by the Sum-Product-Continuity Lemma 2.6. This shows that Θ is continuous.

An elementary argument shows that the restriction of Θ to $\overline{B}^2 \times S^1$ is injective. By the Product Topology-Properties Proposition 5.4 (1) we know that the solid torus $\overline{B}^2 \times S^1$ is compact. Thus we obtain from the Compact-Hausdorff Proposition 2.10 (3) that the restriction of Θ to $\Theta\colon \overline{B}^2 \times S^1 \to \Theta(S^1 \times \overline{B}^2)$ is an embedding. But then it follows that the restrictions of Θ to $S^1 \times S^1$ and $B^2 \times S^1$ are embeddings. ■

5.3. Properties of products of topological spaces II.

Theorem 5.6. (Tychonoff's Theorem) The product of arbitrarily many compact spaces is again a compact space.

Proof. Let $\{X_i\}_{i\in I}$ be a family of topological space. We will prove Tychonoff's Theorem with $I = \mathbb{N}$ in exercise sheet 2, with copious hints. The proof of the general case builds on the Axiom of Choice. We refer to [Fri23] for references to proofs in the literature. ■

Example. We equip $\{0, 2\}$ with the discrete topology. We consider the map

$$\begin{array}{rcl} \varphi\colon \prod_{i\in\mathbb{N}} \{0,2\} & \to & C = \text{Cantor set from page 35} \\ (f\colon \mathbb{N} \to \{0,2\}) & \mapsto & \sum_{i=1}^{\infty} \frac{f(i)}{3^i}. \end{array}$$

It is elementary to see that φ takes values in C. It is a refreshing exercise to show that φ is a bijection. The left-hand side is compact by Tychonoff's Theorem 5.6 and the right-hand is evidently Hausdorff. We leave it to the reader to verify, using the Subbasis-Continuity Proposition 2.2 and the JH-Openness Criterion 1.1, that φ is continuous. Thus it follows from the Compact-Hausdorff Proposition 2.10 (3) that φ is a homeomorphism.

5.4. The pullback of topological spaces.

On page 66 we introduced the general notion of pullbacks in a category. In this section we show that the category of topological spaces always has pullbacks. It is rather reassuring to hear that this is the case, even though the pullback will only really play a role much later when we study bundles. Thus this section can also be skipped without any major loss.

Proposition 5.7. (Topological-Pullback Proposition) Let $p\colon X \to B$ be a continuous map of topological spaces, let C be a topological space and let $f\colon C \to B$ be a continuous map. We consider the topological space

$$f^*X \;:=\; \{(c,x) \in C \times X \mid f(c) = p(x)\}$$

and the maps $\begin{array}{rcl} \varphi\colon f^*X & \to & C \\ (c,x) & \mapsto & c \end{array}$ and $\begin{array}{rcl} \psi\colon f^*X & \to & X \\ (c,x) & \mapsto & x. \end{array}$

The following statements hold:

(1) (a) The maps $\varphi\colon f^*X \to C$ and $\psi\colon f^*X \to X$ are continuous.

(b) Suppose we are given a topological space W and two continuous maps $\alpha\colon W \to C$ and $\beta\colon W \to X$ such that $f \circ \beta = p \circ \alpha\colon W \to B$. The map

$$\begin{array}{rcl} \Phi\colon W & \to & f^*X \\ w & \mapsto & (\alpha(w), \beta(w)). \end{array}$$

is continuous. In particular we obtain the following commutative diagram:

$$\begin{tikzcd} W \arrow[dr, dashed, "\exists!\,\Phi"] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\ & f^*X \arrow[r, "\psi"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\ & C \arrow[r, "f"'] & B. \end{tikzcd}$$

($*$) A map $\Phi\colon W \to f^*X$ is continuous if and only if the maps $\varphi \circ \Phi\colon W \to C$ and $\psi \circ \Phi\colon W \to X$ are continuous.

(2) The topological space f^*X with the natural maps $\varphi\colon f^*X \to C$ and $\psi\colon f^*X \to X$ is the pullback in the category $\mathcal{Top}$ of topological spaces.

Proof.

(1) (1) This statement follows from the Topological-Product Proposition 5.1 (1a) and the Basics-of-Continuity Lemma 2.1 (3).

(2) This statement follows from the Topological-Product Proposition 5.1 (1b) and the Basics-of-Continuity Lemma 2.1 (4).

($*$) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) It remains to show uniqueness of Φ. This follows immediately from the definitions and the fact that a point in $C \times X$ is uniquely determined by its projections. ■

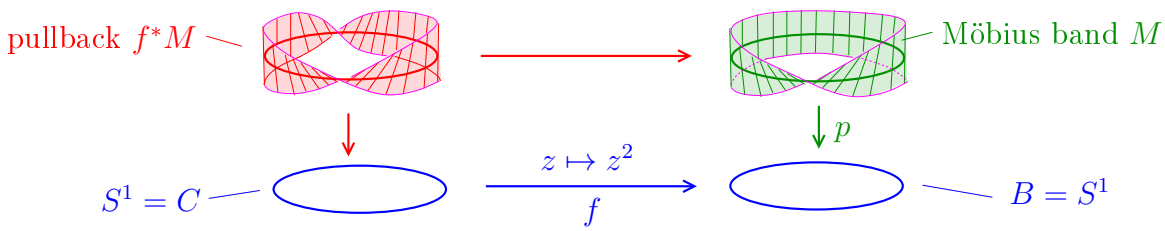
Example.

(1) Let $p\colon X \to B$ be a continuous map of topological spaces and let $C \subset B$ be a subspace. We denote by $\iota\colon C \to B$ the inclusion. One can easily verify that the maps

$$\begin{array}{rcl} p^{-1}(C) & \to & \iota^*X \\ x & \mapsto & (p(x), x) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \iota^*X & \to & p^{-1}(C) \\ (c, x) & \mapsto & x \end{array}$$

are continuous and inverses of one another. Thus we see that ι^*X is naturally homeomorphic to $p^{-1}(C)$.

(2) In the figure below we show the projection $p\colon M \to S^1$ from the Möbius band to S^1 and we see the map $f\colon C = S^1 \to S^1 = B$ that is given by $f(z) = z^2$. It is a wonderful exercise to convince oneself that the pullback f^*M is a "doubly twisted band" and thus homeomorphic to $S^1 \times [-1, 1]$.



5.5. The disjoint union of topological spaces.

Convention. Let $\{A_i\}_{i\in I}$ be a family of sets. For any $i \in I$ we use the natural injection $A_i \to \bigsqcup_{i\in I} A_i$ to think of A_i as a subset of the disjoint union.

Proposition 5.8. (Topological-Coproduct Proposition) Let $\{X_i\}_{i\in I}$ be a family of topological spaces. We consider the disjoint union $\bigsqcup_{i\in I} X_i$.

(0) (a) We say $U \subset \bigsqcup_{i\in I} X_i$ is open if and only if for each $i \in I$ the intersection $X_i \cap U$ is an open subset of X_i. This defines a topology on $\bigsqcup_{i\in I} X_i$. We will work with this topology throughout the proposition.

(b) A subset $A \subset \bigsqcup_{i\in I} X_i$ is closed if and only if for each $i \in I$ the intersection $X_i \cap A$ is a closed subset of X_i.

(1) (a) Each natural injection $\iota_j \colon X_j \to \bigsqcup_{i\in I} X_i$ is continuous. In fact it is an embedding that is open and closed.

(b) If Z is a topological space and if $\{f_i \colon X_i \to Z\}_{i\in I}$ is a family of continuous maps, then the map

$$\begin{array}{rcl} \bigsqcup_{i\in I} X_i & \to & Z \\ (x,i) & \mapsto & f_i(x) \end{array}$$

is continuous.

($*$) A map $f \colon \bigsqcup_{i\in I} X_i \to Z$ to a topological space Z is continuous if and only if for each $i \in I$ the map $f \circ \iota_i \colon X_i \to Z$ is continuous.

(2) The topological space $\bigsqcup_{i\in I} X_i$ together with the natural injections ι_i is a coproduct of $\{X_i\}_{i\in I}$ in the category $\mathcal{Top}$ of topological spaces.

Proof.

(0) (a) This statement follows easily from the definitions.

(b) This statement follows from an elementary argument.

(1) (a) It follows immediately from writing down the definitions that each natural injection $X_j \to \bigsqcup_{i\in I} X_i$ is continuous, and that it is open and closed. It follows from the Open-Injective Lemma 2.9 that the natural injection is an embedding that is open and closed.

(b) Let $\{f_i \colon X_i \to Z\}_{i\in I}$ be a family of continuous maps. We consider the corresponding map $F \colon \bigsqcup_{i\in I} X_i \to Z$. Let $U \subset Z$ be an open subset. For each $i \in I$ we have $F^{-1}(U) \cap X_i = f_i^{-1}(U)$. This shows that $F^{-1}(U)$ is open, i.e. F is continuous.

(∗) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) Let Z be a topological space and let $\{f_i\colon X_i \to Z\}_{i\in I}$ be a family of continuous maps. By the definition of the coproduct on page 63 we have to show that there exists a *unique* continuous map $\Phi\colon \bigsqcup_{i\in I} X_i \to Z$ such that for each $i \in I$ we have $f_i = \Phi \circ \iota_i$. In (1b) we defined such a map Φ and we saw that it is continuous. The proof of uniqueness is basically identical to the argument in the proof of the Coproduct-of-Sets-and-Abelian Groups Lemma 4.4 (1) where we dealt with the coproduct in the category of sets. ■

Definition. Let $\{X_i\}_{i\in I}$ be a family of topological spaces. We refer to the topology on $\bigsqcup_{i\in I} X_i$ that we introduced in the Topological-Coproduct Proposition 5.8 as the disjoint union topology. We refer to this topological space as the disjoint union of the topological spaces X_i.

Lemma 5.9. (Disjoint Union Topology-Properties Lemma) Let $\{X_i\}_{i\in I}$ be a family of topological spaces.

(1) The following two statements are equivalent:
  (a) The disjoint union $\bigsqcup_{i\in I} X_i$ is compact.
  (b) Each X_i is compact and there are only finitely many $i \in I$ such that $X_i \neq \varnothing$.

(2) (a) Each X_i is an open and closed subset of $\bigsqcup_{i\in I} X_i$.
  (b) The path-components of the disjoint union $\bigsqcup_{i\in I} X_i$ are precisely the path-components of the X_i.

(3) The disjoint union $\bigsqcup_{i\in I} X_i$ is Hausdorff if and only if each X_i is Hausdorff.

Proof. We prove the statements of the lemma in a different order than they were stated.

(2) (a) Let $i \in I$. Note that for $j \neq i$ we have $X_i \cap X_j = \varnothing$, which is open and closed. Thus it follows from the Topological-Coproduct Proposition 5.8 (0a) and (0b) that X_i is an open and closed subset.
  (b) This statement follows from (3a) and the Path-Component Lemma 2.21 (4).

(1) We first prove the "(b)⇒(a)"-direction. Thus we assume that each X_i is compact and that there are only finitely many $i_1, \dots, i_k \in I$ such that the corresponding topological space is non-empty. Let $\{U_j\}_{j\in J}$ be an open cover of $\bigsqcup_{i\in I} X_i$. Since each X_i is compact we can find for each $i \in I$ a finite subset $J_i \subset J$ such that $\bigsqcup_{j\in J_i} (U_j \cap X_i) = X_i$. It is now clear that $X = \bigcup_{r\in\{1,\dots,k\}} X_i = \bigcup_{r\in\{1,\dots,k\}} \bigsqcup_{j\in J_{i_r}} U_j$.

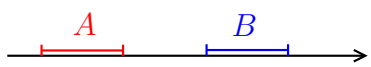
We turn to the proof of the "(a)⇒(b)"-direction. Thus we assume that $\bigsqcup_{i\in I} X_i$ is compact. It follows from (2a) together with the Compact-Closed Lemma 1.9 (1) that each X_i is compact. Next note that by (2a) we also know that the $\{X_i\}_{i\in I}$ form an open cover of the compact topological space $\bigsqcup_{i\in I} X_i$. Thus we see that there exists a finite subset $J \subset I$ with $\bigsqcup_{j\in J} X_j = \bigsqcup_{i\in I} X_i$. In particular $X_i = \varnothing$ for $i \notin J$.

(3) This statement follows easily from the definitions and (2a). ■

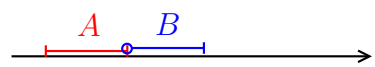
Example. Let X be a topological space and let $A, B \subset X$ be subsets of X. It follows from the Topological-Coproduct Proposition 5.8 (1b) that the obvious map

$$\underbrace{A \sqcup B}_{\text{disjoint union topology}} \quad \to \quad \underbrace{A \cup B}_{\text{subspace topology}}$$

is continuous. If A and B are disjoint, then this map is also a bijection. To be more specific, let $X = \mathbb{R}$. For $A = [0,1]$ and $B = [2,3]$ one can show by hand that the map is open, thus a homeomorphism. On the other hand for $A = [0,1]$ and $(1,2]$ the above map is certainly not a homeomorphism, for example $A \sqcup B$ is not compact whereas $A \cup B = [0,2]$ is compact.



$A \sqcup B$ is homeomorphic to $A \cup B \subset \mathbb{R}$



$A \sqcup B$ is *not* homeomorphic to $A \cup B \subset \mathbb{R}$

6. Quotients of topological spaces

6.1. The quotient topology.

Proposition 6.1. (Topological-Quotient Proposition) Let $\sim$ be an equivalence relation on a topological space X. We denote by $p\colon X \to X/\sim$ the natural projection from X onto the set of equivalence classes $X/\sim$. The following statements hold:

(0) (a) The set
$$\mathcal{T} \;:=\; \{U \subset X/\sim \mid p^{-1}(U) \text{ is open in } X\}$$
is a topology on $X/\sim$. We use this topology throughout this proposition.

(b) A subset $A \subset X/\sim$ is closed if and only if the preimage $p^{-1}(A)$ is closed in X.

(1) (a) The natural projection $p\colon X \to X/\sim$ is continuous.

(b) Let Y be a topological space and let $f\colon X \to Y$ be a continuous map with the property that $f(x) = f(y)$ whenever $x \sim y$. The induced map
$$\begin{array}{rcl} g\colon X/\sim & \to & Y \\ [x] & \mapsto & f(x) \end{array}$$
is also continuous.

($*$) A map $f\colon X/\sim \to Y$ is continuous if and only if the map $f \circ p\colon X \to Y$ is continuous.

(2) If $\mathcal{S}$ is a topology on $X/\sim$ such that $(X/\sim, \mathcal{S})$ has properties (1a) and (1b), then $\mathcal{S}$ equals the topology from (0).

Proof. Let $\sim$ be an equivalence relation on a topological space X.

(0) (a) This statement follows easily from the Image-Preimage Lemma 0.3 (6) and (7).

(b) This statement follows easily from the Image-Preimage Lemma 0.3 (8).

(1) (a) It follows immediately from the definition of the topology on $X/\sim$ that the natural projection $p\colon X \to X/\sim$ is continuous.

(b) Let $f\colon X \to Y$ be a continuous map such that $f(x) = f(y)$ whenever $x \sim y$. We need to show that the induced map $g\colon X/\sim \to Y$ is continuous as well. Let $U \subset Y$ be open. We need to show that $g^{-1}(U) \subset X/\sim$ is open. By definition of the quotient topology on $X/\sim$ we need to show that $p^{-1}(g^{-1}(U))$ is open in X. We observe that
$$p^{-1}(g^{-1}(U)) \;=\; (g \circ p)^{-1}(U) \;\underset{\substack{\uparrow \\ \text{by the commutative diagram}}}{=}\; f^{-1}(U) \;\underset{\substack{\uparrow \\ \text{since } f \text{ continuous}}}{=}\; \text{open subset}\,.$$
We have thus shown that g is continuous.

($*$) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) Let $\mathcal{S}$ be a topology on $X/\sim$ such that the following conditions are satisfied:

(a′) The natural projection $p\colon X \to (X/\sim, \mathcal{S})$ is continuous.

(b′) Whenever we are given a topological space Y and a continuous map $f\colon X \to Y$ with the property that $f(x) = f(y)$ whenever $x \sim y$, then the corresponding induced map $(X/\sim, \mathcal{S}) \to Y$ is also continuous.

We apply (b) and (a′) to obtain that the identity map $\Phi\colon (X/\sim, \mathcal{T}) \to (X/\sim, \mathcal{S})$ is continuous. This implies that $\mathcal{S} \subset \mathcal{T}$. Similarly we apply (a) and (b′) to obtain that

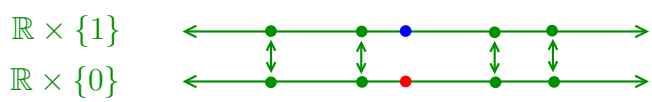
the identity map $\Psi\colon (X/\sim, \mathcal{S}) \to (X/\sim, \mathcal{T})$ is continuous, which tells us that $\mathcal{T} \subset \mathcal{S}$. In summary we see that $\mathcal{S} = \mathcal{T}$. ■

Definition. Let $\sim$ be an equivalence relation on a topological space X.

(1) We refer to the topology on $X/\sim$ from the Topological-Quotient Proposition 6.1 (0a) as the quotient topology and we refer to $X/\sim$ as a quotient space of X.

(2) Let Y be a topological space and let $f\colon X \to Y$ be a continuous map with the property that $f(x) = f(y)$ whenever $x \sim y$.
   (a) We refer to the corresponding map $X/\sim \to Y$ as the induced map.
   (b) At times we denote the induced map by $\overline{f}$. Sometimes, when there is no danger of confusion we denote the induced map just by f itself.
   (c) Sometimes we also say that the map $f\colon X \to Y$ descends to a map $X/\sim \to Y$.

Example. We consider the topological space $X = (\mathbb{R}\times\{0\})\cup(\mathbb{R}\times\{1\})$ and the equivalence relation that is generated by $(x,0) \sim (x,1)$ for all $x \neq 0$. We denote by $p\colon X \to X/\sim$ the natural projection. One can easily show that $p((0,0))$ and $p((0,1))$ cannot be separated by open neighborhoods. This shows that the quotient of a Hausdorff space is not necessarily Hausdorff.



Definition. Let X be a topological space and let $\sim$ be an equivalence relation on X. Recall that an equivalence relation is, by the definition on page 20, a subset of $X \times X$ with suitable properties. We say that the relation $\sim$ is closed if $\sim$ is a closed subset of $X \times X$.

Proposition 6.2. (Quotient Topology-Properties Proposition) Let $\sim$ be an equivalence relation on a topological space X.

(1) If X is compact, then $X/\sim$ is also compact.
(2) If X is (path)-connected, then $X/\sim$ is also (path)-connected.
(3) If X is Hausdorff and compact and if $\sim$ is a closed equivalence relation on X, then $X/\sim$ is Hausdorff.

Proof. By the Topological-Quotient Proposition 6.1 (1a) we know that the natural projection $X \to X/\sim$ is continuous. The first two statements of the lemma thus follow from the Compact Image Lemma 2.8 and the Image-Connected Lemma 2.16. The proof of the third statement is non-trivial but dull, we refer to [Fri23] for details. ■

Lemma 6.3. (Twice Quotient Lemma) Let $f\colon X \to Y$ be a continuous map between two topological spaces and let $\sim_X$ and $\sim_Y$ be equivalence relations on X respectively Y. If $x_1 \sim_X x_2$ implies that $f(x_1) \sim_Y f(x_2)$, then the map

$$\begin{array}{rcl} X/\sim_X & \to & Y/\sim_Y \\ [x] & \mapsto & [f(x)] \end{array}$$

is well-defined and continuous.

Proof. First note that it follows from the Topological-Quotient Proposition 6.1 (1a) that the map $X \xrightarrow{f} Y \xrightarrow{y \mapsto [y]} Y/\sim_Y$ is continuous. Next note that it follows from our hypothesis on f and the Topological-Quotient Proposition 6.1 (1b) that this map descends to a continuous map $X/\sim_X \to Y/\sim_Y$. ■

6.2. Examples: Surfaces I.

(1) We consider $X = [0,1] \times [0,1] \subset \mathbb{R}^2$ and the equivalence relation that is generated by

$$(x,0) \sim (x,1) \quad \text{for all } x \in [0,1].$$

We consider the map

$$\begin{array}{rcl} X/\sim & \to & S^1 \times [0,1] \\ [(s,t)] & \mapsto & (\exp(2\pi \mathrm{i} t), s). \end{array}$$

It follows from the Topological-Product Proposition 5.1 (1b) and the Topological-Quotient Proposition 6.1 (1b) that this map is continuous. One can easily verify that the map is a bijection. By the Quotient Topology-Properties Proposition 6.2 (1) we know that $X/\sim$ is compact and by the Product Topology Properties Proposition 5.4 (3) we know that $S^1 \times [0,1]$ is Hausdorff. Thus we obtain from the Compact-Hausdorff Proposition 2.10 (3) that the above map $X/\sim \to S^1 \times [0,1]$ is a homeomorphism. We refer to $X/\sim$ and also to $S^1 \times [0,1]$ as the cylinder or the annulus.

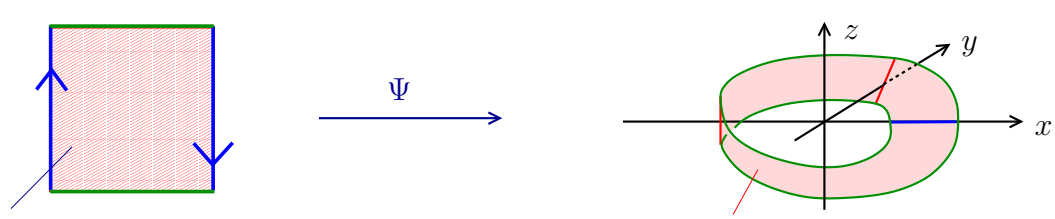
(2) We consider $X = [0,1] \times [0,1] \subset \mathbb{R}^2$, this time with the equivalence relation that is generated by

$$(0,y) \sim (1, 1-y) \quad \text{for all } y \in [0,1].$$

We refer to the quotient $X/\sim$ as the Möbius band. The name "Möbius band" conjures up a certain picture. We will now see that our name is justified. We consider the map

$$\begin{array}{rcl} \Psi\colon ([0,1] \times [0,1])/\sim & \to & \mathbb{R}^3 \\ [(\varphi, r)] & \mapsto & \underbrace{\begin{pmatrix} \cos(2\pi\cdot\varphi) & -\sin(2\pi\cdot\varphi) & 0 \\ \sin(2\pi\cdot\varphi) & \cos(2\pi\cdot\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation around } z\text{-axis}} \cdot \underbrace{\begin{pmatrix} 2 + (r - \frac{1}{2}) \cdot \sin(\pi\cdot\varphi) \\ 0 \\ (r - \frac{1}{2}) \cdot \cos(\pi\cdot\varphi) \end{pmatrix}}_{\substack{\text{describes interval} \\ \text{that makes half a twist} \\ \text{as we go around the } z\text{-axis}}} \end{array}$$

Similar to (1) one can see that Ψ is an embedding.



$M = ([0,1] \times [0,1])/(0,y) \sim (1, 1-y)$ Möbius band viewed as a subspace of $\mathbb{R}^3$

(3) We consider again $X = [0,1] \times [0,1] \subset \mathbb{R}^2$, but this time with the equivalence relation that is generated by

$$(x,0) \sim (x,1) \quad \text{for all } x \in [0,1]$$

and by

$$(0,y) \sim (1,y) \quad \text{for all } y \in [0,1].$$

As in (1) one can write down an obvious homeomorphism $X/\sim \to S^1 \times S^1$.

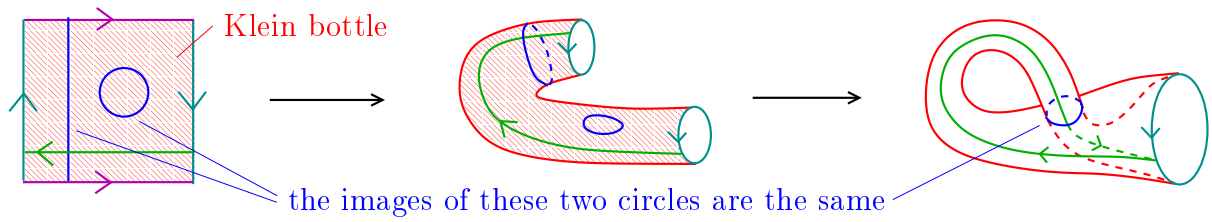
(4) We consider $X = [0,1] \times [0,1] \subset \mathbb{R}^2$, this time with the equivalence relation that is generated by

$$(x,0) \sim (x,1) \qquad \text{for all } x \in [0,1]$$

and by

$$(0,y) \sim (1,1-y) \qquad \text{for all } y \in [0,1].$$

The resulting topological space is called the Klein bottle. We obtain the quotient space $X/\sim$ by gluing the upper edge to the lower edge and by gluing the edge on the left to the edge on the right with a twist. In the figure below we show the Klein bottle to the left. Furthermore we indicate a map from the Klein bottle to $\mathbb{R}^3$. This particular map is not injective.



6.3. Quotients by subspaces.

Definition. Let X be a topological space and let $A \subset X$ be a subset.

(1) First we suppose that A is a non-empty subset. For $P, Q \in X$ we define

$$P \sim_A Q \quad :\Longleftrightarrow \quad P = Q \ \text{ or } \ P, Q \text{ both lie in } A.$$

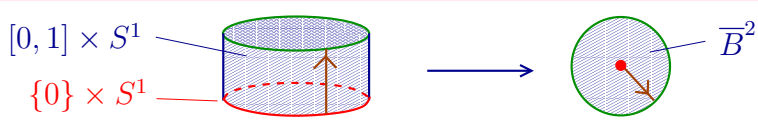
This is easily seen to be an equivalence relation on X. We write $X/A := X/\sim_A$ and we always equip X/A with the quotient topology. We refer to X/A as the quotient of X by A. Note that $A \subset X$ is an equivalence class, thus it defines a point X/A. In an attempt to avoid confusion we write $[A]$ for the point in X/A that is given by the equivalence class A.

(2) If $A = \varnothing$ is the empty set, then we define $X/\varnothing := X \sqcup \{\varnothing\}$, i.e. $X/\varnothing$ equals the disjoint union of the topological space X with the topological space consisting of a single point $\varnothing$. For consistency we denote this point by $[A] = [\varnothing]$ as well.

Lemma 6.4. (Cylinder Quotient-Ball Lemma) The map

$$\begin{array}{rcl} ([0,1] \times S^n)/(\{0\} \times S^n) & \to & \overline{B}^{n+1} \\ [(r,v)] & \mapsto & r \cdot v \end{array}$$

is a homeomorphism.



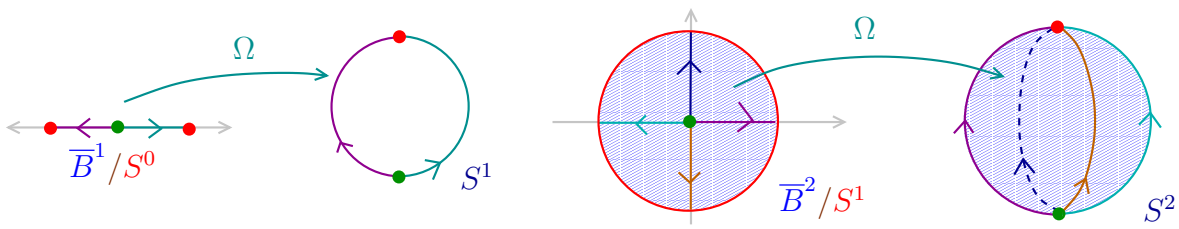
Proof. As in the proof of the Torus Embedding Lemma 5.5 it follows from the Topological-Quotient Proposition 6.1 (1b) that the given map is continuous. It is basically clear that the map is a bijection. The left-hand side is compact by the Topological-Quotient Proposition 6.1 (1b). It follows from the Compact-Hausdorff Proposition 2.10 (3) that the map is a homeomorphism. ■

Lemma 6.5. (Ball Quotient-Sphere Lemma) Let $n \in \mathbb{N}$. The map

$$\begin{array}{rcl} \Omega\colon \overline{B}^n/S^{n-1} & \to & S^n \\ [r\cdot v] & \mapsto & \underbrace{\begin{pmatrix} 0 & (-1)^{n-1} & 0 \\ 0 & 0 & \mathrm{id}_{n-1} \\ -1 & \dots & 0 \end{pmatrix}}_{\in \mathrm{SO}(n+1)} \cdot \underbrace{\begin{pmatrix} \cos(\pi\cdot r) \\ \sin(\pi\cdot r)\cdot v \end{pmatrix}}_{\in \mathbb{R}\times\mathbb{R}^n = \mathbb{R}^{n+1}} \quad \text{with } r\in[0,1] \text{ and } v\in S^{n-1} \end{array}$$

is a homeomorphism.

Since $\overline{B}^0/S^{-1} = \overline{B}^0/\varnothing = \overline{B}^0 \cup \{*\}$ this lemma, with a couple of minor changes, also makes sense for $n=0$.



Identification. We use the homeomorphism from the Ball Quotient-Sphere Lemma 6.5 to make the identification $S^n =_i \overline{B}^n/S^{n-1}$.

Proof of Lemma 6.5. It follows quite easily from the Cylinder Quotient-Ball Lemma 6.4 (2), the Topological-Quotient Proposition 6.1 (1b), the Quotient Topology-Properties Proposition 6.2 (1) and the Compact-Hausdorff Proposition 2.10 (3) that the map $\Omega\colon \overline{B}^n/S^{n-1} \to S^n$ is a homeomorphism. ■

6.4. Group actions on topological spaces.

Definition. Let X be a set and let G be a group with trivial element e.

(1) An action of G on X is a map

$$\begin{array}{rcl} G\times X & \to & X \\ (g,x) & \mapsto & g\cdot x \end{array}$$

with the following properties:

$$\begin{array}{rcll} e\cdot x & = & x, & \text{for all } x\in X, \\ \underset{\substack{\uparrow \\ \text{group action}}}{g\cdot(h\cdot x)} & = & \underset{\substack{\uparrow \\ \text{multiplication in the group } G}}{(g\cdot_G h)}\cdot x, & \text{for all } x\in X \text{ and } g,h\in G. \end{array}$$

(2) The action is called free, if $g\cdot x = x$ for some $x\in X$ implies that $g=e$.

(3) If X is a topological space, then we say that the action is continuous, if for every $g\in G$ the map

$$\begin{array}{rcl} X & \to & X \\ x & \mapsto & g\cdot x \end{array}$$

is continuous.

(4) Let X be a topological space and let G be a group that acts on X. By the Group Action-Equivalence Relation Lemma 0.13 we know that

$$x \sim_G y \quad :\Longleftrightarrow \quad \text{there exists a } g\in G \text{ such that } g\cdot x = y$$

is an equivalence relation on X. We set $X/G := X/\sim_G$ and we view X/G as a topological space equipped with the quotient topology.

Lemma 6.6. (Torus-as-Quotient Lemma)

(1) The map
$$\begin{array}{rcl} \mathbb{Z}^n \times \mathbb{R}^n & \to & \mathbb{R}^n \\ (z, v) & \mapsto & z + v \end{array}$$
defines a free and continuous action of the group $(\mathbb{Z}^n, +)$ on the topological space $X = \mathbb{R}^n$.

(2) The map
$$\begin{array}{rcl} \mathbb{R}^n/\mathbb{Z}^n & \to & (S^1)^n \\ [(t_1, \ldots, t_n)] & \mapsto & \big(\exp(2\pi \mathrm{i} t_1), \ldots, \exp(2\pi \mathrm{i} t_n)\big) \end{array}$$
is a homeomorphism.

Proof of Lemma 6.6. We start out with the following observations:

(1) This statement follows easily from the definitions.

(2) (a) The map is continuous by the Topological-Product Proposition 5.1 (1b) and the Topological-Quotient Proposition 6.1 (1b). Furthermore it is straightforward to see that it is a bijection.

(b) The obvious map $[0,1]^n \to \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is continuous and it is easily seen to be surjective. By the Heine-Borel Theorem 1.11 we know that $[0,1]^n$ is compact. It follows from the Compact Image Lemma 2.8 that $\mathbb{R}^n/\mathbb{Z}^n$ is compact.

(c) From the Product Topology-Properties Proposition 5.4 (3) we obtain that $(S^1)^n$ is Hausdorff.

It follows from the above together with the Compact-Hausdorff Proposition 2.10 (3) that the given map $\mathbb{R}^n/\mathbb{Z}^n \to (S^1)^n$ is indeed a homeomorphism. ■

Example. If X is a Hausdorff space and if G acts continuously on X, then it is unfortunately not necessarily true that the quotient space X/G itself is Hausdorff. For example, consider the obvious action of $G = (\mathbb{Q}, +)$ on $X = \mathbb{R}$. It is a moderately insightful exercise to verify that the quotient space $\mathbb{R}/\mathbb{Q}$ is not Hausdorff.

Definition. Let X be a topological space and let G be a group that acts on X. We say that G acts properly if for every two points x and y in X there exist open neighborhoods U of x and V of y such that the set $\{g \in G \,|\, gU \cap V \neq \varnothing\}$ is finite.

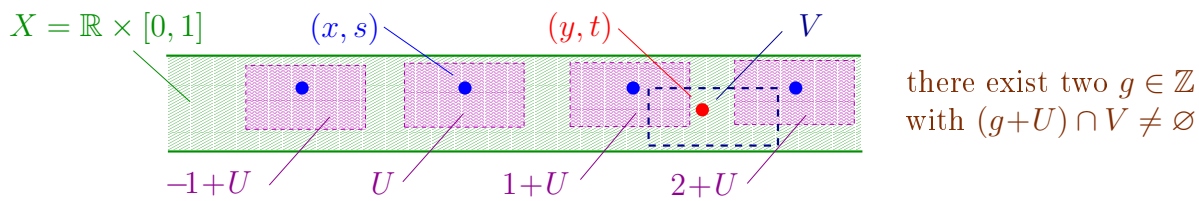
Examples.

(1) The action of any finite group is proper by definition.

(2) We consider the action of the group $G = \mathbb{Z}$ on $X = \mathbb{R} \times [0,1]$ that is given by
$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [0,1]) & \to & \mathbb{R} \times [0,1] \\ (n, (x, y)) & \mapsto & (x + n, y). \end{array}$$
This action is proper. For example if we are given by $(x, s), (y, t) \in \mathbb{R} \times [0,1]$, then $U := \{(a, b) \in \mathbb{R} \times [0,1] \,|\, |a - x| < 1\}$ and $V := \{(a, b) \in \mathbb{R} \times [0,1] \,|\, |a - y| < 1\}$ have the desired properties. With some artistic freedom this situation is shown in the figure below.

(3) Similar to (2) one can easily verify that the action of $G = (\mathbb{Z}^n, +)$ on $X = \mathbb{R}^n$ is proper.

(4) One can easily show that the obvious action of $G = (\mathbb{Q}, +)$ on $X = \mathbb{R}$ is not proper.



Proposition 6.7. (Group Action-Hausdorff Quotient Proposition) Let G be a group that acts on a topological space X. If X is Hausdorff and if the action is proper and continuous, then the quotient space X/G is also Hausdorff.

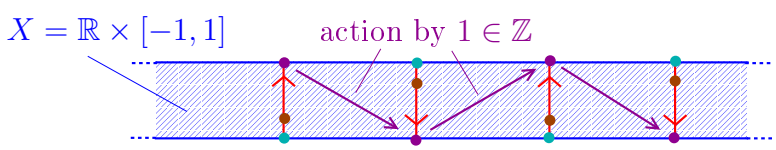
Proof. We outsource the uninspiring proof to [Fri23]. ■

Lemma 6.8. (Möbius Band-as-Quotient Lemma)

(1) The map
$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) & \to & \mathbb{R} \times [-1,1] \\ (n,(x,y)) & \mapsto & (x+n, (-1)^n \cdot y) \end{array}$$
defines an action of the group $(\mathbb{Z},+)$ on the topological space $X = \mathbb{R} \times [-1,1]$ that is free, continuous and proper.

(2) As on page 81 let $([0,1] \times [0,1])/\sim$ be the Möbius band, where $(0,y) \sim (1,1-y)$. The map
$$\begin{array}{rcl} ([0,1] \times [0,1])/\sim & \to & (\mathbb{R} \times [-1,1])/\sim \\ [(x,y)] & \mapsto & [(x,2y-1)] \end{array}$$
is a homeomorphism.

(3) The Möbius band $M = ([0,1] \times [0,1])/\sim$ is compact and Hausdorff.



Identification. We will use the homeomorphism from the Möbius Band-as-Quotient Lemma 6.8 to identify the Möbius band $([0,1] \times [0,1])/\sim$ with $(\mathbb{R} \times [-1,1])/\sim$.

Proof of Lemma 6.8.

(1) It is clear that the action is free and continuous. The fact that the action is proper is proved as in the above example.

(2) It follows from the Twice Quotient Lemma 6.3 that the map is well-defined and continuous. One can easily verify that the map is a bijection. By the Quotient Topology-Properties Proposition 6.2 (1) we know that $([0,1] \times [0,1])/\sim$ is compact. Furthermore it follows from (1) and the Group Action-Hausdorff Quotient Proposition 6.7 that $(\mathbb{R} \times [0,1])/\sim$ is Hausdorff. Thus it follows from the Compact-Hausdorff Proposition 2.10 (3) that the map $([0,1] \times [0,1])/\sim \to (\mathbb{R} \times [0,1])/\sim$ is a homeomorphism.

(3) We showed in the proof of (2) that the Möbius band is compact and Hausdorff. ■

6.5. Projective spaces.

Definition.

(1) The map

$$\begin{array}{rcl}(\mathbb{R}\setminus\{0\})\times(\mathbb{R}^{n+1}\setminus\{0\}) & \to & \mathbb{R}^{n+1}\setminus\{0\}\\ (z,P) & \mapsto & z\cdot P\end{array}$$

defines a continuous action of the multiplicative group $(\mathbb{R}\setminus\{0\},\cdot)$ on the topological space $\mathbb{R}^{n+1}\setminus\{0\}$. The quotient space $(\mathbb{R}^{n+1}\setminus\{0\})/(\mathbb{R}\setminus\{0\})$ is called the n-dimensional real projective space $\mathbb{R}\mathrm{P}^n$. For $n=2$ we sometimes refer to $\mathbb{R}\mathrm{P}^2$ as the real projective plane.

(2) In (1) we can replace "$\mathbb{R}$" by "$\mathbb{C}$" and we obtain the n-dimensional complex projective space $\mathbb{C}\mathrm{P}^n := (\mathbb{C}^{n+1}\setminus\{0\})/(\mathbb{C}\setminus\{0\})$.

Notation. For $(x_0,\ldots,x_n)\in\mathbb{R}^{n+1}\setminus\{0\}$ we denote by $[x_0:\ldots:x_n]$ the corresponding point in $\mathbb{R}\mathrm{P}^n$. We also use the same notation for points in complex projective spaces.

Lemma 6.9. (Projective Space-Sphere Quotient Lemma) Let $n\in\mathbb{N}_0$.

(1) The map

$$\begin{array}{rcl}\{\pm1\}\times S^n & \to & S^n\\ (w,(x_1,\ldots,x_{n+1})) & \mapsto & (w\cdot x_1,\ldots,w\cdot x_{n+1})\end{array}$$

defines a continuous action of the group $\{\pm1\}$ on S^n. The maps

$$\begin{array}{rcl}S^n/\{\pm1\} & \to & \mathbb{R}\mathrm{P}^n\\ [x] & \mapsto & [x]\end{array}\quad\text{and}\quad\begin{array}{rcl}\mathbb{R}\mathrm{P}^n & \to & S^n/\{\pm1\}\\ [x] & \mapsto & \left[\frac{x}{\|x\|}\right]\end{array}$$

are inverses of one another and they are homeomorphisms.

(2) The map

$$\begin{array}{rcl}S^1\times\overbrace{S^{2n+1}}^{\subset\mathbb{R}^{2n+2}=\mathbb{C}^{n+1}} & \to & S^{2n+1}\\ (w,(z_1,\ldots,z_{n+1})) & \mapsto & (w\cdot z_1,\ldots,w\cdot z_{n+1})\end{array}$$

defines a continuous action of the group $(S^1,\cdot)$ on S^{2n+1}. The maps

$$\begin{array}{rcl}S^{2n+1}/S^1 & \to & \mathbb{C}\mathrm{P}^n\\ [z] & \mapsto & [z]\end{array}\quad\text{and}\quad\begin{array}{rcl}\mathbb{C}\mathrm{P}^n & \to & S^{2n+1}/S^1\\ [z_1:\ldots:z_{n+1}] & \mapsto & \left[\frac{(z_1,\ldots,z_{n+1})}{\sqrt{|z_1|^2+\ldots+|z_{n+1}|^2}}\right]\end{array}$$

are inverses of one another and they are homeomorphisms.

Identification. We use the homeomorphisms from the Projective Space-Sphere Quotient Lemma 6.9 to make the identifications $\mathbb{R}\mathrm{P}^n =_i S^n/\{\pm1\}$ and $\mathbb{C}\mathrm{P}^n =_i S^{2n+1}/S^1$.

Proof of Lemma 6.9. It follows easily from the Twice Quotient Lemma 6.3 that all four maps are continuous. It is easy to verify that in (1) and (2) the maps are inverses of one another. ■

Proposition 6.10. (Projective Spaces-Compact+Hausdorff Proposition) The real projective and the complex projective spaces $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ are compact, path-connected and Hausdorff.

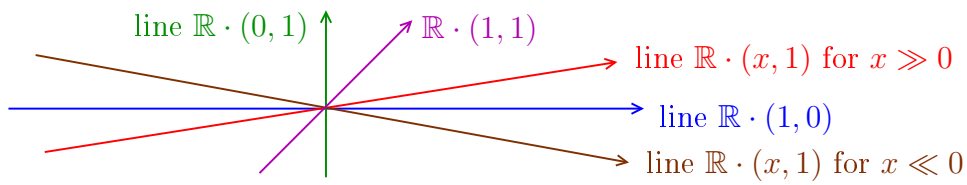
Proof. Let $n\in\mathbb{N}_0$. By the Projective Space-Sphere Quotient Lemma 6.9 it suffices to show that $S^n/\{\pm1\}$ and S^{2n+1}/S^1 are compact, path-connected and Hausdorff. In fact it follows immediately from the fact that S^n and S^{2n+1} are compact and path-connected together with the Quotient Topology-Properties Proposition 6.2 (1) and (2) that both quotients are compact and path-connected. Furthermore note that it follows from the Group Action-Hausdorff Quotient Proposition 6.7 that $S^n/\{\pm1\}$ is Hausdorff. In [Fri23] it is shown, using the Quotient Topology-Properties Proposition 6.2 (3), that S^{2n+1}/S^1 is also Hausdorff. ■

Lemma 6.11. (Projective Space-Avatar Lemma)

(1) The maps

$$\begin{array}{rcl} \overbrace{\mathbb{C}\cup\{\infty\}}^{=_i S^2} & \to & \mathbb{C}\mathrm{P}^1 \\ z & \mapsto & \begin{cases} [z:1], & \text{if } z\in\mathbb{C}, \\ [1:0], & \text{if } z=\infty \end{cases} \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathbb{C}\mathrm{P}^1 & \to & \overbrace{\mathbb{C}\cup\{\infty\}}^{=_i S^2} \\ [z_0:z_1] & \mapsto & \begin{cases} z_0\cdot z_1^{-1}, & \text{if } z_1\neq 0, \\ \infty, & \text{if } [z_0:z_1]=[1:0] \end{cases} \end{array}$$

are inverses of one another and both are homeomorphisms. Similarly we see that $\mathbb{R}\mathrm{P}^1$ is homeomorphic to $\mathbb{R}\cup\{\infty\}$, which in turn is homeomorphic to S^1.



Lemma 6.11. (Projective Space-Avatar Lemma)

(2) Let $n\in\mathbb{N}_0$. If we denote by $\sim$ the equivalence relation on $\overline{B}^n$ that is generated by $P\sim -P$ for $P\in S^{n-1}=\partial\overline{B}^n$, then the maps

$$\begin{array}{rcl} \overline{B}^n/\!\sim & \to & \mathbb{R}\mathrm{P}^n \\ [x] & \mapsto & [(x,\sqrt{1-\|x\|^2})] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathbb{R}\mathrm{P}^n =_i S^n/\{\pm 1\} & \xrightarrow{\cong} & \overline{B}^n/\!\sim \\ [\underbrace{(v,x_{n+1})}_{v\in\mathbb{R}^n,\, x_{n+1}\in\mathbb{R}}] & \mapsto & \begin{cases} [v], & \text{if } x_{n+1}\geq 0, \\ [-v], & \text{if } x_{n+1}\leq 0 \end{cases} \end{array}$$

are homeomorphisms that are inverses of one another.

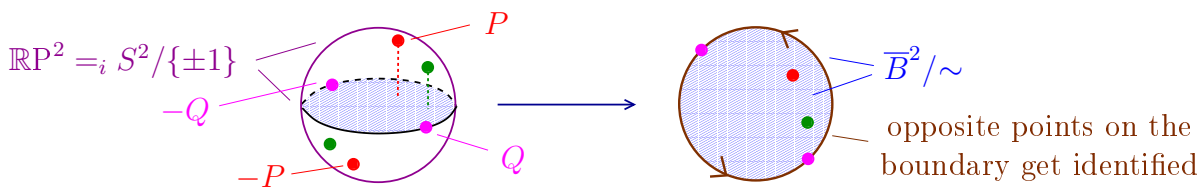


Illustration of the homeomorphism $\mathbb{R}\mathrm{P}^2\to\overline{B}^2/\!\sim$

Lemma 6.11. (Projective Space-Avatar Lemma)

(3) As in (2) let $\sim$ be the equivalence relation on $\overline{B}^3$ that is generated by $x\sim -x$ for $x\in S^2$. The map

$$\begin{array}{rcl} F\colon \overline{B}^3/\!\sim & \to & \mathrm{SO}(3) \\ [\underbrace{r\cdot v}_{r\in[0,1] \text{ and } v\in S^2}] & \mapsto & \text{matrix in } \mathrm{SO}(3) \text{ which represents rotation by the angle } \pi\cdot r \\ & & \text{around the axis } \mathbb{R}_{\geq 0}\cdot v \text{ where the rotation} \\ & & \text{direction is given by the right-hand rule} \end{array}$$

is a homeomorphism. In particular $\mathrm{SO}(3)$ is homeomorphic to $\mathbb{R}\mathrm{P}^3$.

Identification.

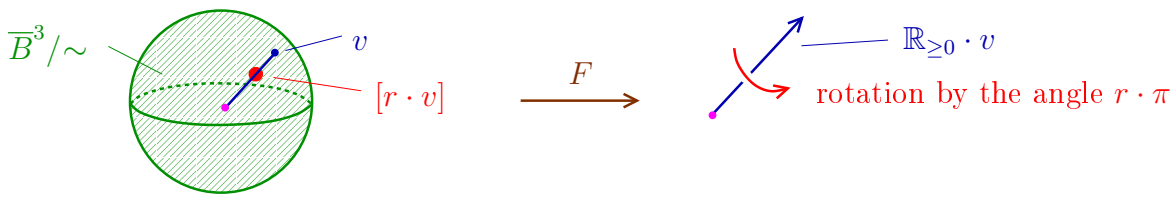


Illustration of the homeomorphism $\overline{B}^3/\sim \to \mathrm{SO}(3)$

(1) We will use the homeomorphism from the Projective Space-Avatar Lemma 6.11 (1) to make the identification $S^1 =_i \mathbb{R}\mathrm{P}^1$.
(2) Recall that in the Stereographic Projection Lemma 2.11 (1) we gave an explicit homeomorphism $S^2 \to \mathbb{R}^2 \cup \{\infty\}$. Combining this homeomorphism with the homeomorphism from the Projective Space-Avatar Lemma 6.11 (1) we obtain a homeomorphism $S^2 \to \mathbb{C}\mathrm{P}^1$ which we use to make the identification $S^2 =_i \mathbb{C}\mathrm{P}^1$.
(3) We will use the homeomorphism from the Projective Space-Avatar Lemma 6.11 (2) to make the identification $\mathbb{R}\mathrm{P}^n =_i \overline{B}^n/\sim$.

Proof of Lemma 6.11.

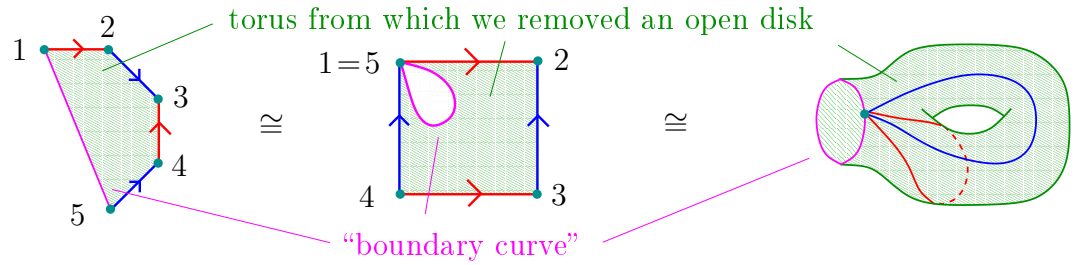
(1) One can easily verify that the given maps are inverses of one another. Next we show that the given map $\mathbb{C}\mathrm{P}^1 \to \mathbb{C} \cup \{\infty\}$ is continuous. First note that it follows from the Topological-Quotient Proposition 6.1 (1b) that it suffices to show that the corresponding map $\Theta\colon \mathbb{C}^2 \setminus \{(0,0\} \to \mathbb{C} \cup \{\infty\}$ is continuous. Next note that for any $C \in \mathbb{R}$ we have

$$\Theta^{-1}(\text{complement of } \overline{B}^n_C) \;=\; \{(z_0, z_1) \in \mathbb{C}^2 \setminus \{(0,0)\} \,|\, |z_0| > |z_1|\}$$

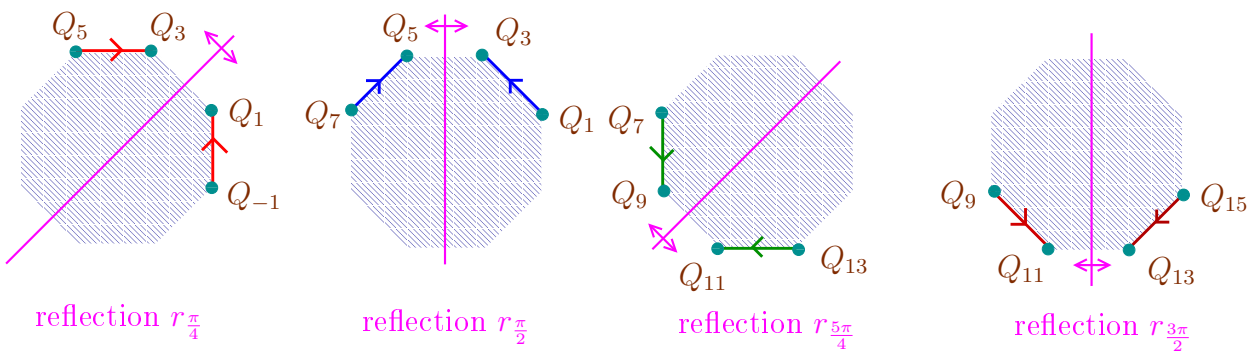
which is an open subset of $\mathbb{C}^2 \setminus \{(0,0)\}$. It follows quite easily from this discussion together with the Subbasis-Continuity Proposition 2.2 and together with the discussion of the basis of $\mathbb{R}^2 \cup \{\infty\}$ on page 36 that the map Θ is continuous. By the Projective Spaces-Compact+Hausdorff Proposition 6.10 we know that each $\mathbb{C}\mathrm{P}^1$ is compact. We leave it to the reader to verify that $\mathbb{R}^2 \cup \{\infty\}$ is Hausdorff. Thus it follows from the Compact-Hausdorff Proposition 2.10 (3) that the map $\mathbb{C}\mathrm{P}^1 \to \mathbb{C} \cup \{\infty\}$ is a homeomorphism.
(2) It is elementary to verify that the given continuous maps are inverses of one another, in particular they are bijections. We still need to show that the maps are continuous.
  (a) We consider the map $S^n/\{\pm 1\} \to \overline{B}^n/\sim$. By the Topological-Quotient Proposition 6.1 (1b) it suffices to show that the corresponding map $S^n \to \overline{B}^n/\sim$ is continuous. One can easily verify that the restrictions of the map $S^n \to \overline{B}^n/\sim$ to the closed subsets $S^n_{\geq 0}$ and $S^n_{\leq 0}$ are continuous. Thus it follows from the Pasting Proposition 2.4 (2) that the map $S^n \to \overline{B}^n/\sim$ is continuous.
  (b) It follows almost immediately from the Twice Quotient Lemma 6.3 that the given map $\overline{B}^n/\sim \to \mathbb{R}\mathrm{P}^n$ is continuous.
(3) The proof that $F\colon \overline{B}^3/\sim \to \mathrm{SO}(3)$ is a homeomorphism is given in [Fri23]. ■

6.6. Examples: Surfaces II.

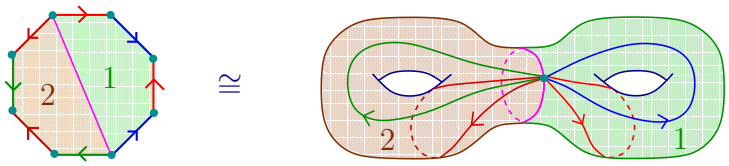
Example. We start out with a slightly informal discussion of an example. In the figure below we see to the left a pentagon, where we identify two pairs of sides. In the figure we illustrate that this topological space is homeomorphic to the torus minus an open disk.



Example. Let E_8 be the regular octagon in $\mathbb{C}$ with the vertices $Q_k = \exp(2\pi i k/16)$, $k = 1, 3, \ldots, 15$. We consider the quotient space $E_8/\sim$ that is illustrated in the figure.



Note that $E_8/\sim$ is given by two copies of the previous example glued together along the "boundary curves". In other words, $E_8/\sim$ is homeomorphic to the actual "physical" surface of genus 2 in $\mathbb{R}^3$.



two copies of torus minus an open disk glued along the boundary

Definition.

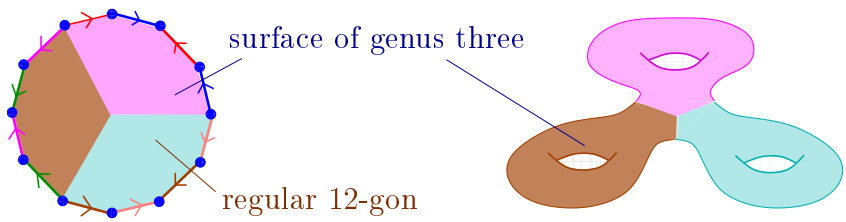
(1) Let $g \in \mathbb{N}$. We take a regular $4g$-gon E_{4g} in $\mathbb{C}$ and for $j = 0, \ldots, g-1$ we use reflections to identify the $(4j+1)$-st edge with the $(4j+3)$-rd edge and the $(4j+2)$-nd edge with the $(4j+4)$-th edge. We refer to the quotient space $E_{4k}/\sim$ as the surface of genus g.

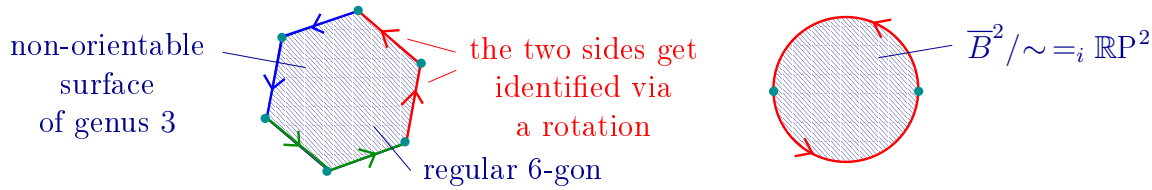
(2) We refer to the sphere S^2 as the surface of genus 0.

Definition.

(1) Given $g \in \mathbb{N}_{\geq 2}$ we consider the regular $2g$-gon E_{2g} with the identification of the boundary segments shown in the figure below. We refer to this topological space as the non-orientable surface N_g of genus g.

(2) We refer to the real projection place $\mathbb{RP}^2$ as the non-orientable surface of genus 1.





6.7. **The pushout in the topological category.**

Proposition 6.12. (Topological-Pushout Proposition) Let $f\colon A \to Y$ and $g\colon A \to Z$ be two continuous maps between topological spaces. We consider the topological space

$$Y \cup_A Z \;:=\; (Y \sqcup Z)/\sim \qquad \text{where } f(a) \sim g(a) \text{ for all } a \in A.$$

and the maps

$$\begin{array}{rcl} i\colon Y & \to & Y \cup_A Z \\ y & \mapsto & [y] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} j\colon Z & \to & Y \cup_A Z \\ z & \mapsto & [z] \end{array}$$

The following statements hold:

(0) A subset $U \subset Y \cup_A Z$ is open if and only if $i^{-1}(U) \subset Y$ and $j^{-1}(U) \subset Z$ are open. The same statement holds for "closed" instead of "open".

(1) (a) The maps $i\colon Y \to Y \cup_A Z$ and $j\colon Z \to Y \cup_A Z$ are continuous.

(b) Suppose we are given two continuous maps $\alpha\colon Y \to W$ and $\beta\colon Z \to W$ to some topological space W such that $\alpha \circ f = \beta \circ g\colon A \to W$. The map

$$\begin{array}{rcl} \Phi\colon Y \cup_A Z & \to & W \\ [P] & \mapsto & \begin{cases} \alpha(y) & \text{if } P = [y] \text{ for some } y \in Y, \\ \beta(z) & \text{if } P = [z] \text{ for some } z \in Z \end{cases} \end{array}$$

is well-defined and it is the unique map $h\colon Y \cup_A Z \to W$ with $\Phi \circ i = \alpha$ and $\Phi \circ j = \beta$. Furthermore, if α and β are continuous, then Φ is also continuous. In particular we obtain the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d, "i", "y\mapsto[y]"'] \arrow[ddr, bend left, "\alpha"] \arrow[dr, phantom, "\lrcorner"] & \\ Z \arrow[r, "j", "z\mapsto[z]"'] \arrow[drr, bend right, "\beta"'] & Y \cup_A Z \arrow[dr, dashed, "\exists!\,\Phi"] & \\ & & W. \end{tikzcd}$$

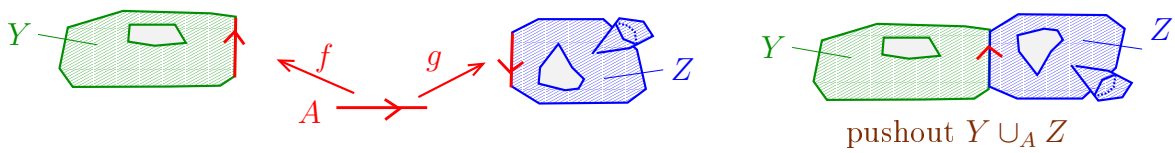
($*$) A map $f\colon Y \cup_A Z \to W$ to some topological space W is continuous if and only if the maps $f \circ i\colon Y \to W$ and $f \circ j\colon Z \to W$ are continuous.

(2) The topological space $Y \cup_A Z$ with the natural maps $Y \to Y \cup_A Z$ and $Z \to Y \cup_A Z$ is the pushout in the category $\mathcal{Top}$ of topological spaces.

Definition. Let $f\colon A \to Y$ and $g\colon A \to Z$ be two continuous maps between topological spaces. We refer to

$$Y \cup_A Z \;:=\; (Y \sqcup Z)/\!\sim \qquad \text{where } f(a) \sim g(a) \text{ for all } a \in A$$

as the pushout of f and g and we refer to the maps $i\colon Y \to Y \cup_A Z$ and $j\colon Z \to Y \cup_A Z$ as the two natural maps.



Proof.

(0) This statement follows from the Topological-Coproduct Proposition 5.8 (0) and the Topological-Quotient Proposition 6.1 (0).

(1) (a) This statement follows immediately from (0). Alternatively this statement follows from the Topological-Coproduct Proposition 5.8 (1a) and the Topological-Quotient Proposition 6.1 (1a).

(b) This existence of the continuous map follows immediately from the Topological-Coproduct Proposition 5.8 (1b) and the Topological-Quotient Proposition 6.1 (1b). We leave the proof of the uniqueness to the reader.

($*$) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) This statement follows easily from the definitions and Statement (1). ■

Proposition 6.13. (Pushout-Properties Proposition) Let $f\colon A \to Y$, $g\colon A \to Z$ be two continuous maps between topological spaces.

(1) If Y and Z are compact, then so is the pushout $Y \cup_A Z$.

(2) If Y and Z are (path-) connected and if A is non-empty, then the pushout $Y \cup_A Z$ is also (path-) connected.

(3) If A, Y and Z are compact, if Y and Z are Hausdorff and if $f\colon A \to Y$ and $g\colon A \to Z$ are injective, then $Y \cup_A Z$ is also Hausdorff.

Proof.

(1) If Y and Z are compact, then it follows from the Disjoint Union Topology-Properties Lemma 5.9 (1) and the Quotient Topology-Properties Proposition 6.2 (1) that $Y \cup_A Z$ is compact.

(2) If Y and Z are (path-) connected and if A is non-empty, then it follows easily from the Union Connected Lemma 2.19 that $Y \cup_A Z$ is also (path-) connected.

(3) Since f and g are injective we see that the relation on $Y \sqcup Z$ is given by

$$\{(f(a), g(a)) \,|\, a \in A\} \cup \{(w, w) \,|\, w \in Y \sqcup Z\} \;\subset\; (Y \sqcup Z) \times (Y \sqcup Z).$$

It follows from the Compact Image Lemma 2.8 and the Compact-Closed Lemma 1.9 (2), that this is a closed subset. It now follows from the Quotient Topology-Properties Proposition 6.2 (3) that $Y \cup_A Z$ is Hausdorff. ■

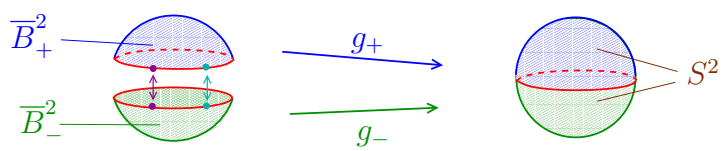
Example. Let $n \in \mathbb{N}_0$. We consider the two obvious inclusion maps $S^{n-1} \to \overline{B}^n_\pm := \overline{B}^n$ and the corresponding pushout $\overline{B}^n_+ \cup_{S^{n-1}} \overline{B}^n_-$. As in the Stereographic Projection Lemma 2.11 we consider for $\epsilon = -1, 1$ the homeomorphisms

$$\begin{array}{rcl} g_\epsilon \colon \overline{B}^n_\epsilon & \to & S^n_{\geq 0} \\ x = (x_1, \ldots, x_n) & \mapsto & \Big(((-1)^{n+1} \cdot x_1, \ldots, x_n) \cdot \frac{2}{\|x\|^2+1}, \epsilon \cdot \frac{\|x\|^2-1}{\|x\|^2+1} \Big). \end{array}$$

Note that $g_+ = g_-$ on S^{n-1}. Thus it follows from the Topological Pushout Proposition 6.12 (1b) that the map

$$\begin{array}{rcl} \overline{B}^n_+ \cup_{S^{n-1}} \overline{B}^n_- & \to & S^n \\ [P] & \mapsto & \begin{cases} g_+(P), & \text{if } P \in \overline{B}^n_+, \\ g_-(P), & \text{if } P \in \overline{B}^n_- \end{cases} \end{array}$$

is continuous. One can easily verify that it is a bijection. It follows from the Pushout-Properties Proposition 6.13 (1) combined with the Compact-Hausdorff Proposition 2.10 (3) that the map is a homeomorphism.



Identification. We use the homeomorphism from the above example to make the identification $\overline{B}^n_+ \cup_{S^{n-1}} \overline{B}^n_- =_i S^n$.

Lemma 6.14. (Topological Pushout-Maps Lemma) Let $f \colon A \to Y$ and $g \colon A \to Z$ be two continuous maps between topological spaces. We obtain the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d, "y \mapsto [y]"] \arrow[dl, phantom, "\lrcorner", very near start] \\ Z \arrow[r, "j", "z \mapsto [z]"'] & Y \cup_A Z. \end{tikzcd}$$

The following statements hold:

(1) If f is a homeomorphism, then the "opposite map" j is a homeomorphism.
(2) If f is injective, then the "opposite map" j is injective. The same statement holds for "surjective" and "bijective".
(3) If f is an embedding, then the "opposite map" j is an embedding.

Proof. Throughout the proof we denote by $p \colon Y \sqcup Z \to Y \cup_A Z$ the natural projection.

(1) This statement is a special case of the Pushout Lemma 4.8 (2).

(2) First we assume that $f\colon A \to Y$ is injective. We need to show that $j\colon Z \to Y \cup_A Z$ is injective. Let $z \in Z$. Note that

$$\begin{aligned} j^{-1}(j(z)) \;=\; \{z' \in Z \,|\, j(z') = j(z)\} \;&=\; \begin{array}{c}\text{set of elements in } Z \text{ equivalent to } z \\ \text{with respect to } \sim \text{ on } Y \sqcup Z\end{array} \\ &= \bigcup_{k\in\mathbb{N}_0} \underbrace{g(f^{-1}(f(g^{-1}(\dots g(f^{-1}(f(g^{-1}}_{\text{applied } k \text{ times}}(\{z\}))))\dots)))) \\ &\underset{\uparrow}{=} \bigcup_{k\in\mathbb{N}_0} g(g^{-1}(\dots g(g^{-1}(\{z\}))\dots)) \;\underset{\uparrow}{=}\; \{z\}. \end{aligned}$$

by the Image-Preimage Lemma 0.3 (1b), since f is injective Image-Preimage Lemma 0.3 (2a)

This shows that the map j is injective.

Now we assume that $f\colon A \to Y$ is surjective. We need to show that $j\colon Z \to Y \cup_A Z$ is surjective. Recall that by definition $Y \cup_A Z := (Y \sqcup Z)/\sim$ where $f(a) \sim g(a)$ for all $a \in A$. Let $[w] \in Y \cup_A Z$. If $w \in Z$, then evidently $j(w) = [w]$. Now consider the case that $w \in Y$. Since $f\colon A \to Y$ is surjective there exists an $a \in A$ with $f(a) = w$. But now we see that $j(g(a)) = i(f(a)) = i(w) = [w]$.

(3) We suppose that $f\colon A \to Y$ is an embedding. We need to show that $j\colon Z \to Y \cup_A Z$ is an embedding. By the Topological Pushout Proposition 6.12 (1a) we know that j is continuous and by (2) we know that $j\colon Z \to Y \cup_A Z$ is injective. By the Open-Injective Lemma 2.9 (2) and by definition of the subspace topology on $j(Z)$ it remains to prove the following claim.

Claim. If U is an open subset of Z, then there exists an open subset W of $Y \cup_A Z$ such that $j(U) = W \cap j(Z)$.

Proof. Let U be an open subset of Z. Since g is continuous we know that $g^{-1}(U)$ is an open subset of A. Since $f\colon X \to Y$ is an embedding we know that there exists an open subset $V \subset Y$ with $f(g^{-1}(U)) = f(A) \cap V$. We set $W := p(U \sqcup V)$. An elementary argument as in (2), that uses that f is injective, shows that $p^{-1}(W) = U \sqcup V$. Thus it follows from the definition of the quotient topology and the disjoint union topology that W is indeed an open subset of $Y \cup_A Z$. Basically by construction we have $j(U) = W \cap j(Z)$. ■

7. Graphs

7.1. Abstract graphs.

Definition. An abstract graph G is a quadruple (V, E, i, t) where V is a set, E is a set and $i, t\colon E \to V$ are maps from E to V.

(1) The elements of V are called the vertices of G and the elements of E are called the edges of G.
(2) Given $e \in E$ the point $i(e)$ is called the initial point of e and $t(e)$ is called the terminal point of e. We refer to $i(e)$ and $t(e)$ as the vertices of e.
(3) We say an edge $e \in E$ is a loop if $i(e) = t(e)$.
(4) We say G is finite (countable) if both V and E are finite (countable).

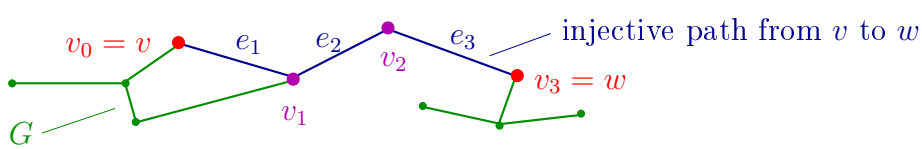
Example. We take $V = \{A, B, C\}$ and $E = \{e, f, g, h\}$. Furthermore let $i\colon E \to V$ and $t\colon E \to V$ be the maps that are given by the following table:

	e	f	g	h
i	A	B	B	A
t	B	C	C	A

The quadruple (V, E, i, t) is an abstract graph.

Definition. Let $G = (V, E, i, t)$ be an abstract graph.

(1) Let $v, w \in V$. A path of length k from v to w is a sequence of edges $e_1, \ldots, e_k$ such that there exist vertices $v = v_0, v_1, \ldots, v_k = w$ such that for each $i \in \{1, \ldots, k\}$ the vertices of e_i are v_{i-1}, v_i. We say that the path $e_1, \ldots, e_k$ is injective if the vertices $v_0, \ldots, v_k$ are distinct.
(2) We say G is connected if for every pair of vertices $v, w \in V$ there exists a path from v to w. One can easily show that if G is connected, then any two vertices can also be connected via an injective path.



Definition.

(1) Let
$$\begin{aligned} G &= (V, E, i\colon E \to V, t\colon E \to V) \\ \text{and} \quad \widetilde{G} &= (\widetilde{V}, \widetilde{E}, \widetilde{i}\colon \widetilde{E} \to \widetilde{V}, \widetilde{t}\colon \widetilde{E} \to \widetilde{V}) \end{aligned}$$

be two abstract graphs in the sense of the definition on page 94. We define a morphism $G \to \widetilde{G}$ of abstract graphs to be a pair $(\alpha\colon V \to \widetilde{V}, \beta\colon E \to \widetilde{E})$ of maps such that for each $e \in E$ we have $\widetilde{i}(\beta(e)) = \alpha(i(e))$ and $\widetilde{t}(\beta(e)) = \alpha(t(e))$.

(2) We refer to the category $\mathcal{AbsGraph}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{AbsGraph}) &:= \text{all abstract graphs,} \\ \mathrm{Mor}(G, \widetilde{G}) &:= \text{all maps from } G \text{ to } \widetilde{G} \text{ in the above sense} \end{aligned}$$

with the usual composition of maps as the category of abstract graphs.

Definition. Let $G = (V, E, i, t)$ be an abstract graph. A subgraph of $G = (V, E, i, t)$ is an abstract graph $G' = (V', E', i', t')$ such that $V' \subset V$ and $E' \subset E$ are subsets such that $i' = i|_{E'}$ and $t' = t|_{E'}$. Since i' and t' are determined by V' and E' we often suppress both of them from the notation.

7.2. The topological realization of an abstract graph.

Definition. Let $G = (V, E, i, t)$ be an abstract graph.

(1) We define the topological realization of G as the topological space

$$|G| := \Big(\underset{\substack{\uparrow \\ \text{here } V \text{ is equipped} \\ \text{with the } \textit{discrete} \text{ topology}}}{V} \sqcup \bigsqcup_{e \in E} [0,1]\Big)\Big/\sim \quad \text{where for any } e \in E \text{ we have } (e,0) \sim i(e) \text{ and } (e,1) \sim t(e).$$

(2) Let $e \in E$.

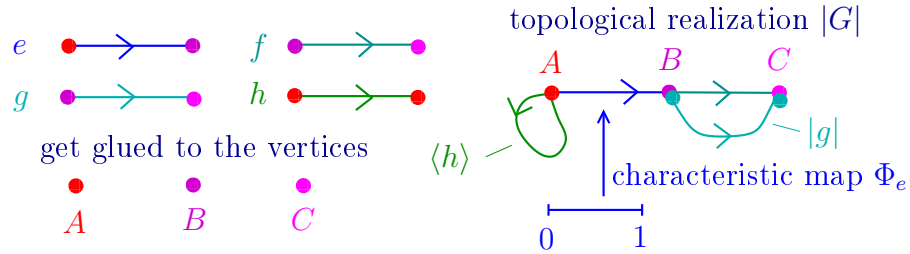
(a) We refer to the map $\Phi_e\colon [0,1] \xrightarrow{x \mapsto (e,x)} \{e\} \times [0,1] \to |G|$ as the characteristic map of the edge e.

(b) We write $|e| = \Phi_e([0,1])$ and we write $\langle e \rangle := \Phi_e((0,1))$.

$V = \{A, B, C\}$

$E = \{e, f, g, h\}$

	e	f	g	h
i	A	B	B	A
t	B	C	C	A



Lemma 7.1. (Abstract Graph-Topology Lemma) Let $G = (V, E, i, t)$ be an abstract graph.

(1) For each $e \in E$ the characteristic map $\Phi_e\colon [0,1] \to |G|$ is continuous.

(2) Let $U \subset |G|$ be a subset.

(a) $U \subset |G|$ is open $\iff$ for each $e \in E$ the preimage $\Phi_e^{-1}(U)$ is open in $[0,1]$.

(b) $U \subset |G|$ is closed $\iff$ for each $e \in E$ the preimage $\Phi_e^{-1}(U)$ is closed in $[0,1]$.

(3) Let $f\colon |G| \to X$ be a map to some topological space X.

$$f \text{ is continuous} \iff \text{for each } e \in E \text{ the map } f \circ \Phi_e\colon [0,1] \to X \text{ is continuous.}$$

Example. Let $G = (V, E, i, t)$ be an abstract graph. The following statements follow easily from the Abstract Graph-Topology Lemma 7.1:

(1) The obvious map $V \to |G|$ is injective and the image is a discrete subset of $|G|$.

(2) Let $e \in E$. Let $\Phi_e \colon [0,1] \to |G|$ be the characteristic map.
  (a) $|e|$ is a closed subset of $|G|$ and $\langle e \rangle$ is an open subset of $|G|$.
  (b) If $i(e) \neq t(e)$, then Φ_e restricts to a homeomorphism $[0,1] \to |e|$.
  (c) If $i(e) = t(e)$, then Φ_e induces a homeomorphism $[0,1]/0 \sim 1 \to |e|$.

Proof of Lemma 7.1.

(1) This statement follows immediately from the Topological-Coproduct Proposition 5.8 (1a) and the Topological-Quotient Proposition 6.1 (1a).
(2) (a) This statement follows easily from the Topological-Coproduct Proposition 5.8 (0a) and the Topological-Quotient Proposition 6.1 (0a) and the observation that any subspace of a topological space consisting of a single point is open.
  (b) This statement follows easily from the Topological-Coproduct Proposition 5.8 (0b) and the Topological-Quotient Proposition 6.1 (0b) and the observation that any subspace of a topological space consisting of a single point is closed.
(3) This statement follows immediately from (1) and (2a). ■

Lemma 7.2. (Topological Realization–Functor Lemma) The following two statements hold:

(1) Let $G = (V, E, i, t)$ and $\widetilde{G} = (\widetilde{V}, \widetilde{E}, \widetilde{i}, \widetilde{t})$ be two abstract graphs. Furthermore let $f := (\alpha \colon V \to \widetilde{V}, \beta \colon E \to \widetilde{E})$ be a map between the two abstract graphs. We consider the following map between the topological realizations of G and $\widetilde{G}$:

$$\begin{array}{rcl} & \text{where } (e,0) \sim i(e) \text{ and } (e,1) \sim t(e) & \qquad \text{similarly } (e,0) \sim \widetilde{i}(e) \text{ and } (e,1) \sim \widetilde{t}(e) \\ & \downarrow & \qquad\qquad \downarrow \\ |f| \colon |G| = \Big(V \sqcup \bigsqcup_{e \in E} [0,1]\Big)/\sim & \to & |\widetilde{G}| = \Big(\widetilde{V} \sqcup \bigsqcup_{e \in \widetilde{E}} [0,1]\Big)/\sim \\ [x] & \mapsto & \begin{cases} \alpha(x), & \text{if } x \in V, \\ (\beta(e), t), & \text{if } x = (e,t) \in E \times [0,1]. \end{cases} \end{array}$$

This map $|f| \colon |G| \to |\widetilde{G}|$ is well-defined and continuous.

(2) The map
$$\begin{array}{rcl} F \colon \mathrm{Ob}(\mathcal{AbsGraph}) & \to & \mathrm{Ob}(\mathcal{Top}) \\ G & \mapsto & |G|, \end{array}$$
and the maps
$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{AbsGraph}}(G, \widetilde{G}) & \to & \mathrm{Mor}_{\mathcal{Top}}(|G|, |\widetilde{G}|) \\ f & \mapsto & |f| \end{array}$$
form a covariant functor from the category $\mathcal{AbsGraph}$ of abstract graphs to the category $\mathcal{Top}$ of topological spaces.

Proof of Lemma 7.2.

(1) It follows almost immediately from the definitions that the map is well-defined. Furthermore it follows easily from the Abstract Graph-Topology Lemma 7.1 (1) and (3) that the map is continuous.
(2) This statement follows easily from the definitions. ■

Lemma 7.3. (Topological Realization–Properties Lemma) Let $G = (V, E, i, t)$ be an abstract graph.

(1) (a) Every compact subspace of $|G|$ is contained in the topological realization of a finite subgraph.
(b) $|G|$ is compact if and only if G is finite.
(2) The following statements are equivalent:
(a) G is connected.
(b) Given any two distinct vertices v, w there exists an embedding $\gamma\colon [0,1] \to |G|$ with $\gamma(0) = v$, $\gamma(1) = w$ and such that $\gamma([0,1]) \subset |G|$ is the union of edges.
(c) The topological realization $|G|$ is path-connected.
(d) The topological realization $|G|$ is connected.
(3) $|G|$ is Hausdorff.

Proof.

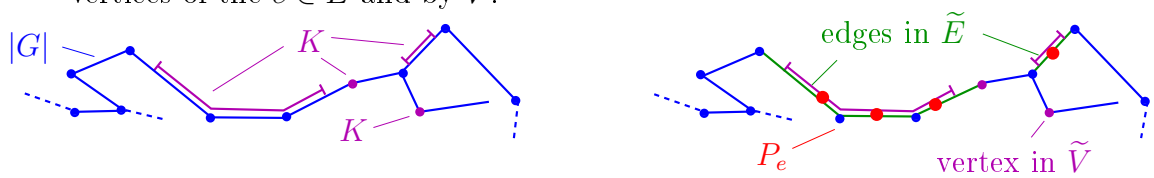
(1) (a) Let $K \subset |G|$ be a compact subspace. We set

$$\begin{aligned} \widetilde{E} &:= \{e \in E \mid K \cap \langle e \rangle \neq \varnothing\} \\ \widetilde{V} &:= \{v \in V \mid v \in K \text{ and there is no } e \in \widetilde{E} \text{ with } v \in |e|\}. \end{aligned}$$

We do the following:

- For each $e \in \widetilde{E}$ we pick a $P_e \in K \cap \langle e \rangle$ and we set $A_e := \langle e \rangle$.
- For each $v \in \widetilde{V}$ we set $B_v := \{v\} \cup \bigcup_e \langle e \rangle$, where we run over all $e \in E$ such that v is a vertex of e.
- We set $C := (|G| \setminus \bigcup_{e \in \widetilde{E}} \{P_e\}) \setminus \widetilde{V}$.

It follows easily from the Topological Realization–Properties Lemma 7.3 that each A_e, B_v and C is an open subset of $|G|$. Note that $\{A_e\}_{e \in \widetilde{E}}$, $\{B_v\}_{v \in \widetilde{V}}$ and C form an open cover of K. Since K is compact we see that we can cover K by finitely many of these open subsets. But each A_e contains a point of K, namely P_e, that does not lie in any of the other sets in the open cover. Similarly we see that each B_v contains a point in K, namely v, that does not lie in any of the other sets in the open cover. This shows that $\widetilde{E}$ and $\widetilde{V}$ are finite. Now we see that K is contained in the topological realization of the finite abstract graph that is formed by $\widetilde{E}$, the vertices of the $e \in \widetilde{E}$ and by $\widetilde{V}$.



(b) If $|G|$ is compact, then it follows from (a) that G is finite. Next we assume that $G = (V, E, i, t)$ is finite. It follows from the Compact Image Lemma 2.8 that each $|e|$ is compact. Since there are only finitely edges we see that $|G|$ is compact.

(2) (a)⇒(b). To save some space we outsource this uninspiring proof to [Fri23].

(3) We leave it to the reader to prove this statement. ■

We conclude this section with the following definition.

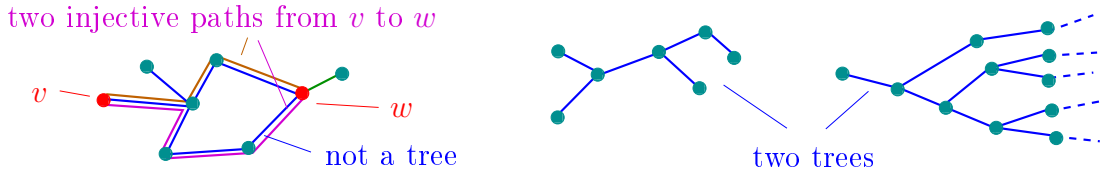
Definition. We say a topological space X is a topological graph if there exists an abstract graph G such that X is homeomorphic to $|G|$.

7.3. Trees.

Definition. Let G be an abstract graph.

(1) We say G is a tree if G is non-empty, if G does not contain a loop, and if given any $v, w \in V$ there exists a unique injective path from v to w.

(2) A subgraph that is a tree is called a subtree.



Lemma 7.4. (Tree–Basics Lemma)

(1) Any connected non-empty subgraph of a tree is again a tree.

(2) Let G be a graph. Let $T = (V, E)$ be a subtree of G and let $\{e_i\}_{i \in I}$ be a family of edges of G with the following properties:

  (a) The edge e_i has two vertices, one vertex t_i of e_i lies in T and the other vertex v_i does not lie in T.

  (b) The vertices v_i are distinct.

  Then $(V \cup \bigcup_{i \in I} \{v_i\}, E \cup \bigcup_{i \in I} \{e_i\})$ is also a subtree of G.

(3) Let G be a graph. If $T_1 \subset T_2 \subset \dots$ is a sequence of subtrees of G, then $\bigcup_{i \in \mathbb{N}} T_i$ is also a subtree of G.

Proof.

(1) This statement follows easily from the definitions.

(2) We set $W := V \cup \bigcup_{i \in I} \{v_i\}$ and $F := E \cup \bigcup_{i \in I} \{e_i\}$. We claim that (W, F) is a subtree. Since none of the e_i's is a loop we see that (W, F) has no loops. Next let $w, w' \in W$ be two vertices. We need to show that there exists a unique injective path from w to w'. There are several cases to distinguish.

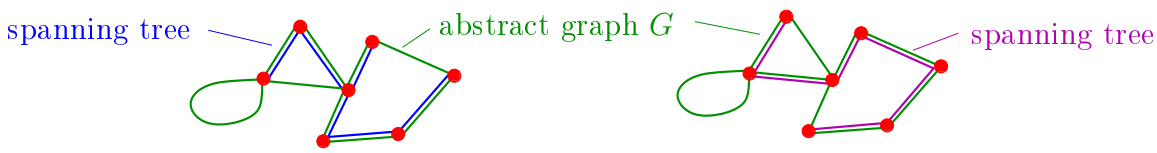
First assume that there exist $i, j \in I$ with $w = v_i$ and $w' = v_j$. Since (V, E) is a subtree we know that there exists an injective path in (V, E) from t_i to t_j. Together with e_i and e_j these form an injective path in (W, F) from $w = v_i$ to $w' = v_j$. We need to show that this is the unique injective path in (W, F) from $w = v_i$ to $w' = v_j$. Thus let $f_1, \dots, f_k$ be an injective path in (W, F) from w to w'. It follows from our hypothesis (a) and (b) that $f_1 = e_i$ and $f_k = e_j$ and that $f_2, \dots, f_{k-1}$ is an injective path which cannot involve any of the other e_l. Thus it is an injective path in (V, E), which is unique since T is a subtree. All other cases are treated in a similar fashion.

(3) We leave the proof of this statement as an instructive exercise to the reader. ■

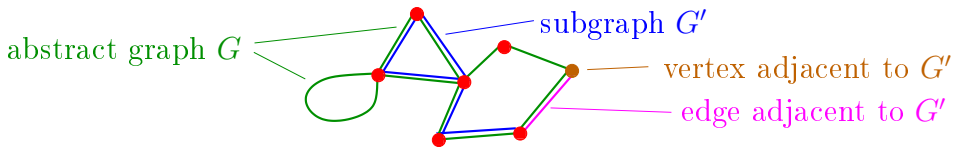
Definition. Let G be an abstract graph. A spanning tree for G is a subtree of G that contains all vertices of G.

Definition. Let G be an abstract graph and let G' be a subgraph.

(1) We say an edge e of G is adjacent to G' if e does not lie in G' and if at least one vertex of e lies in G'.



(2) We saw a vertex v of G is adjacent to G' if v does not lie in G' and if v is a vertex of an edge that is adjacent to G'.



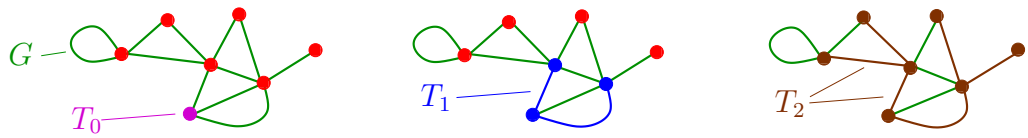
Proposition 7.5. (Spanning Tree–Characterization Proposition) Let G be a connected abstract graph and let $T \subset G$ be a tree. The following statements are equivalent:
(1) T is a spanning tree.
(2) T is maximal in the sense that if T is contained in a subtree T' of G, then $T = T'$.
(3) T does not admit an adjacent vertex.

Proof.
(1)⇒(2). Let T be a spanning tree. We claim that T is maximal. Let H be a subgraph of G with $T \subsetneq H$. We need to show that H cannot be a tree. Since T contains all vertices and since $T \subsetneq H$ we see that there exists an edge e in H that does not lie in T. If e is a loop, then we know already that H is not a tree. Thus it remains to consider the case that e is not a loop. This means that e has two vertices v and w. Since T is connected there exists an injective path from v to w in T. On the other hand e is an injective path from v to w in H. Thus we see that there exist two injective paths in H from v to w. Thus H is not a tree.
(2)⇒(3). Let $T = (W, F)$ be a subtree of G that admits an adjacent vertex x. We need to show that T is not a maximal subtree. Let f be an edge such that one vertex is x and such that the other vertex lies in W. It follows from the Tree–Basics Lemma 7.4 (2) that $(W \cup \{x\}, F \cup \{f\})$ is a subtree of G that contains T.
(3)⇒(1). Let T be a subtree of G. We will show that if T is not a spanning tree, then T admits an adjacent vertex. Thus suppose that there exists a vertex w that does not lie in T. Let v be a vertex in T. Since G is connected there exists a path $e_1, \ldots, e_n$ from v to w. By definition this means that there exist vertices $v = v_0, \ldots, v_n = w$ such that the vertices of each e_i are v_{i-1}, v_i. The first vertex that does not lie in T is adjacent to T. ■

Proposition 7.6. (Spanning Tree–Existence Proposition) Every connected non-empty abstract graph admits a spanning tree.

Proof. Let $G = (V, E)$ be a connected non-empty abstract graph. Since G is non-empty we can pick a vertex $v \in V$. We set $T_0 := \{v\}$ and view T_0 as a subtree of G. Now we construct iteratively a sequence of subtrees $T_0 \subset T_1 \subset \ldots$ as follows: Suppose we have a subtree $T_i = (V_i, E_i)$. We denote by W_i the set of all vertices of G that are adjacent to T_i. For each $w \in W_i$ we pick an edge e_w that is adjacent to T_i and such that w is a vertex of

e_w. We set $V_{i+1} := V_i \cup W_i$ and $E_{i+1} := E_i \cup \bigcup_{w \in W_i} \{e_w\}$. It follows from the Tree–Basics Lemma 7.4 (2) that T_{i+1} is also a tree.

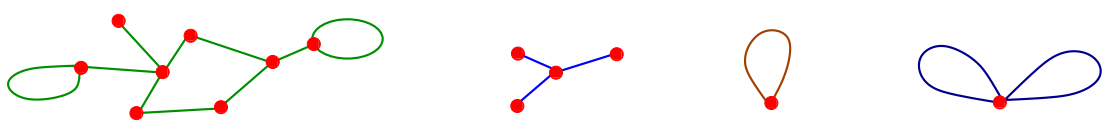


We set $T := \bigcup_{i \in \mathbb{N}_0} T_i$. It follows from the Tree–Basics Lemma 7.4 (3) that T is a tree. Finally note that it follows easily from the explicit construction of the sequence T_i and the (3)⇒(1)-Implication of the Spanning Tree–Characterization Proposition 7.5 that T is a spanning tree. ■

7.4. The Euler characteristic.

Definition. Let $G = (V, E)$ be a finite abstract graph. The Euler characteristic of G is defined as

$$\chi(G) \;:=\; \#V \;-\; \#E \;=\; \text{number of vertices} \;-\; \text{number of edges}.$$



Euler characteristic of G equals $\chi(G) = 8 - 10 = -2$

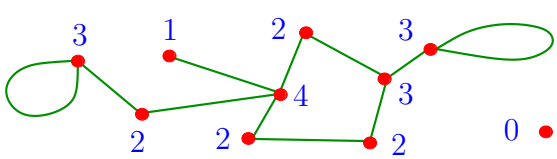
$\chi = 1$ $\chi = 0$ $\chi = -1$

Proposition 7.7. (Tree Characterization Proposition) Let $G = (V, E)$ be a finite abstract graph. The following statements are equivalent:

(1) G is a tree.
(2) G is connected and $\chi(G) = 1$.

Definition. If $G = (V, E, i, t)$ is an abstract graph, then the valence of a vertex v is defined as

$$\text{valence}(v) \;:=\; \#\{e \in E \,|\, i(e) = v\} + \#\{e \in E \,|\, t(e) = v\}.$$



valences of the vertices

Lemma 7.8. (Tree–Valence One Edge Lemma) Let $G = (V, E)$ be a finite abstract graph. If $\chi(G) = 1$ and if G has at least one edge, then G admits at least two vertices of valence one.

Proof of Lemma 7.8. Let $G = (V, E, i, t)$ be a finite abstract graph with $\chi(G) = 1$ that has at least one edge.

(a) Since G has at least one edge and since it is by definition connected, we see that the abstract graph G does not have a vertex of valence 0.

(b) We calculate that

$$\begin{aligned}\sum_{\text{vertex } v} \underbrace{\text{valence}(v)}_{\geq 1 \text{ by (a)}} &= \sum_{\text{vertex } v} (\#\{e \in E \,|\, i(e) = v\} + \#\{e \in E \,|\, t(e) = v\}) \\ &\underset{\substack{\uparrow \\ \text{each edge contributes 2 to the sum}}}{=} 2 \cdot \#\text{ edges} \underset{\substack{\uparrow \\ \text{by hypothesis we have } \chi(G)=1}}{=} 2 \cdot \#\text{vertices} - 2.\end{aligned}$$

By (a) the equality can only hold if there are at least two vertices with valence one. ■

Proof of the Tree Characterization Proposition 7.7. Let $G = (V, E)$ be a finite abstract graph.

(2)⇒(1). We now assume that G is connected and that $\chi(G) = 1$. We need to show that G is a tree. We will do so by inducting on the number of edges of G. If G has no edge, then it follows from $\chi(G) = 1$ that G has one vertex. Clearly this implies that G is a tree.

Now suppose that the statement holds for all graphs with n edges. Let $G = (V, E)$ be a connected graph with $n + 1$ edges and with $\chi(G) = 1$. It follows from the Tree–Valence One Edge Lemma 7.8 that G admits a vertex v of valence one. Let e be the corresponding edge. Note that $G' = (V \setminus \{v\}, E \setminus \{e\})$ is connected and that

$$\chi(G') \;=\; (\#V - 1) - (\#E - 1) \;=\; \#V - \#E \;=\; \chi(G) \;=\; 1.$$

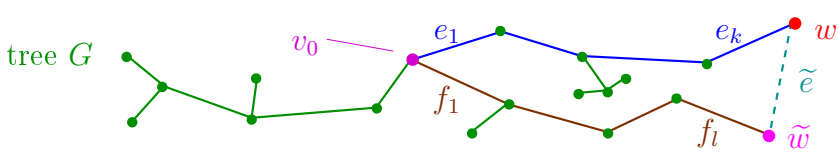
By induction G' is a tree. But then it follows from the Tree–Basics Lemma 7.4 (2) that G itself is also a tree.

(1)⇒(2). We suppose that G is a tree. If follows immediately from the definition of a tree that G is connected. We still need to show that $\chi(G) = 1$. We prove this statement by induction on the number of edges. If there is no edge then it follows from the fact that trees are non-empty and connected that there is precisely one vertex, thus $\chi(G) = 1$.

Next suppose that the statement holds if we have n edges. Let G be a tree with $n + 1$ edges. We pick a vertex $v_0 \in V$. Let $w \in V$ be a vertex of maximal distance to v_0.

Claim. There exists precisely one edge e that has w as a vertex.

Proof. Let $e_1, \ldots, e_k$ be a path of minimal length from v_0 to w. Since the path has minimal length we see that it is injective. Evidently w is a vertex of e_k. Suppose there was a second edge $\widetilde{e}$ for which w is a vertex. As we mentioned before, a tree does not contain a loop. This implies that the edge $\widetilde{e}$ has a second vertex besides w. We denote this second vertex by $\widetilde{w}$. Since G is connected and since w has maximal distance from v_0 there exists an injective path $f_0, \ldots, f_l$ from v_0 to $\widetilde{w}$ of length $\leq k$. Thus this path cannot be the path $e_1, \ldots, e_k, \widetilde{e}$, which has length $k + 1$.



We have thus found two different injective paths from v_0 to $\widetilde{w}$, namely $e_1, \ldots, e_k, \widetilde{w}$ and $f_1, \ldots, f_l$. But this contradicts the fact that we started out with a tree. ⊞

We can now consider the subgraph G' that is given by $V \setminus \{w\}$ and $E \setminus \{e\}$. One can easily show that this graph is again connected, thus it follows from the Tree–Basics

Lemma 7.4 (1) that G' is a tree. We now see that

$$\chi(G) \;=\; \#V - \#E \;=\; (\#V - 1) - (\#E - 1) \;=\; \chi(G') \underset{\substack{\uparrow \\ \text{by induction hypothesis}}}{=} 1.$$

■

7.5. Quotients of abstract graphs.

Definition. Let $G = (V, E, i, t)$ be an abstract graph and let $H = (W, F)$ be a non-empty subgraph.

(1) We define $\sim$ to be the equivalence relation generated by $v \sim v'$ if v and v' are both vertices in H. We denote by $*$ the equivalence class defined by the vertices in H.

(2) We denote by $\pi\colon V \to V/\!\sim$ the map given by $v \mapsto [v]$.

(3) Finally we define the abstract graph

$$G/H \;:=\; (V/\!\sim, E \setminus F, \pi \circ i\colon E \setminus F \to V/\!\sim, \pi \circ t\colon E \setminus F \to V/\!\sim).$$

We refer to G/H as the quotient of the abstract graph G by the abstract graph H.

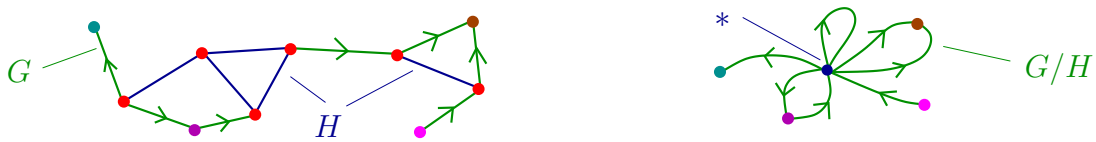
Lemma 7.9. (Abstract Graph–Quotient Lemma) Let $G = (V, E, i, t)$ be an abstract graph and let $H = (W, F)$ be a non-empty subgraph. We continue with the above notation.

(1) If G is finite, then we have $\chi(G/H) = \chi(G) - \chi(H) + 1$.

(2) We consider the map

$$p\colon |G| = (V \sqcup (E \times [0,1]))/\!\sim \;\to\; |G/H| = (V/\!\sim \sqcup ((E \setminus F) \times [0,1]))/\!\sim$$
$$[x] \;\mapsto\; \begin{cases} [*], & \text{if } x \in |H|, \\ [\pi(x)], & \text{if } x \in V, \\ [(e,t)], & \text{if } x = (e,t) \text{ with } e \in E \setminus F \text{ and } t \in [0,1]. \end{cases}$$

The induced map $\overline{p}\colon |G|/|H| \to |G/H|$ is well-defined and it is a homeomorphism.



Proof.

(1) We calculate that

$$\chi(G/H) = \#(V/\!\sim) - \#(E \setminus F) = (\#V - \#W + 1) - (\#E - \#F) = \chi(G) - \chi(H) + 1.$$

(2) First note that it follows easily from the Abstract Graph-Topology Lemma 7.1 (3) and the Topological-Quotient Proposition 6.1 (1b) that the map $\overline{p}\colon |G|/|H| \to |G/H|$ is continuous. One can easily verify that the map is a bijection. We refer to [Fri23] for the proof that the map is actually a homeomorphism. ■

8. Homotopies

8.1. Homotopic maps.

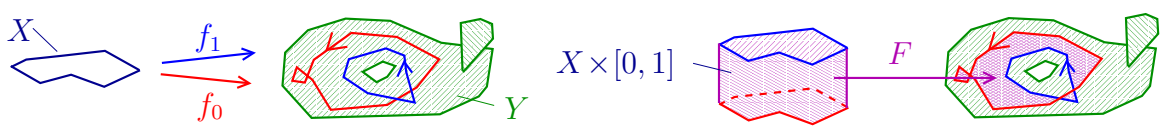
Notation. Let X and Y be two topological spaces and let $I \subset \mathbb{R}$ be a subset. Given a continuous map $F\colon X \times I \to Y$ and given $t \in I$ we denote by F_t the continuous map

$$\begin{array}{rcl} F_t\colon X & \to & Y \\ x & \mapsto & F(x,t). \end{array}$$

Definition. Let X and Y be two topological spaces.

(1) Let $f_0, f_1\colon X \to Y$ be two continuous maps. A homotopy between the maps f_0 and f_1 is a continuous map $F\colon X \times [0,1] \to Y$ with $F_0 = f_0$ and $F_1 = f_1$. If there exists a homotopy between the maps f_0 and f_1, then we say that f_0 and f_1 are homotopic and sometimes we write $f_0 \simeq f_1$. If every F_t is an embedding, then we call F an isotopy.

(2) A continuous map $f\colon X \to Y$ is called null-homotopic if it is homotopic to a constant map.



Examples.

(1) In the figure below we show four maps α, β, γ and δ from the circle S^1 to the annulus $Y = B_2(0) \setminus B_1(0)$. The maps α and β are homotopic whereas the maps γ and δ look as if they are not homotopic.

the maps α and β are homotopic the maps γ and δ appear not to be homotopic

(2) Given an orthogonal matrix $A \in \mathrm{O}(n)$ we denote by $\rho(A)$ the map $S^{n-1} \to S^{n-1}$ that is given $x \mapsto A \cdot x$.

(a) Let $\gamma\colon [0,1] \to \mathrm{SO}(n)$ be a path from some matrix A to some matrix B. The map

$$\begin{array}{rcl} S^{n-1} \times [0,1] & \to & S^{n-1} \\ (x,t) & \mapsto & \gamma(t) \cdot x \end{array}$$

is a homotopy from $\rho(A)$ to $\rho(B)$.

(b) Let $A \in \mathrm{O}(n)$. If $\det(A) = 1$, i.e. if $A \in \mathrm{SO}(n)$, then the Matrix Group-π_0-Proposition 2.23 (3) provides us with a path $\gamma\colon [0,1] \to \mathrm{SO}(n)$ from $\gamma(0) = A$ to $\gamma(1) = \mathrm{id}$. It follows from (a) that $\rho(A)\colon S^{n-1} \to S^{n-1}$ is homotopic to the identity.

Lemma 8.1. (Convex Homotopy Lemma) Let X be a topological space. Given any two continuous maps $f_0, f_1 \colon X \to Y$ to a *convex* subset $Y \subset \mathbb{R}^n$ the map

$$\begin{array}{rcl} F \colon X \times [0,1] & \to & Y \\ (x,t) & \mapsto & \underbrace{f_0(x)\cdot(1-t) + f_1(x)\cdot t}_{\in Y \text{ since } Y \text{ is convex}} \end{array}$$

is a homotopy between f_0 and f_1.

Proof. Basically by definition we have $F_0 = f_0$ and $F_1 = f_1$. The map F can be written as the composition of the following two maps:

$$X \times [0,1] \xrightarrow{(x,t)\mapsto(f_0(x),f_1(x),t)} \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \xrightarrow{(x,y,t)\mapsto(x\cdot(1-t)+y\cdot t)} \mathbb{R}^n.$$

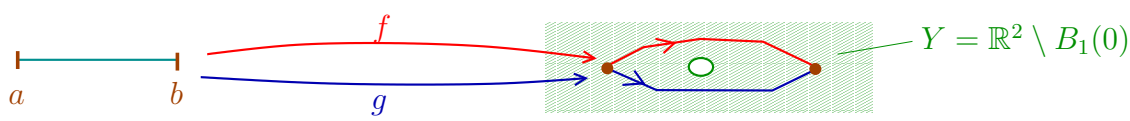
It follows from the Topological-Product Proposition 5.1 (1) that the map to the left is continuous. As in the proof of the Torus Embedding Lemma 5.5 we see that the map to the right is continuous. ■

Definition. Let X and Y be topological spaces and let $A \subset X$ be a subset.

(1) We say a continuous map $F \colon X \times [0,1] \to Y$ is a homotopy rel A if for all $a \in A$ and $s, t \in [0,1]$ we have $F(a,s) = F(a,t)$.

(2) We say that two continuous maps $f, g \colon X \to Y$ with $f|_A = g|_A$ are homotopic rel A if there exists a homotopy F rel A with $F_0 = f$ and $F_1 = g$.

Examples. Let $f, g \colon [a,b] \to Y = \mathbb{R}^2 \setminus B_1(0)$ be the two maps from an interval to the topological space Y that are shown in the figure below. Note that $f(a) = g(a)$ and $f(b) = g(b)$. One can easily show that any two continuous maps from an interval to a path-connected space are homotopic. On other hand it looks rather unlikely that the two maps are homotopic rel $\{a,b\}$, i.e. it does not look like we can deform f into g while keeping the endpoints of the interval fixed.



Lemma 8.2. (Homotopy Combination Lemma) Let X and Z be topological spaces. Let $A \subset X$ be a (possibly empty) subset of X and let $G, H \colon X \times [0,1] \to Z$ be two homotopies rel A. If $G_1 = H_0$, then the map

$$\begin{array}{rcl} G * H \colon X \times [0,1] & \to & Z \\ (x,t) & \mapsto & \begin{cases} G(x,2t), & \text{if } t \in [0,\frac{1}{2}], \\ H(x,2t-1), & \text{if } t \in [\frac{1}{2},1]. \end{cases} \end{array}$$

is a homotopy rel A from G_0 to H_1.

Proof. It follows from the Pasting Proposition 2.4 (2) together with the Product Topology-Basics Lemma 5.3 (1a) that the map is continuous. ■

Lemma 8.3. (Homotopy-Equivalence Relation Lemma) Let X and Y be topological spaces. Then "homotopy of maps" forms an equivalence relation on the set of continuous maps from X to Y.

Proof. We have to show that "homotopy" satisfies the three properties of an equivalence relation. In the following let $f, g, h\colon X \to Y$ be three continuous maps.

(1) The "constant homotopy" given by $F(x,t) := f(x)$ is a homotopy between f and f.
(2) Let F be a homotopy from f to g. Then it is clear that the map F' that is defined by $F'(x,t) := F(x, 1-t)$ is a homotopy from g to f.
(3) Let F be a homotopy from f to g and let G be a homotopy from g to h. We consider the map

$$\begin{array}{rcl} F * G\colon X \times [0,1] & \to & Y \\ (x,t) & \mapsto & \begin{cases} F(x,2t), & \text{if } t \in \left[0, \frac{1}{2}\right], \\ G(x, 2t-1), & \text{if } t \in \left(\frac{1}{2}, 1\right]. \end{cases} \end{array}$$

By the Homotopy Combination Lemma 8.2 (2) this is a homotopy from f to h. ■

Lemma 8.4. (Homotopy Composition Lemma)

(1) Let $f, f'\colon X \to Y$, let $\alpha\colon W \to X$ and $\beta\colon Y \to Z$ be continuous map between topological spaces. Furthermore let $A \subset X$ be a (possibly empty) subset. The following conclusion holds:

$$\begin{array}{rcl} f \text{ is homotopic to } f' \text{ rel } A & \Longrightarrow & f \circ \alpha \text{ is homotopic to } f' \circ \alpha \text{ rel } \alpha^{-1}(A) \\ f \text{ is homotopic to } f' \text{ rel } A & \Longrightarrow & \beta \circ f \text{ is homotopic to } \beta \circ f' \text{ rel } A \end{array}$$

(2) Let X, Y and Z be topological spaces and let $A \subset X$ and $B \subset Y$ be subsets. We suppose that $f, f'\colon X \to Y$ and $g, g'\colon Y \to Z$ are continuous maps such that f and f' are homotopic rel A and such that g and g' are homotopic rel B. If $f(A) = f'(A) \subset B$, then $g \circ f$ is homotopic to $g' \circ f'$ rel A.

Proof.

(1) This statement is basically obvious, we just need to precompose and postcompose a homotopy $H\colon X \times [0,1] \to Y$ between f and g by $\alpha \times \mathrm{id}_{[0,1]}\colon W \times [0,1] \to X \times [0,1]$ respectively by $\beta\colon Y \to Z$.
(2) Note that:

$$g \circ f \underset{\substack{\uparrow \\ \text{both follow from (1)}}}{\simeq} g' \circ f \underset{\uparrow}{\simeq} g' \circ f'.$$

Furthermore one can easily verify that the "rel" business also works out. ■

8.2. Homotopies on quotients.

Theorem 8.5. (Homotopy Quotient Theorem) Let X and Y be topological spaces. Furthermore let $\sim_X$ be an equivalence relation on X and let $\sim_Y$ be an equivalence relation on Y. If $F\colon X \times [0,1] \to Y$ is a homotopy such that $F(x,t) \sim_Y F(x',t)$ whenever $x \sim_X x'$, then the map

$$\begin{array}{rcl} \varphi\colon (X/\sim_X) \times [0,1] & \to & Y/\sim_Y \\ ([x], t) & \mapsto & [F(x,t)] \end{array}$$

is continuous.

Example. Let $n \in \mathbb{N}$ and let $\varphi\colon [0,1] \to \mathrm{GL}(n, \mathbb{R})$ be a continuous map. The map

$$\begin{array}{rcl} F\colon (\mathbb{R}^n \setminus \{0\}) \times [0,1] & \to & \mathbb{R}^n \setminus \{0\} \\ (P, t) & \mapsto & \varphi(t) \cdot P \end{array}$$

is continuous. Next recall that $\mathbb{R}\mathrm{P}^{n-1} = (\mathbb{R}^n \setminus \{0\})/(\mathbb{R} \setminus \{0\})$. It follows from the Homotopy Quotient Theorem 8.5 that the map

$$\begin{array}{rcl} \mathbb{R}\mathrm{P}^{n-1} \times [0,1] & \to & \mathbb{R}\mathrm{P}^{n-1} \\ ([P], t) & \mapsto & [\varphi(t) \cdot P] \end{array}$$

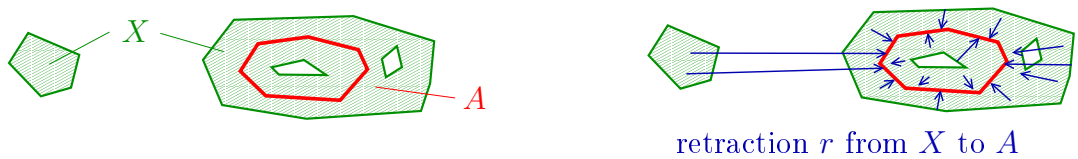
is continuous.

Proof of Theorem 8.5. The result is called a theorem, since the proof is indeed rather tricky. We refer to [Fri23] for details. ■

9. Homotopy equivalences and homotopy equivalent topological spaces

9.1. Deformation retractions.

Definition. Let X be a topological space. We say a subset $A \subset X$ is a retract of X if there exists a retraction $r\colon X \to A$, i.e. a continuous map with $r(a) = a$ for all $a \in A$.



Definition. Let X be a topological space and let A be a subset.

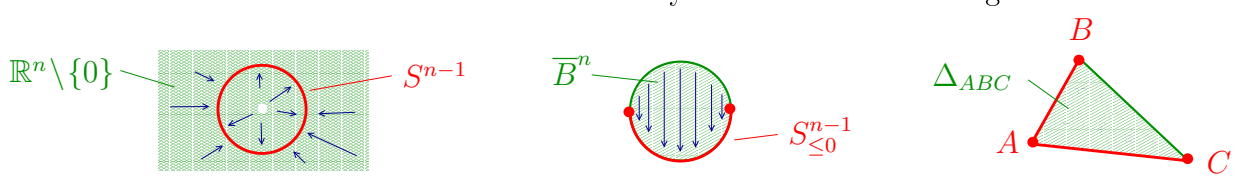
(1) A deformation retraction from X to A is a homotopy $F\colon X \times [0,1] \to X$ which has the following three properties:
  (a) $F_0 = \mathrm{id}_X$,
  (b) F is a homotopy rel A, i.e. for all $a \in A$ and all $t \in [0,1]$ we have $F(a,t) = a$,
  (c) $F_1(X) = A$.

(2) We say A is a deformation retract of X if there exists a deformation retraction from X to A.

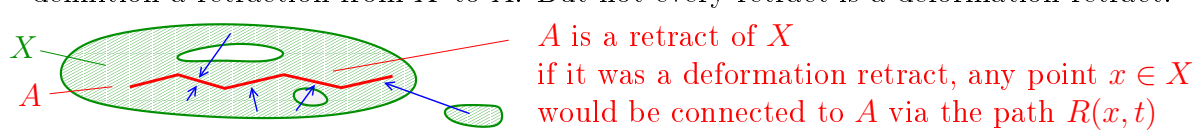
Examples. Let $n \in \mathbb{N}$.

(1) The map
$$\begin{array}{rcl} (\mathbb{R}^n \setminus \{0\}) \times [0,1] & \to & \mathbb{R}^n \setminus \{0\} \\ (v,t) & \mapsto & v \cdot (1 - t + t \cdot \frac{1}{\|v\|}) \end{array}$$
is a deformation retraction from $\mathbb{R}^n \setminus \{0\}$ to the sphere S^{n-1}.

(2) We leave it to the reader to prove the following two facts:
  (a) The hemisphere $S^{n-1}_{\leq 0}$ is a deformation retract of $\overline{B}^n$.
  (b) Given any three non-collinear points $A, B, C \in \mathbb{R}^n$ the triangle Δ_{ABC} admits a deformation retraction to the union of any two sides of the triangle.



(3) If $F\colon X \times [0,1] \to X$ is a deformation retraction from X to A, then $F_1\colon X \to A$ is by definition a retraction from X to A. But not every retract is a deformation retract.



Proposition 9.1. (Matrix Groups Deformation Retract Proposition) Let $n \in \mathbb{N}_0$. There exists a deformation retraction
$$R\colon \mathrm{GL}(n,\mathbb{R}) \times [0,1] \to \mathrm{GL}(n,\mathbb{R})$$
from $\mathrm{GL}(n,\mathbb{R})$ to $\mathrm{O}(n)$ such that $R_1\colon \mathrm{GL}(n,\mathbb{R}) \to \mathrm{O}(n)$ is the map given by the Gram-Schmidt Orthonormalization Process 0.12. The analogous statement holds if we replace

the pair $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{O}(n)$ by one of the following four pairs:

(a)	$\mathrm{GL}_+(n,\mathbb{R})$	and	$\mathrm{SO}(n)$
(b)	$\mathrm{SL}(n,\mathbb{R})$	and	$\mathrm{SO}(n)$
(c)	$\mathrm{GL}(n,\mathbb{C})$	and	$\mathrm{U}(n)$
(d)	$\mathrm{SL}(n,\mathbb{C})$	and	$\mathrm{SU}(n)$.

Proof of Proposition 9.1. For $i = 1,\dots,n$ we consider the following subspaces of $\mathrm{GL}(n,\mathbb{R})$:

$$\begin{aligned} X(i) &:= \{(v_1\dots v_n) \in \mathrm{GL}(n,\mathbb{R}) \,|\, v_1,\dots,v_{i-1} \text{ are orthonormal and } v_i \perp v_1,\dots,v_{i-1}\} \\ Y(i) &:= \{(v_1\dots v_n) \in \mathrm{GL}(n,\mathbb{R}) \,|\, v_1,\dots,v_i \text{ are orthonormal}\}. \end{aligned}$$

In the proof of the Matrix Group Path-Component Proposition 2.23, we implicitly already showed that the $2k$ steps of the Gram-Schmidt Orthogonalization Process define deformation retractions from $X(i)$ to $Y(i)$ and $Y(i)$ to $X(i+1)$. Since $Y(1) = \mathrm{GL}(n,\mathbb{R})$ and since $Y(n) = \mathrm{O}(n)$ we can combine these $2k-1$ deformation retractions to obtain a deformation retraction

$$R\colon\ \mathrm{GL}(n,\mathbb{R}) \times [0,1] \ \to\ \mathrm{GL}(n,\mathbb{R})$$

from $\mathrm{GL}(n,\mathbb{R})$ to $\mathrm{O}(n)$. Furthermore note that it follows from the Gram-Schmidt Orthonormalization Process 0.12 (2) and (3) that this deformation retraction R restricts to a deformation retraction from $\mathrm{GL}_+(n,\mathbb{R})$ to $\mathrm{SO}(n)$ and from $\mathrm{SL}(n,\mathbb{R})$ to $\mathrm{SO}(n)$.

Finally note that the complex cases are treated in a very similar fashion. ■

Theorem 9.2. (Deformation Retraction-Quotient Theorem) Let X be a topological space and let $A \subset X$ be a subset.

(1) Let "$\sim$" be an equivalence relation on X and let $R\colon X \times [0,1] \to X$ be a deformation retraction from X to A. If "$\sim$" is compatible with the deformation retraction R in the sense that $R(x,t) \sim R(x',t)$ whenever $x \sim x'$, then the map

$$\begin{aligned} \overline{R}\colon (X/\sim) \times [0,1] &\ \to\ X/\sim \\ ([x],t) &\ \mapsto\ [R(x,t)] \end{aligned}$$

is a deformation retraction from $X/\sim$ to $A/\sim$.

(2) Let X be a topological space and let A be a subset of X. If A is a deformation retract of X, then the point A/A is a deformation retract of X/A.

Proof.

(1) It follows from the Homotopy Quotient Theorem 8.5 that $\overline{R}$ is continuous. It is now clear that $\overline{R}$ is a deformation retraction from $X/\sim$ to $A/\sim$.

(2) Let $r\colon X \times [0,1] \to X$ be a deformation retraction from X to A. We consider the equivalence relation "$\sim_A$" on X that is generated by the condition that all points in A are equivalent. Note that by definition of a deformation retraction the homotopy r is compatible with the equivalence relation "$\sim_A$". Therefore it follows from (1) that the map

$$\begin{aligned} (X/\sim_A) \times [0,1] &\ \to\ X/\sim_A \\ ([x],t) &\ \mapsto\ [r(x,t)]. \end{aligned}$$

is a deformation retraction from $X/\sim_A = X/A$ to $A/\sim_A = A/A$. ■

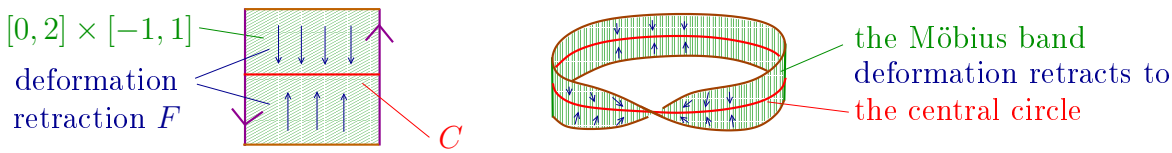
Example. We consider $X = [0,2] \times [-1,1]$ together with the equivalence relation "$\sim$" that is generated by

$$(0,y) \sim (2,-y) \quad \text{for all } y \in [-1,1].$$

It is straightforward to see that $X/\sim$ is homeomorphic to the Möbius band as defined on page 81. We consider the maps

$$\begin{array}{rcl} F\colon X\times[0,1] & \to & X \\ ((x,y),t) & \mapsto & (x, y\cdot(1-t)) \end{array} \quad \text{and} \quad \begin{array}{rcl} \overline{F}\colon (X/\sim)\times[0,1] & \to & X/\sim \\ ([(x,y)],t) & \mapsto & [(x,y\cdot(1-t))]. \end{array}$$

Note that F is a deformation retraction from X to $C := [0,2]\times\{0\}$. One can easily verify that F is compatible with the equivalence relation "$\sim$". Thus we obtain from the Deformation Retraction-Quotient Theorem 9.2 (1) that $\overline{F}$ is a deformation retraction from the Möbius band to $C/\sim$. Note that $C/\sim$ is homeomorphic to S^1.



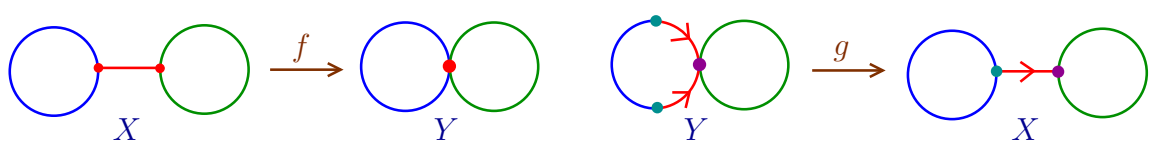
9.2. Homotopy equivalent topological spaces.

Definition. Let X be and Y be topological spaces.

(1) A continuous $f\colon X\to Y$ is a homotopy equivalence between X and Y, if f admits a homotopy inverse, i.e. a continuous map $g\colon Y\to X$ with $g\circ f\simeq \mathrm{id}_X$ and $f\circ g\simeq \mathrm{id}_Y$.
(2) If there exists a homotopy equivalence between X and Y, then we say that X and Y are homotopy equivalent.

Example.

(1) Basically by definition a homeomorphism is a homotopy equivalence.
(2) In the figure we see two subspaces X and Y of $\mathbb{R}^2$ and we indicate continuous maps $f\colon X\to Y$ and $g\colon Y\to X$. With some effort one can show that $g\circ f\simeq \mathrm{id}_X$ and $f\circ g\simeq \mathrm{id}_Y$. Thus f and g are homotopy equivalences and X and Y are homotopy equivalent.



Lemma 9.3. (Deformation Retract–Homotopy Equivalence Lemma) Let X be a topological space and let A be a subset. If $F\colon X\times[0,1]\to A$ is a deformation retraction from X to A, then the inclusion map

$$\begin{array}{rcl} i\colon A & \to & X \\ a & \mapsto & a \end{array} \quad \text{and the map} \quad \begin{array}{rcl} r\colon X & \to & A \\ x & \mapsto & F(x,1) \end{array}$$

are homotopy inverses of one another. In particular both maps are homotopy equivalences and in particular the deformation retract A is homotopy equivalent to the total space X.

Proof. By definition of a deformation retraction the map $r\circ i = F_1\circ i\colon A\to A$ is the identity id_A. Thus it remains to show that $i\circ r\colon X\to X$ is homotopic to the identity id_X. A homotopy between id_X and $i\circ r$ is given by $F\colon X\times[0,1]\to X$. ■

9.3. The homotopy category.

Definition. Given two topological spaces X and Y we set

$$[X,Y] := \{\text{continuous maps } X \to Y\}/\sim$$

where $f \sim g$ if there exists a homotopy between f and g.

Definition. We define the homotopy category $\mathcal{HomTop}$ of topological spaces by

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{HomTop}) & := & \text{all topological spaces,} \\ \mathrm{Mor}(X,Y) & := & [X,Y] \end{array}$$

with the composition

$$\begin{array}{ccc} \mathrm{Mor}(X,Y) \times \mathrm{Mor}(Y,Z) & \to & \mathrm{Mor}(X,Z) \\ ([f],[g]) & \mapsto & [g \circ f]. \end{array}$$

Lemma 9.4. (Homotopy Equivalence-Reformulation Lemma) Let $f\colon X \to Y$ be a continuous map between topological spaces. The following statements are equivalent:

(1) The element $[f] \in \mathrm{Mor}_{\mathcal{HomTop}}(X,Y)$ is an isomorphism in the homotopy category $\mathcal{HomTop}$.
(2) f is a homotopy equivalence.

Proof. This lemma is basically a tautology, it follows immediately from the definitions. ■

Lemma 9.5. (Homotopy-Equivalence Basics Lemma)

(1) The relation "homotopy equivalence" is an equivalence relation on the class of all topological spaces.
(2) Let $f, g\colon X \to Y$ be two continuous maps between topological spaces. If f is a homotopy equivalence and if g is homotopic to f, then g is also a homotopy equivalence.

Proof. All the statements follow easily from the Homotopy Equivalence-Reformulation Lemma 9.4 and the Isomorphism Lemma 3.3. ■

Lemma 9.6. (Homotopy-π_0-Lemma)

(1) The maps
$$\begin{array}{ccc} X & \mapsto & \pi_0(X) \\ [f\colon X \to Y] & \mapsto & [f_*\colon \pi_0(X) \to \pi_0(Y)] \end{array}$$
form a covariant functor from the homotopy category to the category of sets.
(2) Let $f\colon X \to Y$ be a continuous map between topological spaces. If f is a homotopy equivalence, then f induces a bijection $f_*\colon \pi_0(X) \to \pi_0(Y)$.

Proof.

(1) We only really need to show that the functor is well-defined. More precisely, we need to show that if $f_0, f_1\colon X \to Y$ are homotopic, then $f_{0*} = f_{1*}\colon \pi_0(X) \to \pi_0(Y)$. Thus let C be a path-component of X and let D be the unique path-component of Y that contains $f_0(C)$. Since f_0 and f_1 are homotopic we see that for every $x \in X$ the points $f_0(x)$ and $f_1(x)$ are path-equivalent in Y. But that means, by definition of a path-component as a path-equivalence class, that $f_1(C) \subset D$. This implies that $f_{1*}(C) = D = f_{0*}(C)$.
(2) This statement follows from (1), the Homotopy Equivalence-Reformulation Lemma 9.4 and the fact that functors send isomorphisms to isomorphisms. ■

9.4. Contractible topological spaces.

Definition. Let X be a topological space. We say X is contractible, if X is homotopy equivalent to a topological space which consists of precisely one point.

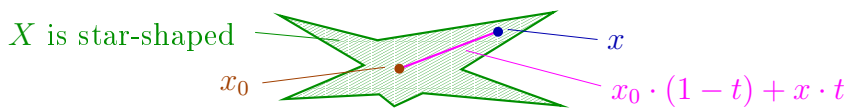
Lemma 9.7. (Point Deformation Retract Lemma) If a topological space X admits a deformation retraction to a point $x_0 \in X$, then X is homotopy equivalent to $\{x_0\}$, in particular X is contractible.

Proof. This lemma is just a special case of the Deformation Retract–Homotopy Equivalence Lemma 9.3. ■

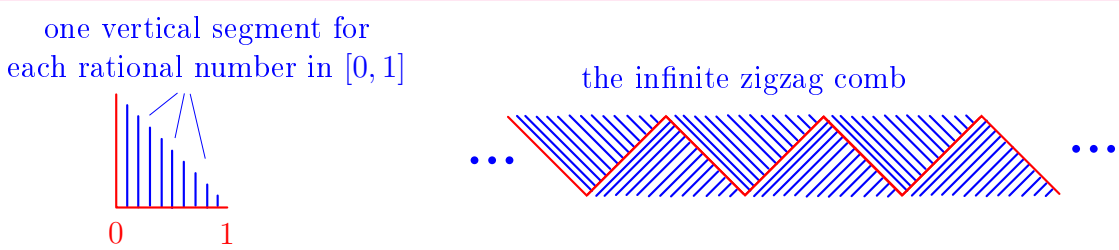
Example. We say a subset X of $\mathbb{R}^n$ is star-shaped if there exists a point $x_0 \in X$ such that for every $x \in X$ the segment from x_0 to x lies in X. Basically by the same argument as in the Convex Homotopy Lemma 8.1 we see that the map

$$\begin{array}{rcl} f\colon X \times [0,1] & \to & X \\ (x,t) & \mapsto & \underbrace{x \cdot (1-t) + x_0 \cdot t}_{\in X,\ \text{since } X \text{ is star-shaped}} \end{array}$$

is deformation retraction from X to the point x_0. It follows from the Point Deformation Retract Lemma 9.7 that X is contractible.



Lemma 9.8. (Zigzag Comb-Lemma) The "infinite zigzag comb" shown in the figure below is contractible, but it does not admit a deformation retraction to a single point.



Proof. We leave it to the reader to provide the proof. ■

Part II

Fundamental groups and higher homotopy groups

10. How can we show that two topological spaces are (not) homeomorphic?

This chatty chapter is only contained in the full lecture notes.

11. The fundamental group

11.1. Path-homotopy.

Definition. Let X be a topological space.

(1) We define a path in X as a continuous map $\gamma\colon [a,b] \to X$. We call $\gamma(a)$ the starting point of γ and we call $\gamma(b)$ the endpoint of γ. Often we say that γ is a path from $\gamma(a)$ to $\gamma(b)$.

(2) A loop is a path for which the starting point and the endpoint coincide.

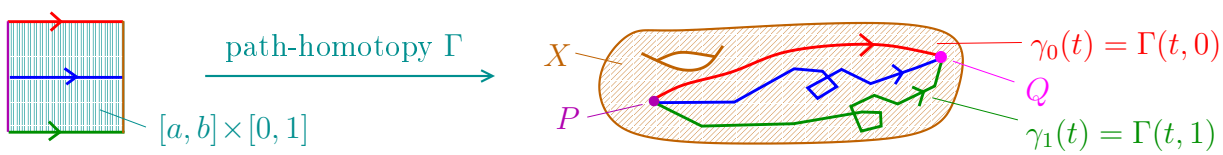
(3) Given a path $\gamma\colon [a,b] \to X$ we refer to the path $\overline{\gamma}\colon [a,b] \to X$ that is defined by $\overline{\gamma}(t) = \gamma(b+a-t)$ as the inverse path of γ.

Definition. Let X be a topological space.

(1) Let $\gamma_0, \gamma_1\colon [a,b] \to X$ be two paths with the same starting point P and the same endpoint Q. A path-homotopy between the paths γ_0 and γ_1 is a homotopy rel $\{a,b\}$ between γ_0 and γ_1.

(2) If there exists a path-homotopy between two paths γ_0 and γ_1, then we say that γ_0 and γ_1 are path-homotopic, and we write $\gamma_0 \simeq_{\mathrm{p}} \gamma_1$.

(3) We say a loop $\gamma\colon [a,b] \to X$ is null-homotopic, if it is path-homotopic to the constant path δ given by $\delta(t) := \gamma(a)$ for all $t \in [a,b]$.



Example. Let C be a convex subset of $\mathbb{R}^n$ and let $\gamma_0, \gamma_1\colon [a,b] \to C$ be two paths in C with same starting point and the same endpoint. As in the Convex Homotopy Lemma 8.1 we see that

$$\begin{array}{rcl} \Gamma\colon [a,b] \times [0,1] & \to & C \\ (t,s) & \mapsto & \underbrace{\gamma_0(t)\cdot(1-s) + \gamma_1(t)\cdot s}_{\in C,\ \text{since } C \text{ is convex}} \end{array}$$

is a path-homotopy between γ_0 and γ_1.

Definition. Let $\gamma\colon [a,b] \to X$ be a path in a topological space X and let $\varphi\colon [a,b] \to [a,b]$ be a continuous map with $\varphi(a) = a$ and $\varphi(b) = b$. We refer to $\gamma \circ \varphi\colon [a,b] \to X$ as a reparametrization of γ.

Lemma 11.1. (Reparametrization-Homotopy Lemma) Let $\gamma\colon [a,b] \to X$ be a path in a topological space X. If a path $\delta\colon [a,b] \to X$ is obtained from γ by a reparametrization, then γ and δ are path-homotopic.

Proof. A path-homotopy between γ and $\gamma \circ \varphi$ is given by

$$\begin{array}{rcl} P\colon [a,b] \times [0,1] & \to & X \\ (t,s) & \mapsto & \gamma(t\cdot s + \varphi(t)\cdot(1-s)). \end{array}$$

■

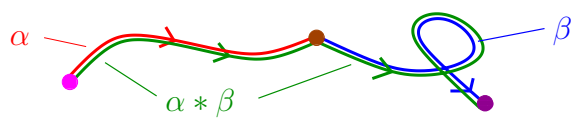
Lemma 11.2. (Path-Homotopy–Equivalence Relation Lemma) Let X be a topological space and let $x, y \in X$. Then "path-homotopy" is an equivalence relation on the set of paths $\gamma\colon [a,b] \to X$ from x to y.

Proof. This lemma is just the special case of the Homotopy-Equivalence Relation Lemma 8.3 with $X = [a,b]$ and $A = \{a,b\}$. ■

Definition. Given a path $f\colon [a,b] \to X$ in a topological space X we denote by $[f]$ the equivalence class of f with respect to the equivalence relation which is given by path-homotopies. We call $[f]$ the path-homotopy class of f.

Definition. Let X be a topological space and let $\alpha\colon [a,b] \to X$ and $\beta\colon [c,d] \to X$ be two paths with $\alpha(b) = \beta(c)$. We define the concatenation of α and β as the path $\alpha * \beta$ which is given by

$$\begin{array}{rcl} \alpha * \beta\colon [0,1] & \to & X \\ t & \mapsto & \begin{cases} \alpha(a + 2t(b-a)), & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta(c + (2t-1)(d-c)), & \text{if } t \in \left(\frac{1}{2}, 1\right]. \end{cases} \end{array}$$



Lemma 11.3. (Path-Homotopy–Concatenation Lemma) Let X be a topological space. Let $\alpha, \alpha'\colon [a,b] \to X$ and $\beta, \beta'\colon [c,d] \to X$ be paths with $\alpha(b) = \alpha'(b) = \beta(c) = \beta'(c)$. Then

$$\alpha \simeq_{\mathrm{p}} \alpha' \quad \text{and} \quad \beta \simeq_{\mathrm{p}} \beta' \quad \Longrightarrow \quad \alpha * \beta \simeq_{\mathrm{p}} \alpha' * \beta'.$$

Proof. We leave it to the reader to provide the fairly elementary proof. ■

Definition. Let $[\alpha]$ and $[\beta]$ be two equivalence classes of paths in a topological space X with $\alpha(1) = \beta(0)$. We define the product of the equivalence classes $[\alpha]$ and $[\beta]$ as

$$[\alpha] \cdot [\beta] \ := \ [\alpha * \beta].$$

Proposition 11.4. (Path-Homotopy Product Proposition) Let X be a topological space. Given any three paths $\alpha, \beta, \gamma\colon [0,1] \to X$ the following hold:

(1) If $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$, then

$$[\alpha] \cdot ([\beta] \cdot [\gamma]) \ = \ ([\alpha] \cdot [\beta]) \cdot [\gamma].$$

(2) For $x \in X$ let e_x be the constant path that is given by $e_x(t) := x$. The following equality holds:

$$[e_{\alpha(0)}] \cdot [\alpha] \ = \ [\alpha] \ = \ [\alpha] \cdot [e_{\alpha(1)}].$$

(3) We denote by $\overline{\alpha}$ the inverse path. Then the following equalities hold:

$$[\alpha] \cdot [\overline{\alpha}] \ = \ [e_{\alpha(0)}] \quad \text{and} \quad [\overline{\alpha}] \cdot [\alpha] \ = \ [e_{\alpha(1)}].$$

Proof. Let $\alpha, \beta, \gamma\colon [0,1] \to X$ be three paths in a topological space X.

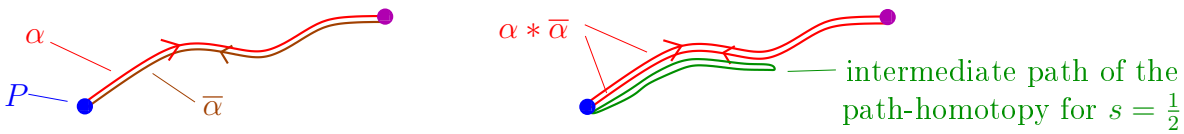
(1) Suppose that $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$. It follows easily from the Reparametrization-Homotopy Lemma 11.1 that $\alpha * (\beta * \gamma) \simeq_{\mathrm{p}} (\alpha * \beta) * \gamma$.

(2) Again this statement follows easily from the Reparametrization-Homotopy Lemma 11.1.

(3) We write $P = \alpha(0)$. Note that $\alpha * \overline{\alpha}$ is a loop in P. We consider the map

$$\begin{array}{rcl} F\colon [0,1]\times[0,1] & \to & X \\ (t,s) & \mapsto & \begin{cases} \alpha(2t\cdot s), & \text{if } t\in[0,\frac{1}{2}], \\ \alpha(2s-2t\cdot s), & \text{if } t\in[\frac{1}{2},1]. \end{cases} \end{array}$$

It follows from the Homotopy Combination Lemma 8.2 (1) that this map is indeed continuous. For $s=0$ we obtain the constant path at P and for $s=1$ we obtain the path $\alpha*\overline{\alpha}$. For any $s\in[0,1]$ we furthermore have $F(0,s)=F(1,s)=P$. Thus F is a path-homotopy from the constant path to $\alpha*\overline{\alpha}$. This implies that $[e_{\alpha(0)}]=[\alpha*\overline{\alpha}]=[\alpha]\cdot[\overline{\alpha}]$. Almost the same argument shows that $[\overline{\alpha}*\alpha]=[e_{\alpha(1)}]$. ■



11.2. The fundamental group of a pointed topological space.

Definition. Let X be a topological space and let $x_0\in X$ be a point. A loop in (X,x_0) is a path $f\colon[0,1]\to X$ with $f(0)=f(1)=x_0$.

Proposition 11.5. (π_1–Group Proposition) Let X be a topological space and $x_0\in X$. The set

$$\pi_1(X,x_0) \;:=\; \{\text{path-homotopy classes of loops in } (X,x_0)\}$$

together with the product map $[\alpha]\cdot[\beta]:=[\alpha*\beta]$ forms a group where the neutral element is represented by the constant path at x_0 and where the inverse of $[\gamma]$ is given by $[\overline{\gamma}]$.

Proof. It is clear that the concatenation of any two loops in (X,x_0) is defined. According to the Path-Homotopy–Concatenation Lemma 11.3 the concatenation of paths descends to a well-defined map

$$\begin{array}{rcl} \pi_1(X,x_0)\times\pi_1(X,x_0) & \to & \pi_1(X,x_0) \\ ([\alpha],[\beta]) & \mapsto & [\alpha]\cdot[\beta]:=[\alpha*\beta]. \end{array}$$

The Path-Homotopy Product Proposition 11.4 (1)–(3) gives us the following:

(1) The above product map satisfies the associativity axiom.
(2) The path-homotopy class of the constant loop $[e_{x_0}]$ is a neutral element.
(3) The inverse of the path-homotopy class of a loop $\gamma\colon[0,1]\to X$ in x_0 is given by the path-homotopy class of the inverse loop defined by $\overline{\gamma}(t):=\gamma(1-t)$, $t\in[0,1]$. ■

Definition. Let (X,x_0) be a pointed topological space. We call

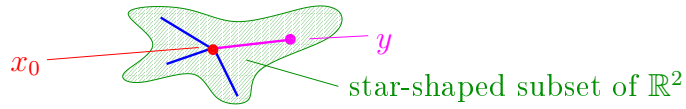
$$\pi_1(X,x_0) \;:=\; \{\text{path-homotopy classes of loops in } (X,x_0)\}$$

the fundamental group of X with respect to the base point x_0. If $\pi_1(X,x_0)$ is the trivial group, then we usually write $\pi_1(X,x_0)=0$.

Example. We say that a subset $X\subset\mathbb{R}^n$ is star-shaped with respect to $x_0\in X$ if for all $y\in X$ the segment $\{x_0\cdot(1-t)+y\cdot t\,|\,t\in[0,1]\}$ lies in X. If $\gamma\colon[0,1]\to\mathbb{R}^n$ is a loop in (X,x_0), then

$$\begin{array}{rcl} F\colon[0,1]\times[0,1] & \to & X \\ (t,s) & \mapsto & \gamma(t)\cdot(1-s)+x_0\cdot s \end{array}$$

is a path-homotopy in X between the loop γ and the constant loop e_{x_0}. Thus it follows that $\pi_1(X, x_0) = 0$.

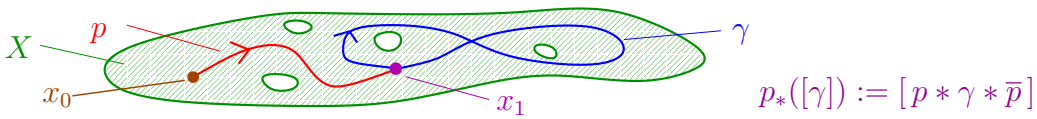


11.3. Change of base points.

Proposition 11.6. (Change-of-Base Point Proposition) Let X be a topological space, let x_0 and x_1 be two points in X and let $p\colon [0,1] \to X$ be a path from x_0 to x_1. The map

$$\begin{array}{rcl} p_*\colon \pi_1(X, x_1) & \to & \pi_1(X, x_0) \\ [\gamma] & \mapsto & [p * \gamma * \overline{p}\,] \end{array}$$

is well-defined and it is a group isomorphism.



Proof. Let X be a topological space and let x_0 and x_1 be two points in X that are connected via a path $p\colon [0,1] \to X$ from x_0 to x_1. Let γ be a loop in (X, x_1). Then $p * \gamma * \overline{p}$ is evidently a loop in (X, x_0). From the Path-Homotopy–Concatenation Lemma 11.3 and the Path-Homotopy Product Proposition 11.4 it follows that the path-homotopy class of $p*\gamma*\overline{p}$ depends only on the choice of the path-homotopy class of γ, i.e. the map

$$\begin{array}{rcl} \Phi\colon \pi_1(X, x_1) & \to & \pi_1(X, x_0) \\ [\gamma] & \mapsto & [p * \gamma * \overline{p}\,] \end{array}$$

is well-defined. Furthermore for two loops γ, δ in (X, x_1) we have

$$\begin{array}{rcl} \Phi([\gamma]) \cdot \Phi([\delta]) & = & [p * \gamma * \overline{p}\,] \cdot [p * \delta * \overline{p}\,] \;=\; [p * \gamma * \overline{p} * p * \delta * \overline{p}] \\ & \underset{\uparrow}{=} & [p * \gamma * e_{x_1} * \delta * \overline{p}\,] \;\underset{\uparrow}{=}\; [p * (\gamma * \delta) * \overline{p}\,] \;=\; \Phi([\gamma * \delta]) \;=\; \Phi([\gamma] \cdot [\delta]). \end{array}$$

the two statements follow from the Path-Homotopy Concatenation Proposition 11.4 (3) and (2) together with the Path-Homotopy–Concatenation Lemma 11.3

Thus we have shown that Φ is a group homomorphism. Furthermore Φ is even a group isomorphism, since an inverse map is given by

$$\begin{array}{rcl} \Psi\colon \pi_1(X, x_0) & \to & \pi_1(X, x_1) \\ [\delta] & \mapsto & [\overline{p} * \delta * p]. \end{array}$$

■

Corollary 11.7. (Base Point Independence Corollary) Let X be a topological space and let $x_0, x_1 \in X$. If X is path-connected, then the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic.

Proof. This statement follows easily from the Change-of-Base Point Proposition 11.6. ■

Definition. Let X be a path-connected non-empty topological space.
(1) We denote by $\pi_1(X)$ the isomorphism type of the fundamental group.
(2) If $\pi_1(X) = 0$, then we say that X is simply connected.

Lemma 11.8. (Paths-in-Simply Connected Space Lemma) Let X be a topological space and let P, Q be two points in X. If X is simply connected, then any two paths $\alpha, \beta\colon [0,1] \to X$ from P to Q are path-homotopic.

Proof. We leave the proof as an exercise to the reader. ■

11.4. The fundamental group of the circle.

Theorem 11.13. (Circle-π_1-Theorem) The map

$$\begin{array}{rcl} \Psi\colon \mathbb{Z} & \to & \pi_1(S^1, 1) \\ n & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & S^1 \\ t & \mapsto & \exp(2\pi\,\mathrm{i}\cdot n\cdot t) \end{array}\right] \end{array}$$

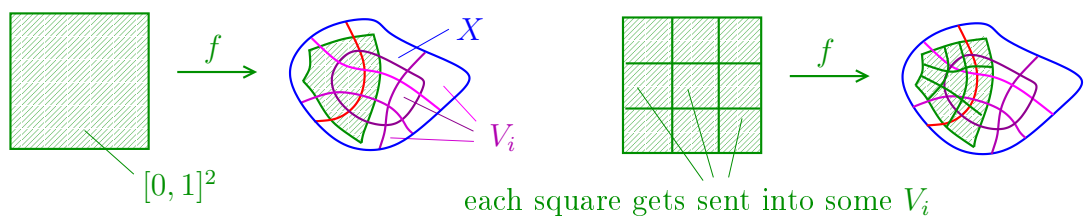
is an isomorphism of groups.

Lemma 11.9. (Lebesgue Lemma)

(1) Let $K \subset \mathbb{R}^n$ be a compact subset and let $\{U_i\}_{i\in I}$ be an open cover of K. There exists a $\delta > 0$ such that for every subset A of K with diameter $\operatorname{diam}(A) < \delta$ there exists an $i \in I$ with $A \subset U_i$.

(2) Let $f\colon [0,1]^m \to X$ be a continuous map from the cube $[0,1]^m$ to a topological space X and let $\{V_i\}_{i\in I}$ be an open cover of X. Then there exists an $N \in \mathbb{N}$ such that for any $a_1, \ldots, a_m \in \{0, \ldots, N-1\}$ there exists an $i \in I$ such that

$$f\big(\big[\tfrac{a_1}{N}, \tfrac{a_1+1}{N}\big] \times \cdots \times \big[\tfrac{a_m}{N}, \tfrac{a_m+1}{N}\big]\big) \;\subset\; V_i.$$



Proof. We proved Statement (1) in Analysis III. Statement (2) is a straightforward consequence of Statement (1). Full details can also be found in [Fri23]. ■

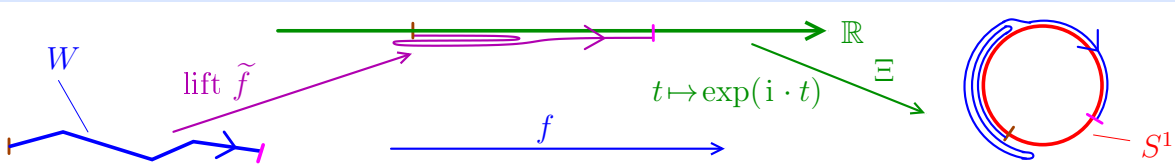
Definition.

(1) We work with the map

$$\begin{array}{rcl} \Xi\colon \mathbb{R} & \to & S^1 \\ t & \mapsto & \exp(\mathrm{i}\cdot t). \end{array}$$

(2) Let W be a topological space and let $f\colon W \to S^1$ be a continuous map. A lift of f is a continuous map $\widetilde{f}\colon W \to \mathbb{R}$ such that $\Xi \circ \widetilde{f} = f$, in other words, such that for all $w \in W$ we have

$$f(w) \;=\; \exp\big(\mathrm{i}\cdot \widetilde{f}(w)\big).$$



Example. For $n \in \mathbb{Z}$ we consider the map $f\colon [0,1] \to S^1$ with $f(t) := \exp(2\pi\,\mathrm{i} \cdot n \cdot t)$. A lift of f is given by the map

$$\begin{array}{rcl} \widetilde{f}\colon [0,1] & \to & \mathbb{R} \\ t & \mapsto & 2\pi \cdot n \cdot t. \end{array}$$

This lift is not unique though since we can "shift" the lift by any $k \cdot 2\pi$ with $k \in \mathbb{Z}$.

Lemma 11.10. (Lift Uniqueness Lemma) Let W be a connected topological space, let $f\colon W \to S^1$ be a continuous map and let $\widetilde{\varphi}, \widetilde{\psi}\colon W \to \mathbb{R}$ be two lifts of f. If $\widetilde{\varphi}$ and $\widetilde{\psi}$ agree at some point $w_0 \in W$, then they agree everywhere.

Proof. We consider the map

$$\begin{array}{rcl} \theta\colon W & \to & \mathbb{R} \\ w & \mapsto & \theta(w) := \widetilde{\varphi}(w) - \widetilde{\psi}(w). \end{array}$$

By hypothesis we have $\theta(w_0) = 0$. Next note that for each $w \in W$ we have

$$\underbrace{\exp\big(\mathrm{i} \cdot (\widetilde{\varphi}(w) - \widetilde{\psi}(w))\big)}_{=\exp(\mathrm{i}\cdot\theta(w))=\Xi(\theta(w))} = \exp\big(\mathrm{i} \cdot \widetilde{\varphi}(w)\big) \cdot \exp\big(\mathrm{i} \cdot \widetilde{\psi}(w)\big)^{-1} \underset{\substack{\uparrow \\ \text{since } \widetilde{\varphi}, \widetilde{\psi}\colon W \to \mathbb{R} \text{ are lifts of } f}}{=} f(w) \cdot f(w)^{-1} = 1.$$

This shows that the continuous map $s \mapsto \theta(w) := \widetilde{\varphi}(w) - \widetilde{\psi}(w)$ takes values in the discrete subspace $2\pi \cdot \mathbb{Z} = \Xi^{-1}(1) \subset \mathbb{R}$. Since W is connected it follows from the Connected-to-Discrete Lemma 2.18 that the map θ is constant. Since the map θ vanishes at w_0 we see that it is zero everywhere. This implies that $\widetilde{\varphi} = \widetilde{\psi}$. ■

Lemma 11.11. (Lift-Extension Lemma) Let X be a topological space. We suppose that we have a decomposition $X = A \cup B$ such that A and B are closed subsets. Let $f\colon X \to S^1$ be a continuous map and let $\widetilde{f}\colon A \to \mathbb{R}$ be a lift of $f|_A$. We suppose that the following statements hold:

(1) There exists a $P \in S^1$ with $f(B) \subset S^1 \setminus \{P\}$.
(2) The intersection $A \cap B$ is connected and non-empty.

Then there exists a lift $\widetilde{f}\colon X \to \mathbb{R}$ of f that agrees with the given lift on A.

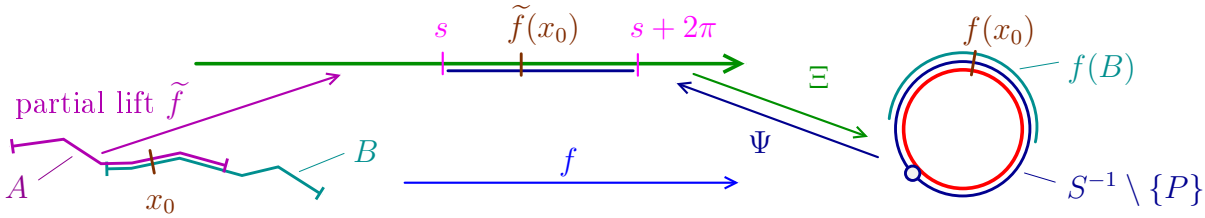
Proof. We start with a few preparations:

(1) We pick $x_0 \in A \cap B$ and we pick $s \in \mathbb{R}$ with $\Xi(s) = P$.
(2) Since $f(x_0) \in f(A \cap B) \subset f(B) \subset S^1 \setminus \{P\}$ and since $\widetilde{f}\colon A \to \mathbb{R}$ is a lift of $f\colon A \to S^1$ we see that $\widetilde{f}(x_0) \notin \Xi^{-1}(P)$.
(3) It follows from (2) that there exists an $m \in \mathbb{Z}$ such that $\widetilde{f}(x_0) \in (s+m\cdot 2\pi, s+m\cdot 2\pi+2\pi)$.
(4) One can easily verify that the restriction of Ξ to $(s+m\cdot 2\pi, s+m\cdot 2\pi+2\pi) \to S^1 \setminus \{P\}$ is a homeomorphism. We denote by Ψ the inverse.

We set

$$\begin{array}{rcl} \widetilde{f}\colon X & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} \widetilde{f}(x), & \text{if } x \in A, \\ \Psi(\underbrace{f(x)}_{\in S^1\setminus\{P\}}), & \text{if } x \in B. \end{cases} \end{array}$$

Note that on $A \cap B$ we have a priori two different definitions of $\widetilde{f}$. But both definitions are lifts of $f|_{A\cap B}$ and by design they agree on x_0. By hypothesis the intersection $A \cap B$



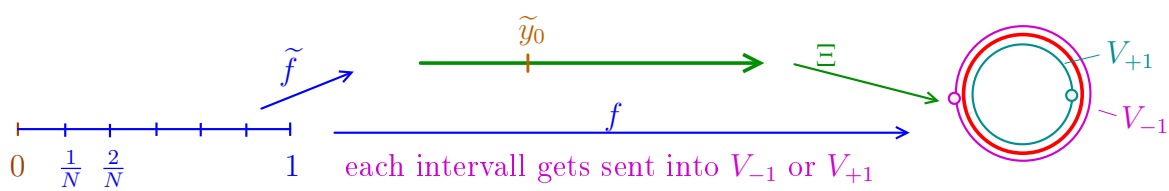
is connected. Thus we obtain from the Lift Uniqueness Lemma 11.10 that the two maps actually agree on the overlap $A \cap B$. This shows that $\widetilde{f}\colon X \to \mathbb{R}$ is well-defined.

Next note that since A and B are closed it follows from the Pasting Proposition 2.4 that this map $\widetilde{f}\colon X \to \mathbb{R}$ is continuous. It is now clear that it is a lift of $f\colon X \to \mathbb{R}$. ■

Lemma 11.12. (Lift-from-S^1-to-$\mathbb{R}$ Lemma) Given any continuous map $f\colon [0,1] \to S^1$ and given any $\widetilde{y}_0 \in \mathbb{R}$ with $f(0) = \exp(\mathrm{i}\cdot\widetilde{y}_0) =: \Xi(\widetilde{y}_0)$ there exists a *unique* lift $\widetilde{f}\colon [0,1] \to \mathbb{R}$ of f with $\widetilde{f}(0) = \widetilde{y}_0$.

Proof of Lemma 11.12. The uniqueness statement follows immediately from the Lift Uniqueness Lemma 11.10. Thus it remains to show existence. We set $V_{\pm 1} := S^1 \setminus \{\pm 1\}$. By the Lebesgue Lemma 11.9 there exists an $N \in \mathbb{N}$ such that for every $i \in \{0, \dots, N-1\}$ the image $f([\frac{i}{N}, \frac{i+1}{N}])$ is contained in V_{-1} or V_{+1}. It remains to prove the following claim.

Claim. For each $i \in \{0, \dots, N\}$ there exists a lift $\widetilde{f}\colon [0, \frac{i}{N}] \to \mathbb{R}$ of f with $\widetilde{f}(0) = \widetilde{y}_0$.



Proof. For $i = 0$ we just set $\widetilde{f}(0) := \widetilde{y}_0$. Now suppose that we have already defined $\widetilde{f}$ on some interval $[0, \frac{i}{N}]$ with $i < N$. Note that by construction there exists an $\epsilon \in \{\pm 1\}$ with $f([\frac{i}{N}, \frac{i+1}{N}]) \in V_\epsilon = S^1 \setminus \{\epsilon\}$. Thus it follows from the Lift-Extension Lemma 11.11 that we can extend the lift $\widetilde{f}$ to the interval $[0, \frac{i+1}{N}]$. ■

Theorem 11.13. (Circle-π_1-Theorem) Let $z_0 = \exp(2\pi\,\mathrm{i}\cdot t_0) \in S^1$. Given any continuous map $\gamma\colon [0,1] \to S^1$ with $\gamma(0) = z_0$ we denote by $\widetilde{\gamma}\colon [0,1] \to \mathbb{R}$ the unique lift with $\widetilde{\gamma}(0) = t_0 \in \mathbb{R}$. The maps

$$\Psi\colon \mathbb{Z} \to \pi_1(S^1, z_0) \qquad n \mapsto \left[\begin{array}{rcl} [0,1] & \to & S^1 \\ t & \mapsto & z_0\cdot\exp(2\pi\,\mathrm{i}\cdot n\cdot t) \end{array}\right] \quad \text{and} \quad \Theta\colon \pi_1(S^1, z_0) \to \mathbb{Z} \qquad [\gamma\colon [0,1] \to S^1] \mapsto \tfrac{1}{2\pi}\cdot(\widetilde{\gamma}(1) - \widetilde{\gamma}(0))$$

are well-defined, they are isomorphisms and they are inverses of one another.

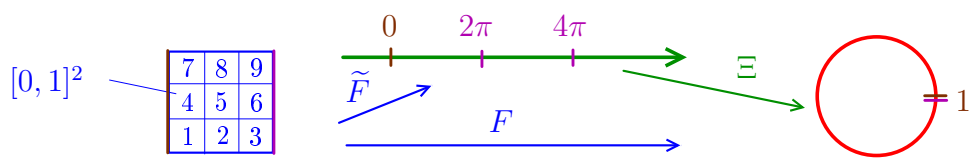
Proof. To simplify the notation we only consider the case that $z_0 = 1$ and $t_0 = 0$. The following claim shows that $\Theta\colon \pi_1(S^1, 1) \to \mathbb{Z}$ is well-defined.

Claim 1. Let $\alpha, \beta\colon [0,1] \to S^1$ be two loops in $(S^1, 1)$. If α and β are path-homotopic, then $\widetilde{\alpha}(1) = \widetilde{\beta}(1) \in \mathbb{R}$.

Proof. Let $F\colon [0,1]\times[0,1]\to S^1$ be a path-homotopy between α and β. First we want to show that F lifts to a map $[0,1]\times[0,1]\to\mathbb{R}$. As in the proof of the Lift-from-S^1-to-$\mathbb{R}$ Lemma 11.12 we set $V_{\pm 1} := S^1\setminus\{\pm 1\}$. By the Lebesgue Lemma 11.9 there exists an $N\in\mathbb{N}$ such that for any $i,j\in\{0,\dots,N-1\}$ there exists an $\epsilon\in\{\pm 1\}$ with

$$f\big(\big[\tfrac{i}{N},\tfrac{i+1}{N}\big]\times\big[\tfrac{j}{N},\tfrac{j+1}{N}\big]\big)\ \subset\ V_\epsilon.$$

The same way as we lifted the map $[0,1]\to S^1$ to $\mathbb{R}$ by lifting the map "intervall-by-intervall" starting from the left, we now lift the map $F\colon[0,1]^2\to S^1$ to $\mathbb{R}$ by lifting the map "square-by-square" starting from the "bottom left". We leave it to the reader to fill in the details.



Finally note that the fact that F is a path-homotopy means that $F(1,t)=1$ for all $t\in[0,1]$. This in turn implies that $\widetilde{F}(1,t)\in 2\pi\cdot\mathbb{Z}$ for all $t\in[0,1]$. It follows from the Connected-to-Discrete Lemma 2.18 the map $t\mapsto\widetilde{F}(1,t)$ is constant. Therefore we have shown that $\widetilde{\alpha}(1)=\widetilde{F}(1,0)=\widetilde{F}(1,1)=\widetilde{\beta}(1)$. ⊞

Next note that it follows almost immediately from the definitions that $\Psi\colon\mathbb{Z}\to\pi_1(S^1,1)$ is a homomorphism. Furthermore note that it follows easily from the definitions that $\Theta\circ\Psi\colon\mathbb{Z}\to\mathbb{Z}$ is the identity. Thus it remains to prove the following claim.

Claim 2. The map $\Theta\colon\pi_1(S^1,1)\to\mathbb{Z}$ is a bijection.

Proof. Since $\Theta\circ\Psi=\mathrm{id}_{\mathbb{Z}}$ we see that Θ is surjective. It remains to show that Θ is injective. So suppose we are given $[\alpha],[\beta]\in\pi_1(S^1,1)$ with $\Theta([\alpha])=\Theta([\beta])$, i.e. with $\widetilde{\alpha}(1)=\widetilde{\beta}(1)$. Note that the maps $\widetilde{\alpha},\widetilde{\beta}\colon[0,1]\to\mathbb{R}$ have the same starting point and the same endpoint in $\mathbb{R}$. *Since* $\mathbb{R}$ *is convex* we see, as in the Convex Homotopy Lemma 8.1, that

$$\begin{aligned} F\colon[0,1]\times[0,1] &\to \mathbb{R}\\ (t,s) &\mapsto \widetilde{\alpha}(t)\cdot(1-s)+\widetilde{\beta}(t)\cdot s\end{aligned}$$

is a path-homotopy in $\mathbb{R}$ between $\widetilde{\alpha}$ and $\widetilde{\beta}$. But this implies that

$$\begin{aligned} \Xi\circ F\colon[0,1]\times[0,1] &\to S^1\\ (t,s) &\mapsto \Xi(\widetilde{\alpha}(t)\cdot(1-s)+\widetilde{\beta}(t)\cdot s)\end{aligned}$$

is a path-homotopy in S^1 between $\alpha=\Xi\circ\widetilde{\alpha}$ and $\beta=\Xi\circ\widetilde{\beta}$. Thus $[\alpha]=[\beta]\in\pi_1(S^1,1)$. ■

11.5. The fundamental group of spheres.

Proposition 11.14. (Sphere-π_1-Proposition) For any $n\geq 2$ the sphere S^n is simply connected, i.e. we have $\pi_1(S^n)=0$.

Lemma 11.15. (Sphere-Minus-Point Lemma) For every $P\in S^n$ the topological space $S^n\setminus\{P\}$ is simply connected.

Proof of Lemma 11.15. Recall that on page 118 we saw that $\mathbb{R}^n$ is simply connected. Thus it suffices to show that $S^n\setminus\{P\}$ is homeomorphic to $\mathbb{R}^n$.

Recall that in the Stereographic Projection Lemma 2.11 we showed that there exists a homeomorphism $f\colon S^n \to \mathbb{R}^n \cup \{\infty\}$ with $f^{-1}(\infty) = N := (0,\dots,0,1)$. Thus f restricts to a homeomorphism $f\colon S^n \setminus \{N\} \to \mathbb{R}^n$. Now let P be any other point on S^n. There exists an orthogonal matrix A with $A \cdot P = N$. This shows that there exists a homeomorphism $g\colon S^n \to S^n$ with $g(P) = N$. Then $f \circ g\colon S^n \to \mathbb{R}^n \cup \{\infty\}$ restricts to a homeomorphism $S^n \setminus \{P\} \to \mathbb{R}^n$. ■

Proof of the Sphere-π_1-Proposition 11.14. Let $n \geq 2$. We write

$$N := (0,\dots,0,1) \quad \text{“North Pole”} \quad \text{and} \quad U := S^n \setminus \{N\},$$

and also

$$S := (0,\dots,0,-1) \quad \text{“South Pole”} \quad \text{and} \quad V := S^n \setminus \{S\}.$$

We fix a base point $x_0 \in U \cap V$. In the following we will only use the following three facts about U and V:

(I) U and V form an open cover of S^n.
(II) By the Sphere-Minus-Point Lemma 11.15 we know that U and V are simply connected.
(III) Our hypothesis $n \geq 2$ implies that the intersection $U \cap V = S^n \setminus \{N, S\}$ is path-connected.

It follows from (II) that it suffices to show that any loop in (S^n, x_0) is path-homotopic to a loop in U. Thus let $\gamma\colon [0,1] \to S^n$ be a loop in (S^n, x_0).

Claim 1. There exists a subdivision

$$0 = t_0 < t_1 < \ldots < t_{k-1} < t_k = 1,$$

such that the following hold:
(1) for each $i \in \{0,\dots,k-1\}$ we have $\gamma([t_i, t_{i+1}]) \subset U$ or $\gamma([t_i, t_{i+1}]) \subset V$, and
(2) for each $i \in \{1,\dots,k-1\}$ we have $\gamma(t_i) \in U \cap V$.

Proof. By (I) we can apply the Lebesgue Lemma 11.9 to $S^n = U \cup V$ and to $\gamma\colon [0,1] \to S^n$. Thus there exists a $k \in \mathbb{N}$ such that for any $i \in \{0,\dots,k-1\}$ the image $\gamma([\frac{i}{k}, \frac{i+1}{k}])$ lies in U or in V. In particular the subdivision $t_i = \frac{i}{k}$, $i = 0,\dots,k$ satisfies condition (1).

Now suppose that there exists an $i \in \{1,\dots,k-1\}$ with $\gamma(t_i) \notin U$. Then neither $\gamma([t_{i-1}, t_i])$ nor $\gamma([t_i, t_{i+1}])$ are contained in U. So by (1) both have to be contained in V, in particular we have $\gamma([t_{i-1}, t_{i+1}]) \subset V$. But then we can remove t_i from the subdivision. The same way we deal with the case that $\gamma(t_i) \notin V$. Iterating this process we can assume that for each every i we have $\gamma(t_i) \in U \cap V$. ⊞

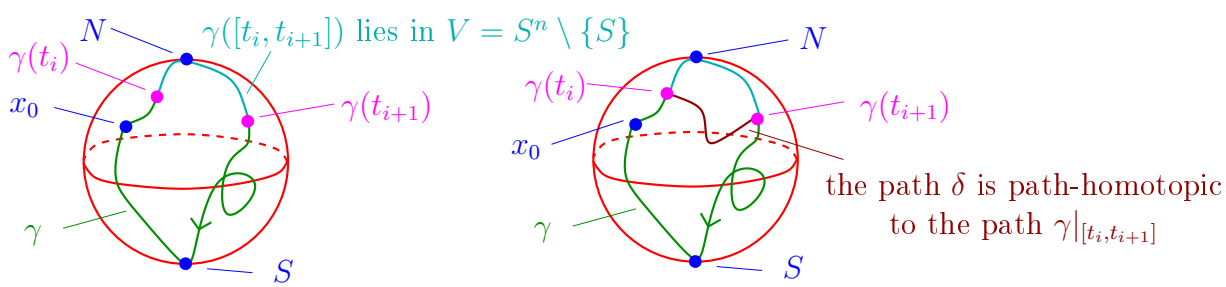


illustration for Claim 1 illustration for the proof of Claim 2

Since U is simply connected, i.e. since any loop in U is null-homotopic it now, suffices to prove the following claim.

Claim 2. The loop $\gamma\colon [0,1] \to S^n$ in (S^n, x_0) is path-homotopic to a path in U.

Proof. We pick a subdivision of $[0,1]$ as in Claim 1. Suppose there exists an $i \in \{0, \dots, k-1\}$ such that $\gamma([t_i, t_{i+1}])$ is not contained in U. By the choice of our subdivision we know that this implies that $\gamma([t_i, t_{i+1}])$ lies entirely in V. Since $U \cap V$ is path-connected and since $\gamma(t_i)$ and $\gamma(t_{i+1})$ both lie in $U \cap V$ we can pick a path $\delta\colon [t_i, t_{i+1}] \to U \cap V$ that connects $\gamma(t_i)$ and $\gamma(t_{i+1})$. Since the paths $\gamma|_{[t_i,t_{i+1}]}$ and δ both lie in the simply connected set V they are path-homotopic by the Paths-in-Simply Connected Space Lemma 11.8. In the loop γ we now replace $\gamma|_{[t_i,t_{i+1}]}$ by δ. It is straightforward to see that this way we obtain a new loop that is path-homotopic to the original loop and that stays within $U \cap V \subset U$ on the interval $[t_i, t_{i+1}]$. Iterating this procedure gives us the desired path-homotopy to a loop that is contained in U. ■

11.6. Alternative description of fundamental groups.

Identification. Note that the map

$$\begin{array}{rcl} [0,1]/0 \sim 1 & \to & S^1 \\ [t] & \mapsto & \exp(2\pi \mathrm{i} \cdot t) \end{array}$$

is a homeomorphism. We use it to make the identification $S^1 =_i [0,1]/0 \sim 1$.

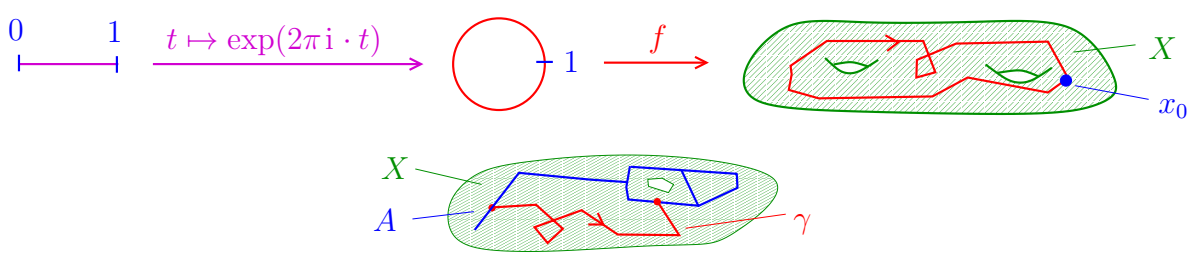
Convention. Let X be a topological space and let $x_0 \in X$.

(1) Using the above identification, using the Topological-Quotient Proposition 6.1 (1b) and using the Homotopy Quotient Theorem 8.5 we see that we can make the natural identification

$$\pi_1(X, x_0) \ =_i \ \{\text{all continuous maps } (S^1, 1) \to (X, x_0)\}/\sim$$

where $f \sim g$ if there exists a homotopy $F\colon S^1 \times [0,1] \to X$ such that for all $t \in [0,1]$ we have $F_t(1) = x_0$.

(2) We refer to continuous maps $(S^1, 1) \to (X, x_0)$ also as loops in (X, x_0).



12. Applications of fundamental groups

12.1. The Fundamental Theorem of Algebra.

Theorem 12.1. (Fundamental Theorem of Algebra) Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a zero in $\mathbb{C}$.

Proof. We start out with some notation: Let $q(z)$ be a polynomial with complex coefficients. If $q(z) \neq 0$ for all $z \in \mathbb{C}$ with $|z| = r$, then we can consider the path

$$\begin{array}{rcl} \gamma_r^{q(z)} \colon [0,1] & \to & S^1 \\ s & \mapsto & \dfrac{q(r \cdot \exp(2\pi \mathrm{i} \cdot s))}{\left|q(r \cdot \exp(2\pi \mathrm{i} \cdot s))\right|} \cdot \dfrac{|q(r)|}{q(r)}. \end{array}$$

Note that $\gamma_r^{q(z)}(0) = \gamma_r^{q(z)}(1) = 1$, i.e. $\gamma_r^{q(z)}$ is a loop in $(S^1, 1)$.

Now we start out with the actual proof of the theorem. Thus let

$$p(z) \;=\; a_n \cdot z^n + a_{n-1} \cdot z^{n-1} + \cdots + a_1 \cdot z + a_0$$

be a polynomial with $a_n \neq 0$. We assume that $p(z)$ has no zeros in $\mathbb{C}$. We need to show that $n = 0$. For $t \in [0,1]$ we also consider the polynomial

$$p_t(z) \;:=\; a_n \cdot z^n + t \cdot (a_{n-1} \cdot z^{n-1} + \cdots + a_1 \cdot z + a_0).$$

Note that $p_1(z) = p(z)$ and $p_0(z) = a_n \cdot z^n$. Since $a_n \neq 0$ there exists an $R > 0$ such that

$$|a_n \cdot z^n| \;>\; |\underbrace{a_{n-1} \cdot z^{n-1} + \cdots + a_1 \cdot z + a_0}_{=0 \text{ if } n=0}|$$

for any $z \in \mathbb{C}$ with $|z| = R$. This inequality implies that for any $t \in [0,1]$ and any $z \in \mathbb{C}$ with $|z| = R$ we have $p_t(z) \neq 0$, i.e. $p_t(z)$ has no zeros on the circle $|z| = R$.
We obtain the following sequence of path-homotopic loops:

$$\text{constant loop at } 1 \;\overset{\substack{\text{follows from the definition} \\ \downarrow}}{=}\; \gamma_0^{p(z)} \;\underset{\substack{\uparrow \\ \text{maps } \gamma_r^{p(z)} \text{ with } r \in [0,R] \text{ are defined,} \\ \text{since the polynomials } p(z) \text{ has no zeros,} \\ \text{they form a path-homotopy from } \gamma_0^{p(z)} \text{ to } \gamma_R^{p(z)}}}{\simeq_{\mathrm{p}}}\; \gamma_R^{p(z)} \;=\; \gamma_R^{p_1(z)} \;\underset{\substack{\uparrow \\ \text{maps } \gamma_R^{p_t(z)} \text{ with } t \in [0,1] \text{ are defined, since} \\ \text{polynomials } p_t(z) \text{ have no zeros on } |z|=R, \text{ and} \\ \text{and they form a path-homotopy from } \gamma_R^{p_1(z)} \text{ to } \gamma_R^{p_0(z)}}}{\simeq_{\mathrm{p}}}\; \gamma_R^{p_0(z)} \;\overset{\substack{\text{follows from } p_0(z) = a_n z^n \\ \downarrow}}{=}\; \text{the loop } \begin{array}{l} [0,1] \to S^1 \\ t \mapsto \exp(2\pi \mathrm{i} n t). \end{array}$$

This shows that the constant loop is path-homotopic to the loop on the right hand side. By the Circle-π_1-Theorem 11.13 this is only possible if $n = 0$. ■

12.2. The 2-dimensional Borsuk-Ulam Theorem.

Theorem 12.2. (Borsuk-Ulam Theorem) For every continuous map $f \colon S^2 \to \mathbb{R}^2$ there exists a pair of antipodal points x and $-x$ in S^2 with $f(x) = f(-x)$.

Example. It follows from the Borsuk-Ulam Theorem 12.2 that there exists a pair of antipodal points on earth with the same temperature and the same barometric pressure.

Proof. We prove the theorem by contradiction. So suppose there exists a continuous map $f \colon S^2 \to \mathbb{R}^2$ such that $f(x) \neq f(-x)$ for all $x \in S^2$. This allows us to define the map

$$\begin{array}{rcl} g \colon S^2 & \to & S^1 \\ x & \mapsto & \frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}. \end{array}$$

Note that $g(-x) = -g(x)$ for all $x \in S^2$. Now let η be the loop that is defined by

$$\begin{array}{rcl} \eta\colon [0,1] & \to & S^2 \\ s & \mapsto & \eta(s) = (\cos(2\pi s), \sin(2\pi s), 0) \end{array}$$

and let $h = g \circ \eta\colon [0,1] \to S^1$ be the composition of η with g. The argument now breaks up into two parts:

(1) By the Sphere-π_1-Proposition 11.14 the loop $\eta\colon [0,1] \to S^2$ is null-homotopic in S^2. Note that this implies that $h = g \circ \eta$ is null-homotopic in S^1.

(2) As on page 119 we consider the map $\Xi\colon \mathbb{R} \to S^1$ that is given by $t \mapsto \exp(\mathrm{i}t)$. We note that for any $z, w \in \mathbb{R}$ we have

$$(*) \qquad \Xi(z) = -\Xi(w) \iff z = w + (2q+1)\cdot\pi \quad \text{for some } q \in \mathbb{Z}.$$

We pick a $c \in \mathbb{R}$ such that $h(0) = \Xi(c)$. By the Lift-from-S^1-to-$\mathbb{R}$ Lemma 11.12 we can lift the path $h\colon [0,1] \to S^1$ to a path $\widetilde{h}\colon [0,1] \to \mathbb{R}$ with $\widetilde{h}(0) = c$. We make the following observations:

(a) It follows immediately from the definition of g that $g(-x) = -g(x)$ for all $x \in S^2$.

(b) It follows from (a) that $g(\eta(s+\frac{1}{2})) = -g(\eta(s))$ for all $s \in [0, \frac{1}{2}]$.

(c) It follows from (b) and the fact that $\Xi \circ \widetilde{h} = h = g \circ \eta$ that we have the equality $\Xi(\widetilde{h}(s+\frac{1}{2})) = -\Xi(\widetilde{h}(s))$ for all $s \in [0, \frac{1}{2}]$.

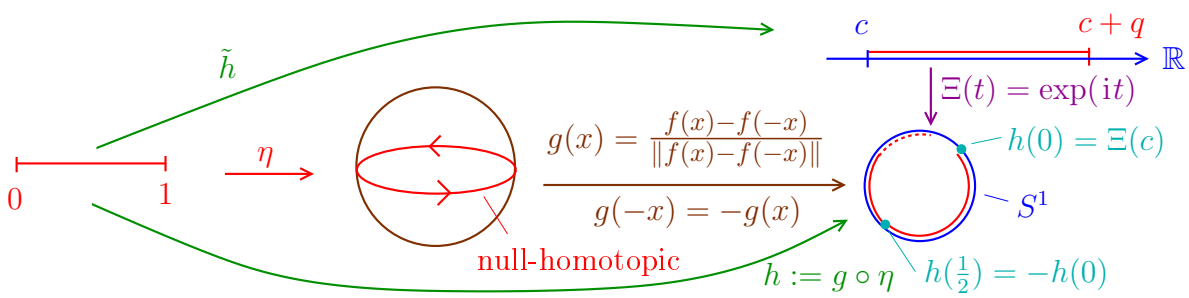
(d) It follows from (c) and $(*)$ that for each $s \in [0, \frac{1}{2}]$ there exists a $q(s) \in \mathbb{Z}$ with $\widetilde{h}(s+\frac{1}{2}) = \widetilde{h}(s) + (2q(s)+1)\cdot\pi$.

(e) The map $q\colon [0, \frac{1}{2}] \to \mathbb{Z}$ is the difference of two continuous maps, hence continuous. It follows from the Connected-to-Discrete Lemma 2.18 that it is constant. We denote this constant value by q.

It now follows that

$$\widetilde{h}(1) \underset{\substack{\uparrow \\ \text{by (e)}}}{=} \widetilde{h}\big(\tfrac{1}{2}\big) + (2q{+}1){\cdot}\pi \underset{\substack{\uparrow \\ \text{by (e)}}}{=} \widetilde{h}(0) + (2q{+}1){\cdot}\pi + (2q{+}1){\cdot}\pi = \widetilde{h}(0) + (4q{+}2){\cdot}\pi \underset{\substack{\uparrow \\ \text{since } q \in \mathbb{Z} \text{ we see that } 4q+2 \neq 0}}{\neq} \widetilde{h}(0).$$

Since $\widetilde{h}(1) \neq \widetilde{h}(0)$ we obtain the loop $h = g \circ \eta\colon [0,1] \to S^1$ is not null-homotopic. But this contradicts (1). ■



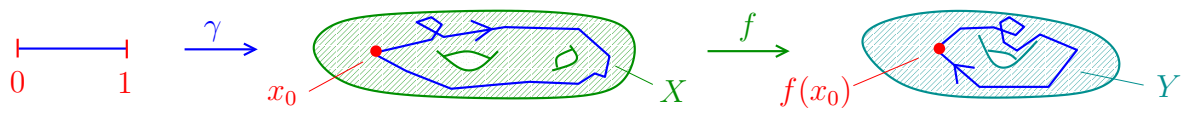
13. Functoriality of fundamental groups

13.1. The fundamental group as functor.

Definition. Let $f\colon X \to Y$ be a continuous map between topological spaces and let $x_0 \in X$ be a base point. It follows from the Homotopy Composition Lemma 8.4 that

$$\begin{array}{rcl} f_*\colon \pi_1(X,x_0) & \to & \pi_1(Y,f(x_0)) \\ [\gamma] & \mapsto & [f\circ\gamma] \end{array}$$

is a well-defined map We call f_* the induced map.



Definition. A pointed topological space is a pair (X,x_0), where X is a topological space and x_0 is a point in X. We refer to the category $\mathcal{PTop}$ with

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{PTop}) & := & \text{all pointed topological spaces,} \\ \mathrm{Mor}((X,x_0),(Y,y_0)) & := & \text{all continuous maps } f \text{ from } X \text{ to } Y \text{ with } f(x_0)=y_0 \end{array}$$

with the usual composition of maps as the category of pointed topological spaces.

Proposition 13.1. (π_1-Functor Proposition)

(1) Let $\mathcal{PTop}$ be the category of pointed topological spaces and let $\mathcal{Group}$ be the category of groups. The map

$$(X,x_0) \mapsto \pi_1(X,x_0)$$

and the maps

$$(f\colon (X,x_0)\to(Y,y_0)) \mapsto \left(\begin{array}{rcl} f_*\colon \pi_1(X,x_0) & \to & \pi_1(Y,y_0) \\ [\gamma] & \mapsto & [f\circ\gamma] \end{array}\right)$$

form a covariant functor.

(2) If $f\colon X\to Y$ is a homeomorphism between two topological spaces, then for any $x_0\in X$ the induced map $f_*\colon \pi_1(X,x_0)\to\pi_1(Y,f(x_0))$ is a group isomorphism.

Proof. The proposition follows easily from the definitions. ■

Proposition 13.2. (Change-of-Base Point Proposition) Let $f\colon X\to Y$ be a continuous map between topological spaces and let $x_0\in X$.

(1) If $\alpha\colon [0,1]\to X$ is a path from x_0 to some point x_1, then the following diagram commutes

$$\begin{tikzcd} \pi_1(X,x_1) \arrow[r, "f_*"] \arrow[d, "\alpha_*", "\cong"'] & \pi_1(Y,f(x_1)) \arrow[d, "(f\circ\alpha)_*", "\cong"'] \\ \pi_1(X,x_0) \arrow[r, "f_*"] & \pi_1(Y,f(x_0)). \end{tikzcd}$$

(The vertical maps are the isomorphisms that we introduced in the Change-of-Base Point Proposition 11.6 (1).)

(2) If $f_*\colon \pi_1(X,x_0)\to\pi_1(Y,f(x_0))$ is an isomorphism (epimorphism, monomorphism) for some base point $x_0\in X$, then for any other base point x_1 in the same path component of X the map $f_*\colon \pi_1(X,x_1)\to\pi_1(Y,f(x_1))$ is also an isomorphism (epimorphism, monomorphism).

Proof.

(1) This statement follows immediately from the definitions.

(2) This statement follows from (1) and the fact that by the Change-of-Base Point Proposition 11.6 (1) the vertical maps are isomorphisms. ■

13.2. Retractions and fundamental groups.

Proposition 13.3. (Circle–Non Retract Proposition) The circle S^1 is not a retract of $\overline{B}^2$.

Proof. Let $i\colon S^1 \to \overline{B}^2$ be the inclusion map. We now suppose there exists a retraction $r\colon \overline{B}^2 \to S^1$. By definition of a retraction this means that $r \circ i = \mathrm{id}_{S^1}$. We obtain the following commutative diagram of continuous maps between topological spaces

$$\begin{tikzcd} & \overline{B}^2 \arrow[dr, "r"] & \\ S^1 \arrow[ur, "i"] \arrow[rr, "r\circ i = \mathrm{id}_{S^1}"'] & & S^1 \end{tikzcd} \quad \text{which induces} \quad \begin{tikzcd} & \pi_1(\overline{B}^2, 1) \arrow[dr, "r_*"] & \\ \pi_1(S^1, 1) \arrow[ur, "i_*"] \arrow[rr, "(r\circ i)_* = (\mathrm{id}_{S^1})_*"'] & & \pi_1(S^1, 1). \end{tikzcd}$$

It follows from the functoriality of fundamental groups that $r_* \circ i_* = (r \circ i)_*$, i.e. the diagram on the right-hand side commutes. By the functoriality we also know that $(\mathrm{id}_{S^1})_*$ is the identity of $\pi_1(S^1, 1)$.

We saw on page 118 that $\pi_1(\overline{B}^2, 1)$ is the trivial group whereas we saw in the Circle-π_1-Theorem 11.13 that $\pi_1(S^1, 1)$ is not the trivial group. The lower horizontal map in the second diagram is thus an isomorphism of a non-trivial group, but the upper map factors through the trivial group, i.e. the composition of i_* and r_* cannot be an isomorphism of a non-trivial group. Thus we have obtained a contradiction. ■

13.3. The fundamental groups of homotopy equivalent topological spaces.

Proposition 13.4. (Homotopy-π_1-Proposition)

(1) Let $f, g\colon X \to Y$ be two continuous maps between topological spaces and let $x_0 \in X$. If f and g are homotopic, then there exists a path $\alpha\colon [0,1] \to Y$ from $f(x_0)$ to $g(x_0)$ such that the following diagram commutes:

$$\begin{tikzcd} & \pi_1(Y, g(x_0)) \arrow[d, "\alpha_*"', "{[\gamma] \mapsto [\alpha * \gamma * \overline{\alpha}]}"] \\ \pi_1(X, x_0) \arrow[ur, "g_*"] \arrow[r, "f_*"'] & \pi_1(Y, f(x_0)). \end{tikzcd}$$

In particular

$$\begin{array}{c} f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0)) \\ \text{is an isomorphism} \end{array} \iff \begin{array}{c} g_*\colon \pi_1(X, x_0) \to \pi_1(Y, g(x_0)) \\ \text{is an isomorphism.} \end{array}$$

(2) (a) Let $f\colon X \to Y$ be a continuous map between topological spaces and let $x_0 \in X$. If f is a homotopy equivalence, then the induced map

$$f_*\colon \pi_1(X, x_0) \;\to\; \pi_1(Y, f(x_0))$$

is an isomorphism.

(b) Let X be a topological space and let $A \subset X$ be a subspace. We denote by $i\colon A \to X$ the inclusion. If A is a deformation retract of X, then for any $a_0 \in A$ the induced map
$$i_*\colon \pi_1(A, a_0) \;\to\; \pi_1(X, a_0)$$
is an isomorphism.

(3) If X is a contractible topological space, then for any $x_0 \in X$ we have $\pi_1(X, x_0) = 0$.

Example. For any $n \geq 2$ and any $Q \in \mathbb{R}^n$ we have

$$\pi_1(\mathbb{R}^n \setminus \{Q\}) \underset{\substack{\uparrow \\ \text{follows from the } \pi_1\text{-Functor Proposition 13.1 (2)} \\ \text{since } \mathbb{R}^n \setminus \{Q\} \text{ is homeomorphic to } \mathbb{R}^n \setminus \{0\}}}{\cong} \pi_1(\mathbb{R}^n \setminus \{0\}) \overset{\substack{\text{on page 107 we saw that } S^{n-1} \text{ is a deformation retract of } \mathbb{R}^n \setminus \{0\}, \text{ thus we get an isomorphism} \\ \text{from the Homotopy-}\pi_1\text{-Proposition 13.4 (2b)} \\ \downarrow}}{\cong} \pi_1(S^{n-1}) \underset{\substack{\uparrow \\ \text{by the Circle-}\pi_1\text{-Theorem 11.13 and the} \\ \text{Sphere-}\pi_1\text{-Proposition 11.14}}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } n = 2, \\ 0, & \text{if } n \geq 3. \end{cases}$$

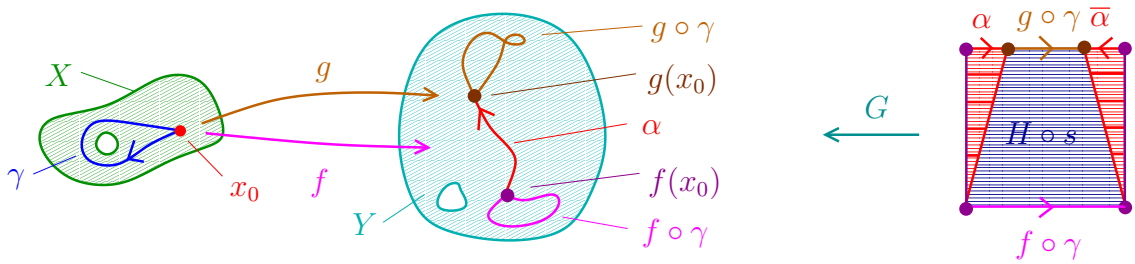
Proof.

(1) Let $f, g\colon X \to Y$ be two continuous maps between topological spaces, let $x_0 \in X$ and let $H\colon X \times [0,1] \to Y$ be a homotopy between f and g. Let
$$\begin{aligned} \alpha\colon [0,1] &\to Y \\ t &\mapsto H(x_0, t) \end{aligned}$$
be the path with which the homotopy H connects the two points $H(x_0, 0) = f(x_0)$ and $H(x_0, 1) = g(x_0)$.

Claim. The maps
$$\begin{aligned} f_*\colon \pi_1(X, x_0) &\to \pi_1(Y, f(x_0)) \\ [\gamma] &\mapsto [f \circ \gamma] \end{aligned} \qquad \text{and} \qquad \begin{aligned} \pi_1(X, x_0) &\to \pi_1(Y, f(x_0)) \\ [\gamma] &\mapsto [\alpha * (g \circ \gamma) * \overline{\alpha}] \end{aligned}$$
are identical.

Proof. Let $\gamma\colon [0,1] \to X$ be a loop in (X, x_0). We need to show that the loops $f \circ \gamma$ and $\alpha * (g \circ \gamma) * \overline{\alpha}$ are path-homotopic. We consider the map
$$\begin{aligned} G\colon [0,1] \times [0,1] &\to Y \\ (r,t) &\mapsto \begin{cases} \alpha(4r), & \text{if } r \in \left[0, \frac{1}{4}t\right], \\ H\left(\gamma\left(\frac{r - \frac{1}{4}t}{1 - \frac{1}{2}t}\right), t\right), & \text{if } r \in [\frac{1}{4}t, 1 - \frac{1}{4}t], \\ \overline{\alpha}(4r - 3), & \text{if } r \in \left[1 - \frac{1}{4}t, 1\right]. \end{cases} \end{aligned}$$
Using the Pasting Proposition 2.4 (2) it is straightforward to verify that G is indeed a continuous map. Furthermore it follows easily from the definition that the map G satisfies $G_0(t) = G_1(t) = f(x_0)$ for all $t \in [0,1]$ and that it satisfies
$$G(r, 0) \;=\; (f \circ \gamma)(r) \quad \text{and} \quad G(r, 1) \;=\; (\alpha * (g \circ \gamma) * \overline{\alpha})(r) \quad \text{for all } r \in [0,1].$$

This observation shows that G is in fact a path-homotopy between the loops $f \circ \gamma$ and $\alpha * (g \circ \gamma) * \overline{\alpha}$ in $(Y, f(x_0))$. ⊞



As in the statement of the proposition we consider the following diagram:

$$\begin{tikzcd} & \pi_1(Y, g(x_0)) \arrow[d, "\alpha_*"', "{[\gamma] \mapsto [\alpha * \gamma * \overline{\alpha}]}"] \\ \pi_1(X, x_0) \arrow[ur, "g_*"] \arrow[r, "f_*"'] & \pi_1(Y, f(x_0)). \end{tikzcd}$$

We make the following observations:

(i) It follows from the claim that the diagram commutes.

(ii) By the Change-of-Base Point Proposition 11.6 (1) we know that the vertical map is an isomorphism.

It follows from (i) and (ii) that f_* is an isomorphism if and only if g_* is an isomorphism.

(2) Let $f\colon X \to Y$ be a continuous map between topological spaces which is a homotopy equivalence. Let $x_0 \in X$ be a point. We pick a homotopy inverse $g\colon Y \to X$ of f. By definition of a homotopy inverse this means that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

Next note that it follows from (1) and the functoriality of fundamental groups that the induced maps

$$\begin{aligned} \alpha := g_* \circ f_* = (g \circ f)_* \colon \pi_1(X, x_0) &\to \pi_1(X, g(f(x_0)) \\ \beta := f_* \circ g_* = (f \circ g)_* \colon \pi_1(Y, f(x_0)) &\to \pi_1(Y, f(g(f(x_0)))) \end{aligned}$$

are isomorphisms. This shows that f_* has the left inverse $\alpha^{-1} \circ g_*$ and it has the right inverse $g_* \circ \beta^{-1}$. It follows from the Inverse Uniqueness Lemma 3.2 (1) that f_* is an isomorphism. Similarly we see that g_* is an isomorphism.

(3) This statement is just a frequently used special case of the second statement. ■

Lemma 13.5. ($\mathbb{R}^2$-$\mathbb{R}^3$-Non Homeomorphism–Lemma) The topological spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ are not homeomorphic.

Proof. Let us suppose that there exists a homeomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^3$. Let $P \in \mathbb{R}^2$ be a point. We put $Q := f(P)$. Note that f restricts to a homeomorphism from $\mathbb{R}^2 \setminus \{P\}$ to $\mathbb{R}^3 \setminus \{Q\}$. But then we obtain that

$$\mathbb{Z} \underset{\substack{\uparrow \\ \text{page 129}}}{\cong} \pi_1(\mathbb{R}^2 \setminus \{P\}) \underset{\substack{\uparrow \\ \text{since } f \text{ is a homeomorphism}}}{\cong} \pi_1(\mathbb{R}^3 \setminus \{Q\}) \underset{\substack{\uparrow \\ \text{page 129}}}{=} 0.$$

Thus we obtained a contradiction. ■

Corollary 13.6. (Möbius Band-π_1-Corollary) We have

$$\pi_1(\text{Möbius band }) \cong \mathbb{Z}.$$

Proof. On page 109 we saw that the Möbius band admits a deformation retraction onto the "central circle" C. Thus we see that

$$\pi_1(\text{Möbius band }) \underset{\substack{\uparrow \\ \text{Homotopy-}\pi_1\text{-Proposition 13.4}}}{\cong} \pi_1(C) \cong \pi_1(S^1) \underset{\substack{\uparrow \\ \text{Circle-}\pi_1\text{-Theorem 11.13}}}{\cong} \mathbb{Z}.$$

■

13.4. The fundamental group of the product of two topological spaces.

Proposition 13.7. (Product-π_1-Proposition) Let $n \in \mathbb{N}$. Let $(X_i, x_i)_{i\in I}$ be a family of pointed topological spaces. For each $j \in I$ we denote by $p_j \colon \prod_{i\in I} X_i \to X_j$ the natural projection. The induced natural homomorphism

$$\begin{aligned} \Phi \colon \pi_1\Big(\prod_{i\in I} X_i, (x_i)_{i\in I}\Big) &\to \prod_{i\in I} \pi_1(X_i, x_i) \\ [f] &\mapsto (p_{i*}([f]))_{i\in I} \end{aligned}$$

is an isomorphism.

Example. For $n \in \mathbb{N}$ we obtain from the Product-π_1-Proposition 13.7 together with the Circle-π_1-Theorem 11.13 that $\pi_1((S^1)^n) \cong \mathbb{Z}^n$.

Proof. First we introduce some general notation:

- Given two pairs of topological spaces (W, A) and (X, B) we denote by $C((W, A), (X, B))$ the set of all continuous maps $f \colon W \to X$ with $f(A) \subset B$.
- We say two continuous maps $f, g \in C((W, A), (X, B))$ are *homotopic* if there exists a homotopy $H \colon W \times [0, 1] \to X$ with $H_0 = f$ and $H_1 = g$ such that for all $t \in [0, 1]$ we have $H_t(A) \subset B$.
- We denote by $[(W, A), (X, B)]$ the set of all homotopy classes of continuous maps $(W, A) \to (X, B)$.
- By definition we have $\pi_1(Z, z_0) = [([0, 1], \{0, 1\}), (Z, \{z_0\})]$ for any pointed topological space (Z, z_0).

Let $(X_i, x_i)_{i\in I}$ be a family of pointed topological spaces. We set $x := (x_i)_{i\in I}$. It follows from the Topological-Product Proposition 5.1 (1) that the map

$$\begin{aligned} C\Big(([0, 1], \{0, 1\}), \Big(\prod_{i\in I} X_i, x\Big)\Big) &\to \prod_{i\in I} C(([0, 1], \{0, 1\}), (X_i, x_i)) \\ f &\mapsto (p_i \circ f)_{i\in I} \end{aligned}$$

is a bijection. The analogous statement holds if we replace the pair $([0, 1], \{0, 1\})$ by the pair $([0, 1] \times [0, 1], \{0, 1\} \times [0, 1])$. This shows that the map

$$\begin{aligned} \pi_1\Big(\prod_{i\in I} X_i, x\Big) = \Big[([0, 1], \{0, 1\}), \Big(\prod_{i\in I} X_i, x\Big)\Big] &\to \prod_{i\in I} [([0, 1], \{0, 1\}), (X_i, x_i)] = \prod_{i\in I} \pi_1(X_i, x_i) \\ [f] &\mapsto \underbrace{[(p_i \circ f)_{i\in I}]}_{=p_{i*}(f)} \end{aligned}$$

is a bijection. But this is precisely the homomorphism that we are considering. ■

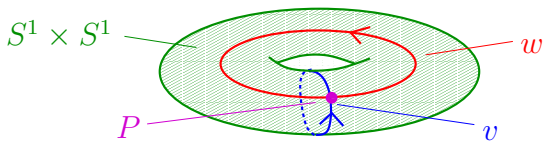
Corollary 13.8. (Torus-π_1-Corollary) Let $n \in \mathbb{N}$. We consider the n-dimensional torus $(S^1)^n = S^1 \times \cdots \times S^1$. Given $j \in \{1, \ldots, n\}$ we consider the loop

$$\begin{array}{rcl} v_j \colon [0,1] & \to & S^1 \times \cdots \times S^1 \\ t & \mapsto & (1, \ldots, 1, \underbrace{\exp(2\pi \mathrm{i} \cdot t)}, 1, \ldots, 1). \end{array}$$

The map

$$\begin{array}{rcl} \mathbb{Z}^n & \mapsto & \pi_1(S^1 \times \cdots \times S^1, (1, \ldots, 1)) \\ (r_1, \ldots, r_n) & \mapsto & [v_1]^{r_1} \cdot \ldots \cdot [v_n]^{r_n} \end{array}$$

is an isomorphism.



the map

$$\begin{array}{rcl} \mathbb{Z}^2 & \to & \pi_1(S^1 \times S^1, P) \\ (r, s) & \mapsto & [v]^r \cdot [w]^s \end{array}$$

is an isomorphism

Proof. The statement follows immediately from the Product-π_1-Proposition 13.7 and the explicit isomorphism $\Psi \colon \mathbb{Z} \to \pi_1(S^1, 1)$ given in the Circle-π_1-Theorem 11.13. ■

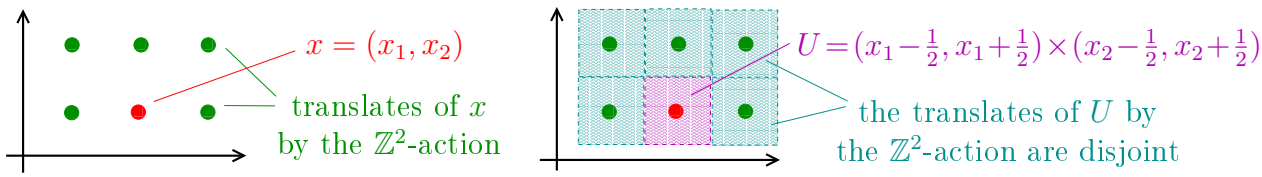
14. Coverings and fundamental groups

14.1. Group actions.

Definition. Let G be a group that acts continuously on a topological space X. We say G acts discretely, if for each $x \in X$ there exists an open neighborhood U of x so that
$$U \cap g \cdot U = \varnothing \quad \text{for all } g \neq e.$$

Examples.

(1) The group $G = \mathbb{Z}^n$ acts on $X = \mathbb{R}^n$ by addition. This action is discrete. Indeed, for any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ the open subset $U := (x_1 - \frac{1}{2}, x_1 + \frac{1}{2}) \times \cdots \times (x_n - \frac{1}{2}, x_n + \frac{1}{2})$ has the desired property.



(2) The action
$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) & \to & \mathbb{R} \times [-1,1] \\ (n,(x,y)) & \mapsto & (x+n, (-1)^n \cdot y) \end{array}$$
of the group $\mathbb{Z}$ on the topological space $\mathbb{R} \times [-1,1]$ is discrete. Indeed, for any (x,y) in X the open neighborhood $U := (x - \frac{1}{2}, x + \frac{1}{2}) \times [-1,1]$ has the desired property.

(3) The map
$$\begin{array}{rcl} \{\pm 1\} \times S^n & \to & S^n \\ (\epsilon, P) & \mapsto & \epsilon \cdot P \end{array}$$
defines an action of the group $\{\pm 1\}$ on S^n that is discrete. Indeed, for any $P \in S^n$ the open hemisphere $U := \{Q \in S^n \mid \underbrace{P \cdot Q}_{\text{scalar product}} > 0\}$ has the desired property.

Lemma 14.1. (Klein Bottle-as-Quotient Lemma) We consider the following two self-homeomorphisms of $\mathbb{R}^2$:
$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x, y+1). \end{array}$$
We denote by G the subgroup of all homeomorphisms of $\mathbb{R}^2$ that is generated by A and B. This means that
$$G := \begin{array}{l} \text{all self-homeomorphisms of } \mathbb{R}^2 \text{ that can be written as} \\ \text{a finite composition of the maps } A, B, A^{-1} \text{ and } B^{-1}. \end{array}$$
The group G acts by definition on $\mathbb{R}^2$.

(1) The elements A and B of G do not commute, in particular the group G is non-abelian.
(2) The action of G on $\mathbb{R}^2$ is proper and discrete.
(3) We consider the Klein bottle $K = ([0,1] \times [0,1])/\sim$ as defined on page 82. The map
$$\begin{array}{rcl} f\colon K = ([0,1] \times [0,1])/\sim & \to & \mathbb{R}^2/G \\ [(x,y)] & \mapsto & [(x,y)] \end{array}$$
is a homeomorphism.

Proof. In the group G we use the product symbol "$\cdot$" to indicate the group structure instead of the "$\circ$" symbol.

(1) We calculate that

$$(A \cdot B)(0,0) \;=\; A(0,1) \;=\; (1,0) \;\neq\; (1,2) \;=\; B(1,1) \;=\; (B \cdot A)(0,0).$$

This shows that A and B do not commute in G. This shows that G is not abelian.

(2) We start out with the following claim.

Claim 1. Given $\epsilon, \eta \in \{-1, 1\}$ we have $A^\epsilon \cdot B^\eta = B^{-\eta} \cdot A^\epsilon$.

Proof. First let $\epsilon = 1$. We perform the following calculation:

$$\begin{aligned}(A \cdot B)(x,y) &= A(x, y+1) = (x+1, -y) &&= B^{-1}(x+1, 1-y) = (B^{-1} \cdot A)(x,y),\\ (A \cdot B^{-1})(x,y) &= A(x, y-1) = (x+1, 2-y) &&= B(x+1, 1-y) = (B \cdot A)(x,y).\end{aligned}$$

We just showed that $A \cdot B^\eta = B^{-\eta} \cdot A$. By taking inverses on both sides we get the promised equalities $A^{-1} \cdot B^\eta = B^{-\eta} \cdot A^{-1}$ for $\epsilon = -1$. ⊞

Claim 2.

(a) Any $g \in G$ is of the form $A^k \cdot B^l$ for some $k, l \in \mathbb{Z}$.

(b) For $k, l \in \mathbb{Z}$ we have

$$(A^k \cdot B^l)(x,y) \;=\; \begin{cases} (x+k, y+l), & \text{if } k \text{ is even,}\\ (x+k, 1-y-l), & \text{if } k \text{ is odd.}\end{cases}$$

Proof.

(a) Let $g \in G$. By definition we can write g as a product of powers of A and B. Using the equality $A^\epsilon \cdot B^\eta = B^{-\eta} \cdot A^\epsilon$ from Claim 1 we can swap the order of powers of A with powers of B, at the price of changing the exponent of B. This means that we can bring all $A^{\pm 1}$ to the left and get the desired result.

(b) This statement follows from an elementary straightforward calculation. ⊞

Claim 3. The group G acts properly on $\mathbb{R}^2$.

Proof. Let $P, Q \in \mathbb{R}^2$. We set $U := B^2_{\frac{1}{2}}(P)$ and $V := B^2_{\frac{1}{2}}(Q)$. It follows easily from Claim 2 (a) and (b) that the set $\{g \in G \,|\, g \cdot U \cap V \neq \varnothing\}$ is finite. ⊞

Claim 4. The group G acts discretely on $\mathbb{R}^2$.

Proof. Let $(x,y) \in \mathbb{R}^2$. We set $U := (x - \frac{1}{2}, x + \frac{1}{2}) \times (y - \frac{1}{2}, y + \frac{1}{2})$. It follows easily from Claim 2 (a) and (b) that for any non-trivial $g \in G$ we have $U \cap g \cdot U = \varnothing$. ⊞

(3) Using Claim 2 from (2) one can easily show that the map $f\colon ([0,1] \times [0,1])/\sim \to \mathbb{R}^2/G$ is a bijection. It follows from the Twice Quotient Lemma 6.3 that f is continuous. By the Quotient Topology-Properties Proposition 6.2 (1) we know that $([0,1] \times [0,1])/\sim$ is compact. By (2) we know that the action of G is proper. Thus it follows from the Group Action-Hausdorff Quotient Proposition 6.7 that $\mathbb{R}^2/\sim$ is Hausdorff. In summary it follows from the Compact-Hausdorff Proposition 2.10 (3) that f is a homeomorphism. ■

Lemma 14.2. (Lens Space-Discrete Action Lemma) We consider the 3-dimensional sphere $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\}$. Let $p \in \mathbb{Z} \setminus \{0\}$ and let $q \in \mathbb{Z}$ with $\gcd(p,q) = 1$. We set $\xi := \exp(2\pi \mathrm{i}/p)$.

(1) The map

$$\begin{aligned}\mathbb{Z}_p \times S^3 &\to S^3\\ (k + p\mathbb{Z}, (z_1, z_2)) &\mapsto (\xi^k \cdot z_1, \xi^{q \cdot k} \cdot z_2)\end{aligned}$$

defines a smooth, orientation-preserving and discrete action of $\mathbb{Z}_p$ on S^3.

(2) The quotient space $S^3/\mathbb{Z}_p$ is a closed 3-dimensional topological manifold that has a unique structure of an oriented smooth manifold such that the projection $S^3 \to S^3/\mathbb{Z}_p$ is a local diffeomorphism and orientation-preserving.

Proof.

(1) It is straightforward to verify that the map $\mathbb{Z}_p \times S^3 \to S^3$ does indeed define an action of the group $\mathbb{Z}_p$ on the topological space S^3. Evidently the action is smooth and a direct calculation shows that the action is orientation-preserving.

Claim. The action is discrete.

Proof. Let $(z_1, z_2) \in S^3$.

(a) If $z_1 \neq 0$, then we set $U := \{(u_1, v_1) \in S^3 \,|\, u_1 = z_1 \cdot \exp(2\pi \mathrm{i} t) \text{ with } t \in (-\frac{1}{2p}, \frac{1}{2p})\}$.

(b) If $z_1 = 0$, then we set $U := \{(u_1, v_1) \in S^3 \,|\, u_2 = z_2 \cdot \exp(2\pi \mathrm{i} t) \text{ with } t \in (-\frac{1}{2p}, \frac{1}{2p})\}$.

One can easily verify that in both cases U is an open neighborhood of (z_1, z_2). Next note that $\xi^k = 1$ if and only if $k \in p \cdot \mathbb{Z}$ and note that our hypothesis $\gcd(p, q) = 1$ implies that $\xi^{k \cdot q} = 1$ if and only if $k \in p \cdot \mathbb{Z}$. Using these two observations one can easily show that all non-trivial translates of U are disjoint. ⊞

(2) We refer to [Fri23] for details. ■

Definition. Let $p \in \mathbb{Z} \setminus \{0\}$ and let $q \in \mathbb{Z}$ with $\gcd(p, q) = 1$. We refer to

$$L(p, q) \;:=\; S^3/\mathbb{Z}_p,$$

with the action from the previous lemma, as a lens space.

Lemma 14.3. (Lens Spaces-Diffeomorphisms Lemma) Let $p \in \mathbb{N}$ and furthermore let $q, r \in \mathbb{Z}$ with $\gcd(p, q) = \gcd(p, r) = 1$. If $q \equiv \pm r^{\pm 1} \bmod p$, then there exists a diffeomorphism $L(p, q) \to L(p, r)$.

Proof. Let $p \in \mathbb{N}$ and let $q, r \in \mathbb{Z}$ with $\gcd(p, q) = \gcd(p, r) = 1$. We pick $s \in \mathbb{Z}$ with $r \cdot s \equiv 1 \bmod p$. We suppose that $q \equiv \pm r \bmod p$ or that $q \equiv \pm s \bmod p$.

Claim. The map

$$\begin{array}{rcl} \Phi\colon S^3 & \to & S^3 \\ (z_1, z_2) & \mapsto & \begin{cases} (z_1, z_2), & \text{if } q \equiv r \bmod p, \\ (z_1, \overline{z_2}), & \text{if } q \equiv -r \bmod p, \\ (z_2, z_1), & \text{if } q \equiv s \bmod p, \\ (z_2, \overline{z_1}), & \text{if } q \equiv -s \bmod p \end{cases} \end{array}$$

descends to a well-defined map $\overline{\Phi}\colon L(p, q) \to L(p, r)$ and this map is a bijection.

Proof. The details are worked out in [Fri23]. ⊞

One can now easily verify that the induced map $\overline{\Phi}\colon L(p, q) \to L(p, r)$ is a diffeomorphism. We refer to [Fri23] for full details. ■

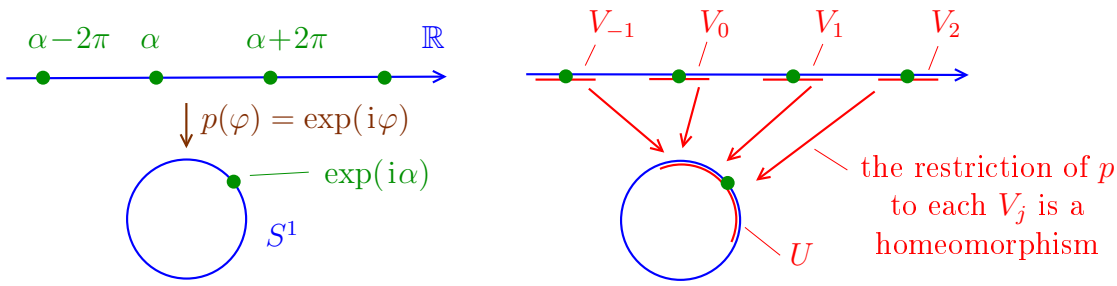
14.2. Coverings of topological spaces.

Definition. Let $p\colon X \to B$ be a continuous map between topological spaces.

(1) An open subset $U \subset B$ is called uniformly covered, if $p^{-1}(U) \subset X$ is the union of disjoint open subsets $\{V_i\}_{i \in I}$ of X with the property such that for each $i \in I$ the map $p|_{V_i}\colon V_i \to U$ is a homeomorphism.

(2) We say the map $p\colon X \to B$ is a covering, if p is surjective and if for every $b \in B$ there exists an open neighborhood U of b which is uniformly covered.
(3) Sometimes we call X a covering space of B.

Example. The map
$$\begin{array}{rcl} p\colon \mathbb{R} & \to & S^1 \\ \varphi & \mapsto & \exp(\mathrm{i}\varphi) \end{array}$$
is a covering. Indeed, let $P = \exp(\mathrm{i}\alpha)$ be a point in S^1. We pick the open neighborhood
$$U := \left\{ \exp(\mathrm{i}\varphi) \,\middle|\, \varphi \in \left(\alpha - \tfrac{\pi}{4}, \alpha + \tfrac{\pi}{4}\right) \right\}.$$
Then
$$p^{-1}(U) = \bigsqcup_{j\in\mathbb{Z}} \underbrace{\left(\alpha - \tfrac{\pi}{4} + 2\pi j, \alpha + \tfrac{\pi}{4} + 2\pi j\right)}_{=:V_j},$$
and for each $j \in \mathbb{Z}$ the restriction of p to $V_j \to U$ is a homeomorphism.

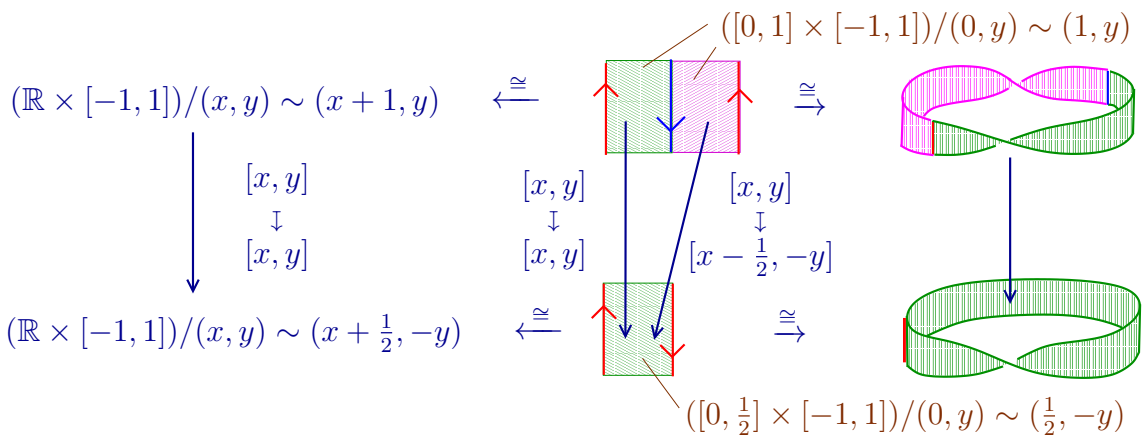


Lemma 14.4. (Covering–Preimage Lemma) Let $p\colon X \to B$ be a covering of topological spaces. If B is a connected topological space, then for any two points P and Q in B we have
$$\#p^{-1}(P) = \#p^{-1}(Q).$$

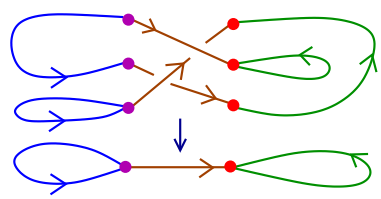
Proof. The proof of the lemma is left as an exercise to the reader. ■

Definition. Let $p\colon X \to B$ be a covering of a connected non-empty topological space.
(1) We define the degree $[X : B]$ of p as the cardinality of $\#p^{-1}(b)$ for some $b \in B$.
(2) If the degree is a finite number n, then we say that p is an n-fold covering. Otherwise we say that p is an infinite covering.

Examples.
(1) As usual we consider $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$ as a subset of $\mathbb{C}$. Given any $n \in \mathbb{N}$ the map $p\colon S^1 \to S^1$ given by $p(z) := z^n$ is an n-fold covering.
(2) We consider the map
$$\begin{array}{rcl} p\colon \overbrace{(\mathbb{R}\times[-1,1])/(x,y)\sim(x+1,y)}^{\text{annulus}} & \to & \overbrace{(\mathbb{R}\times[-1,1])/(x,y)\sim(x+\frac{1}{2},-y)}^{\text{Möbius band}} \\ [(x,y)] & \mapsto & [(x,y)]. \end{array}$$
It is straightforward to show that p is a 2-fold covering. Thus we can view the annulus as a 2-fold covering of the Möbius band.
(3) If in the previous example we replace the interval $[-1,1]$ by the circle $[-1,1]/-1 \sim 1$, then on the left-hand side we obtain the torus and on the right-hand side we obtain the Klein bottle. This shows that there exists a 2-fold covering from the torus to the Klein bottle.



(4) In the figure below we sketch three coverings of topological graphs. The coverings are hereby given by sending edges of a certain color, with the indicated direction, to the edge of the same color with the indicated direction.



$\widetilde{G}=(\widetilde{V},\widetilde{E},\widetilde{i},\widetilde{t})$

$G=(V,E,i,t)$

Proposition 14.5. (Action–Covering Proposition) Let X be a topological space together with a discrete and continuous action by a group G.

(1) The natural projection $p\colon X\to X/G$ is a covering where the degree is given by $\#G$.

(2) Let H be a subgroup of G. The natural projection

$$\begin{array}{rcl} p\colon X/H & \to & X/G \\ [x] & \mapsto & [x] \end{array}$$

is a covering of degree $[G:H]$.

Example. The Action–Covering Proposition 14.5 (1), applied to the examples on page 133, shows that

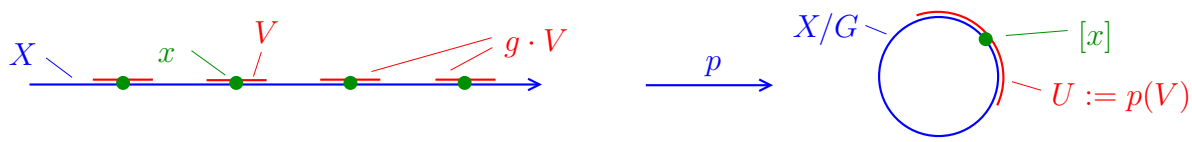
$$\begin{array}{rcl} p\colon \mathbb{R}^n & \to & \mathbb{R}^n/\mathbb{Z}^n \\ x & \mapsto & [x]=x+\mathbb{Z}^n \end{array} \quad \text{and} \quad \begin{array}{rcl} p\colon S^n & \to & \mathbb{R}\mathrm{P}^n=_i S^n/x\sim -x \\ x & \mapsto & [x] \end{array}$$

is an infinite respectively 2-fold covering.

Proof of Proposition 14.5. In the following we prove Statement (1). We leave it to the reader to generalize the proof to the more general setting of Statement (2).

Let G be a group that acts discretely and continuously on a topological space X. We want to show that the projection $p\colon X\to X/G$ is a covering. Evidently the map is surjective. Next let $[x]\in X/G$. Since the group acts discretely there exists, by definition, an open neighborhood V of $x\in X$ such that $V\cap g{\cdot}V=\varnothing$ for all $g\neq e$. We put $U:=p(V)$.

Claim. U is a uniformly covered open neighborhood of $[x]$.



Proof. Using the fact that G acts continuously one can easily show that $p\colon X \to X/G$ is an open map. Thus it follows that the set U is an open neighborhood of $[x] = p(x)$.

Next note that the preimage $p^{-1}(U)$ is the disjoint union of the subsets $g \cdot V, g \in G$. Since G acts continuously we know that each $g \cdot V$ is again open. The restriction of the map p to any $g \cdot V$ is a continuous and bijective map $g \cdot V \to U$. Since $p\colon X \to X/G$ is open and since $g \cdot V \subset X$ is open we see that the restriction of p to $g \cdot V$ is also open. It follows from the Open-Injective Lemma 2.9 that the map $p|_{g \cdot V}\colon g \cdot V \to U$ is in fact a homeomorphism. Thus we have shown that U is uniformly covered. ⊞

Since this holds for any point in X/G we have shown that the projection $p\colon X \to X/G$ is a covering. Finally note that $[X : X/G] = p^{-1}([x]) = \#G$. ■

14.3. Basic properties of coverings.

Lemma 14.6. (Covering–Basics Lemma) Given any covering $p\colon X \to B$ of topological spaces the following statements hold:

(1) The map $p\colon X \to B$ is open.
(2) For every $b \in B$ the preimage $p^{-1}(b)$ is a discrete subset of X.
(3) If B is Hausdorff, then so is X.
(4) The following two statements are equivalent:
  (a) X is compact.
  (b) p is a finite covering and B is compact.

Proof. Let $p\colon X \to B$ be a covering of topological spaces.

(1) We will not really make use of this statement, thus we refer to [Fri23] for the proof.
(2) We will not really make use of this statement, thus we refer to [Fri23] for the proof.
(3) This statement is relatively easy to prove. We leave it to the reader to fill in the details.
(4) We first prove the (a)⇒(b)-direction. Thus we assume that X is compact. By definition the covering $p\colon X \to B$ is surjective and continuous. Thus it follows from the Compact Image Lemma 2.8 that B is also compact.

It remains to show that the covering has finite degree. By definition we can cover the topological space B by open subsets that are uniformly covered. Since B is compact we can cover B by finitely many open subsets $U_1, \ldots, U_k$ that are uniformly covered. Without loss of generality we can assume that U_k is not contained in $U_1 \cup \cdots \cup U_{k-1}$. Now let P be a point in $U_k \setminus (U_1 \cup \cdots \cup U_{k-1})$. Furthermore let V_i, $i \in I$ be the disjoint open subsets of X such that $p^{-1}(U_k)$ is the union of the V_i's and such that each $p|_{V_i}\colon V_i \to U_k$ is a homeomorphism. For $i \in I$ we denote by $Q_i \in V_i$ the unique point with $p(Q_i) = P$. Now we consider the open cover

$$X \;=\; p^{-1}(U_1) \cup \ldots \cup p^{-1}(U_{k-1}) \cup \bigcup_{i \in I} V_i.$$

Note that for $i \in I$ the point Q_i is contained only in V_i. This means that we cannot remove any of the open subsets V_i, $i \in I$ from the open cover. Since X is compact we

see that I is finite. But $\#I$ is precisely the degree of the covering. Thus we have shown that the covering has finite degree.
The proof of the (b)⇒(a)-direction is provided in [Fri23]. ■

14.4. Lifts of maps.

Definition. Let $p\colon X \to B$ be a covering and let $f\colon Y \to B$ be a continuous map between topological spaces. A lift of f to X is a continuous map $\widetilde{f}\colon Y \to X$ such that $p \circ \widetilde{f} = f$, i.e. such that the following diagram of maps commutes:

$$\begin{tikzcd} & X \arrow[d, "p"] \\ Y \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & B. \end{tikzcd}$$

Proposition 14.7. (Lift Uniqueness Lemma) Let $p\colon X \to B$ be a covering and let Y be a connected topological space. Furthermore let $f\colon Y \to B$ be a continuous map and let $\widetilde{f}, \widehat{f}\colon Y \to X$ be two lifts of f. If $\widetilde{f}$ and $\widehat{f}$ agree at some point $y_0 \in Y$, then they agree everywhere.

Proof. We set

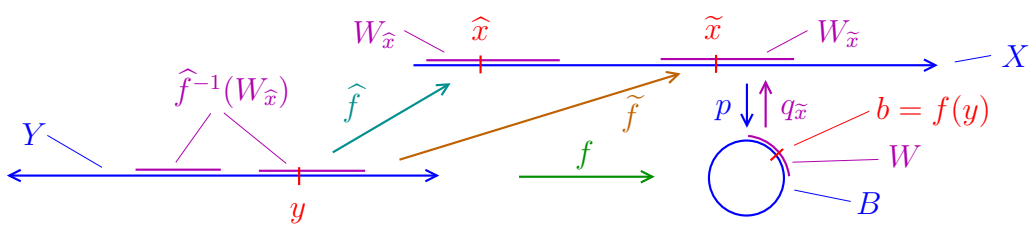
$$\begin{aligned} U &:= \{y \in Y \mid \widetilde{f}(y) = \widehat{f}(y)\} &&\subset Y \\ V &:= \{y \in Y \mid \widetilde{f}(y) \neq \widehat{f}(y)\} &&\subset Y. \end{aligned}$$

Note that by hypothesis $U \neq \varnothing$. We want to show that $U = Y$. Since Y is connected it suffices to prove that U and V are open subsets of Y. By the JH-Openness Criterion 1.1 it suffices to prove the following claim.

Claim. Every $y \in Y$ admits an open neighborhood N such that $N \subset U$ or $N \subset V$.

Proof. Let $y \in Y$. We set $b := f(y)$ and we set $\widetilde{x} := \widetilde{f}(y)$ and $\widehat{x} := \widehat{f}(y)$. Since $p\colon X \to B$ is a covering we know that b admits an open neighborhood W that is uniformly covered, i.e. we know that we can write $p^{-1}(W) = \bigsqcup_{x \in p^{-1}(b)} W_x$ such that for each $x \in p^{-1}(b)$ the following statements hold:

(1) W_x is an open neighborhood of $x \in X$.
(2) The map $p|_{W_x}\colon W_x \to W$ is a homeomorphism. We denote by $q_x\colon W \to W_x$ the inverse.



Note that on $\widetilde{f}^{-1}(W_{\widetilde{x}})$ we have

$$\widetilde{f} = \underbrace{q_{\widetilde{x}} \circ p|_{W_{\widetilde{x}}}}_{=\mathrm{id}_{W_{\widetilde{x}}}} \circ \widetilde{f} = q_{\widetilde{x}} \circ \underbrace{p|_{W_{\widetilde{x}}} \circ \widetilde{f}}_{=f} = q_{\widetilde{x}} \circ f\colon \widetilde{f}^{-1}(W_{\widetilde{x}}) \to W_{\widetilde{x}}.$$

Similarly we see that on $\widehat{f}^{-1}(W_{\widehat{x}})$ we have $\widehat{f} = q_{\widehat{x}} \circ f$. The claim follows from the following observations:

- If $y \in U$, i.e. if $\widetilde{x} = \widehat{x}$, then we see that $\widetilde{f} = \widehat{f}$ on $\widetilde{f}^{-1}(W_{\widetilde{x}}) \cap \widehat{f}^{-1}(W_{\widehat{x}})$, i.e. we see that $\widetilde{f}^{-1}(W_{\widetilde{x}}) \cap \widehat{f}^{-1}(W_{\widehat{x}}) \subset U$.
- If $y \notin U$, i.e. if $\widetilde{x} \neq \widehat{x}$, then we see that the images of $\widetilde{f}$ and $\widehat{f}$ on $\widetilde{f}^{-1}(W_{\widetilde{x}}) \cap \widehat{f}^{-1}(W_{\widehat{x}})$ are disjoint, thus we see that $\widetilde{f}^{-1}(W_{\widetilde{x}}) \cap \widehat{f}^{-1}(W_{\widehat{x}}) \subset V$. ■

Lemma 14.8. (Lift-Extension Lemma) Let $p\colon X \to B$ be a covering and let Y be a topological space. We suppose that we have a decomposition $Y = C \cup D$ such that C and D are closed subsets. Let $f\colon Y \to B$ be a continuous map and let $\widetilde{f}\colon C \to X$ be a lift of $f|_C$. We suppose that the following statements hold:

(1) The image $f(D)$ is contained in a uniformly covered open subset of B.

(2) The intersection $C \cap D$ is connected and non-empty.

Then there exists a lift $\widetilde{f}\colon Y \to X$ of f that agrees with the given lift on C.

Proof. We start with a few preparations:

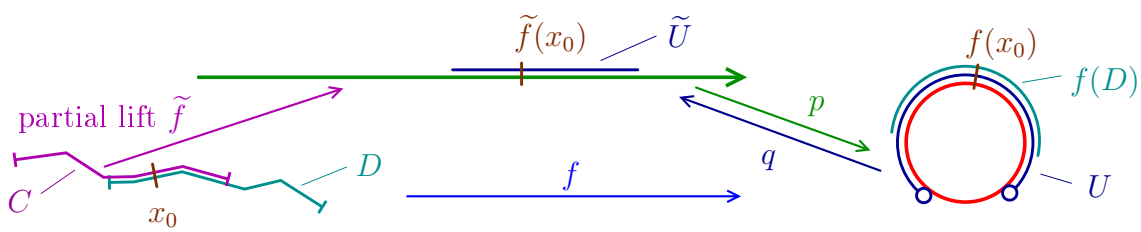
(a) Since $C \cap D = \varnothing$ we can pick $x_0 \in C \cap D$.

(b) We pick a uniformly covered open subset U of B that contains $f(D)$.

(c) Since U is uniformly covered open subset it thus follows from $p(\widetilde{f}(x_0)) = f(x_0) \in f(D)$ that there exists an open neighborhood $\widetilde{U}$ of $\widetilde{f}(x_0)$ such that $p\colon \widetilde{U} \to U$ is a homeomorphism.

(d) We denote by $q\colon U \to \widetilde{U}$ the inverse of $p\colon \widetilde{U} \to U$.

We set
$$\begin{array}{rcl} \widetilde{f}\colon Y & \to & X \\ x & \mapsto & \begin{cases} \widetilde{f}(x), & \text{if } x \in C, \\ q(\underbrace{f(x)}_{\in U}), & \text{if } x \in D. \end{cases} \end{array}$$



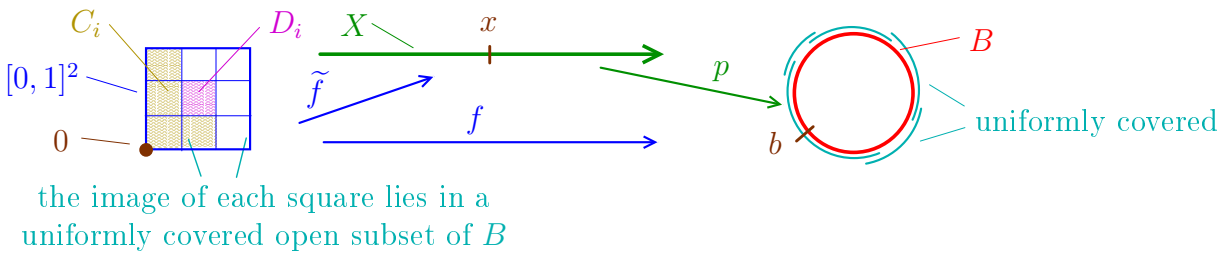
Note that on $C \cap D$ we have a priori two different definitions of $\widetilde{f}$. But both definitions are lifts of $f|_{C\cap D}$ and by design they agree on x_0. By hypothesis the intersection $C \cap D$ is connected. Thus we obtain from the Lift Uniqueness Lemma 11.10 that the two maps actually agree on the overlap $C \cap D$. This shows that $\widetilde{f}\colon Y \to X$ is well-defined.

Finally note that since C and D are closed it follows from the Pasting Proposition 2.4 that this map $\widetilde{f}\colon Y \to X$ is continuous. It is now clear that it is a lift of $f\colon Y \to X$. ■

Lemma 14.9. (Lift-Existence Lemma) Let $p\colon X \to B$ be a covering of topological spaces. Given any continuous map $f\colon [0,1]^m \to B$ and given any $x \in X$ with $p(x) = f(0)$ there exists a unique lift $\widetilde{f}\colon [0,1]^m \to X$ with $\widetilde{f}(0) = x$.

Proof. Let $p\colon X \to B$ be a covering of topological spaces, let $f\colon [0,1]^m \to B$ be a continuous map and let $x \in X$ with $p(x) = f(0)$. Since $p\colon X \to B$ is a covering there exists an open cover of B by uniformly covered open subsets of B. It follows from the Lebesgue Lemma 11.9 that there exists an $N \in \mathbb{N}$ such that for any $(i_1, \dots, i_m) \in \{0, \dots, N-1\}^m$ the image $f([\frac{i_1}{N}, \frac{i_1+1}{N}] \times \cdots \times [\frac{i_m}{N}, \frac{i_m+1}{N}])$ is contained in a uniformly covered open subset of B. We now to lift the map $f\colon [0,1]^m \to B$ as follows:

(1) We set $\widetilde{f}(0) := x$.
(2) We apply the Lift-Extension Lemma 14.8 iteratively to the N^m cubes of $[0,1]^m$ starting from the "bottom left". ■



14.5. **Lifts of paths.**

Proposition 14.10. (Path-Lifting Proposition) Let $p\colon X \to B$ be a covering.

(1) Let $f\colon [0,1] \to B$ be a path. Given any $\widetilde{b} \in X$ with $p(\widetilde{b}) = f(0)$ there exists a unique lift $\widetilde{f}\colon [0,1] \to X$ of f to the starting point $\widetilde{b}$.
(2) Let $f, g\colon [0,1] \to B$ be two paths and let $\widetilde{f}, \widetilde{g}\colon [0,1] \to X$ be two lifts with the same starting point. If f and g are path-homotopic, then the endpoints of $\widetilde{f}$ and $\widetilde{g}$ agree and the paths $\widetilde{f}$ and $\widetilde{g}$ are path-homotopic.

Proof.

(1) This statement follows immediately from the Lift-Existence Lemma 14.9.
(2) We set $b := f(0) = g(0)$ and $c := f(1) = g(1)$. Furthermore let $\widetilde{b} := \widetilde{f}(0) = \widetilde{g}(0)$ be the common starting point of the lifts. Let

$$\begin{array}{rcl} H\colon [0,1] \times [0,1] & \to & B \\ (t,s) & \mapsto & H(t,s) \end{array}$$

be a path-homotopy between the paths f and g. By the Lift-Existence Lemma 14.9 we can lift H to a continuous map $\widetilde{H}\colon [0,1] \times [0,1] \to X$ with $\widetilde{H}(0,0) = \widetilde{f}(0)$.

Claim. The map $\widetilde{H}\colon [0,1] \times [0,1] \to X$ is a path-homotopy from $\widetilde{f}$ to $\widetilde{g}$. In particular the endpoints of $\widetilde{f}$ and $\widetilde{g}$ agree.

Proof.

(a) For each $s \in [0,1]$ we have $p(\widetilde{H}(0,s)) = H(0,s) = b$, thus the map $s \mapsto \widetilde{H}(0,s)$ takes values in $p^{-1}(b)$. By the Covering–Basics Lemma 14.6 we know that $p^{-1}(b)$ is a discrete subspace. It follows from the Connected-to-Discrete Lemma 2.18 that the map $s \mapsto \widetilde{H}(0,s)$ is constant. Similarly we see that the $s \mapsto \widetilde{H}(1,s)$ is constant. Thus we have shown that $\widetilde{H}$ is a path-homotopy.

(b) Since $\widetilde{H}_0$ and $\widetilde{f}$ are both lifts of $H_0 = f$ to the starting point $\widetilde{b}$ we obtain from the Lift Uniqueness Proposition 14.7 that $\widetilde{H}_0 = \widetilde{f}$.

(c) It follows from (a) that $\widetilde{H}_1$ and $\widetilde{g}$ are both lifts of $H_1 = g$ to the starting point $\widetilde{b}$. Thus we obtain again from the Lift Uniqueness Proposition 14.7 that $\widetilde{H}_1 = \widetilde{g}$. ■

Proposition 14.11. (Covering-π_1-Monomorphism Proposition) Let $p\colon X \to B$ be a covering. Given any $x_0 \in X$ the induced map $p_*\colon \pi_1(X, x_0) \to \pi_1(B, p(x_0))$ is a monomorphism.

Proof. We set $b_0 := p(x_0)$. We see that

$$\begin{aligned}[f] \in \ker(p_*\colon \pi_1(X,x_0) \to \pi_1(B,b_0)) \implies [p \circ f] = e \in \pi_1(B,b_0) \implies p \circ f \simeq_{\mathrm{p}} \mathrm{const}_{b_0} \\ \underset{\uparrow}{\implies} f \simeq_{\mathrm{p}} \mathrm{const}_{x_0} \implies [f] = e \in \pi_1(X,x_0).\end{aligned}$$

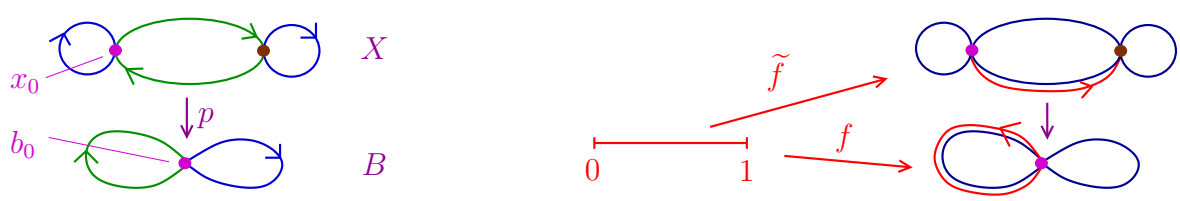
this follows from the Path-Lifting Proposition 14.10 (2), since
f is the lift of $p \circ f$ to the starting point x_0 and const_{x_0} is the lift of const_{b_0} to the starting point x_0 ■

Proposition 14.12. (Loop–Lifting Proposition) Let $p\colon X \to B$ be a covering. Let $b_0 \in B$ and let $x_0 \in X$ be a point with $p(x_0) = b_0$. Let $f\colon [0,1] \to B$ be a loop in (B, b_0). We denote by $\widetilde{f}\colon [0,1] \to X$ the lift of f to the starting point x_0. The following holds:

$$\widetilde{f} \text{ is a loop in } (X, x_0) \iff [f] \text{ lies in } p_*(\pi_1(X,x_0)) \subset \pi_1(B,b_0).$$

In particular, if $\widetilde{f}$ is not a loop in (X, x_0), then $[f]$ is a non-trivial element of $\pi_1(B, b_0)$.

Example. We consider the pointed topological space (B, b_0) and the loop $f\colon [0,1] \to B$ that are shown in the figure below. We also show a covering $p\colon X \to B$ together with the lift $\widetilde{f}$ of f to the starting point x_0. Since $\widetilde{f}$ is not a loop it follows from the Loop–Lifting Proposition 14.12 that $[f] \in \pi_1(B, b_0)$ is non-trivial.



Proof. Let f be a loop in (B, b_0). We denote by $\widetilde{f}$ the lift of f to the starting point x_0.

($\Rightarrow$) If $\widetilde{f}$ is a loop in (X, x_0), then it defines an element in $\pi_1(X, x_0)$. Therefore we see that $[f] = [p \circ \widetilde{f}] = p_*([\widetilde{f}]) \in p_*(\pi_1(X, x_0))$.

($\Leftarrow$) We assume that $[f]$ lies in $p_*(\pi_1(X, x_0)) \subset \pi_1(B, b_0)$. We make two observations:

- The hypothesis that $[f] \in p_*(\pi_1(X, x_0))$ implies that there exists a loop g in (X, x_0) with $[f] = p_*([g]) = [p \circ g]$. In particular we see that f and $p \circ g$ are path-homotopic.
- Note that $\widetilde{f}$ is a lift of f and that g is a lift of $p \circ g$ and note that these two lifts have the same starting point x_0.

It follows from these two observations and the Path-Lifting Proposition 14.10 (2) that the endpoint of $\widetilde{f}$ agrees with the endpoint of $\widetilde{g}$, which is x_0. Thus we have shown that $\widetilde{f}$ is a loop in (X, x_0). ■

Definition. Let G be a group and let H be a subgroup of G.

(1) We get an equivalence relation $\sim$ on G by setting $g_1 \sim g_2$ if there exists an $h \in H$ with $g_1 \cdot h = g_2$. We denote by G/H the set of equivalence classes.
(2) If in (1) we use left multiplication by $h \in H$, then also get an equivalence relation and we denote the set of equivalence classes by $H\backslash G$. Note that the map $[g] \mapsto [g^{-1}]$ defines a bijection $H\backslash G \to G/H$.
(3) We write $[G : H] := \#G/H = \#H\backslash G$.

Proposition 14.13. (Covering-Degree Proposition) Let $p\colon X \to B$ be a covering. Let $b_0 \in B$ and let $x_0 \in X$ be a point with $p(x_0) = b_0$. If X is path-connected, then

$$[X : B] \;=\; [\pi_1(B, b_0) : p_*(\pi_1(X, x_0))].$$

Example. Let $n \geq 2$.

(1) On page 137 we saw that we have a 2-fold covering $p\colon S^n \to S^n/\{\pm 1\} =_i \mathbb{R}\mathrm{P}^n$.
(2) It follows from the Covering-Degree Proposition 14.13 that $p_*(\pi_1(S^n))$ is subgroup of index two of $\pi_1(S^n/\{\pm 1\})$.
(3) Since $n \geq 2$ we know by the Sphere-π_1-Proposition 11.14 that $\pi_1(S^n) = 0$.
(4) It follows from (1)–(3) that $\pi_1(S^n/\{\pm 1\})$ is a group with two elements.
(5) It is elementary to see that (4) implies that $\pi_1(\mathbb{R}\mathrm{P}^n) = \pi_1(S^n/\{\pm 1\}) \cong \mathbb{Z}_2$.

In the next chapter we will return to this calculation in a more systematic fashion.

Lemma 14.14. (Covering–Bijection Lemma) Let $p\colon X \to B$ be a covering. Let $b_0 \in B$ and let $x_0 \in X$ be a point with $p(x_0) = b_0$. If X is path-connected, then the map

$$\begin{aligned} \Theta\colon p^{-1}(b_0) &\to p_*(\pi_1(X, x_0))\backslash\pi_1(B, b_0) \\ y_0 &\mapsto [\, p \circ (\underbrace{\text{path } f \text{ in } X \text{ from } x_0 \text{ to } y_0}_{\text{exists since } X \text{ is path-connected}})\,] \end{aligned}$$

is well-defined and it is a bijection.

Proof of Lemma 14.14.

(1) First we show that the map Θ is well-defined. Thus let $f, g\colon [0,1] \to X$ be two paths from x_0 to y_0. Note that

$$\underbrace{[p \circ f]}_{\in \pi_1(B,b_0)} \;=\; [p \circ f * \underbrace{p \circ \overline{g} * p \circ g}_{\substack{\text{null-homotopic by} \\ \text{the Path-Homotopy} \\ \text{Product Proposition 11.4}}}] \;=\; [p \circ (\underbrace{f * \overline{g}}_{\substack{\text{loop in } (X,x_0) \\ \text{since the endpoint} \\ \text{of } f \text{ equals the} \\ \text{starting point of } \overline{g}}})] \cdot [p \circ g] \;=\; p_*(\underbrace{[f * \overline{g}]}_{\in \pi_1(X,x_0)}) \cdot \underbrace{[p \circ g]}_{\in \pi_1(B,b_0)}.$$

(2) Next we show that the map Θ is surjective. Thus let $[f] \in \pi_1(B, b_0)$. By the Path-Lifting Proposition 14.10 (1) there exists a lift $\widetilde{f}$ of f to the starting point x_0. Note that $\Theta(\widetilde{f}(1)) = [p \circ \widetilde{f}] = [f]$.

(3) Finally we show that the map Θ is injective. Suppose we have $y_0, z_0 \in p^{-1}(b_0)$ such that $\Theta(y_0) = \Theta(z_0)$.
   (a) We pick a path $f_0\colon [0,1] \to X$ from x_0 to y_0 and we pick a path $g_0\colon [0,1] \to X$ from x_0 to z_0.
   (b) The fact that $\Theta(y_0) = \Theta(z_0) \in p_*(\pi_1(X, x_0))\backslash\pi_1(B, b_0)$ implies that there exists a loop $h\colon [0,1] \to X$ in (X, x_0) such that $[p \circ h] \cdot [p \circ f_0] = [p \circ g_0]$.
   (c) Note that $h * f_0$ is defined since the endpoint of h equals x_0, which is the starting point of f_0. Now we see that

$$[p \circ (h * f_0)] \;=\; [(p \circ h) * (p \circ f_0)] \;=\; [p \circ h] \cdot [p \circ f_0] \;=\; [p \circ g_0].$$

   (d) By (c) we know that $p \circ (h * f_0) \simeq_{\mathrm{p}} p \circ g_0$.
   (e) Since $h * f_0$ is a lift of $p \circ (h * f_0)$ to the starting point x_0 and since g_0 is a lift of $p \circ g_0$ to the starting point x_0 we obtain from (d) and the Path-Lifting Proposition 14.10 (2) that the endpoints of $h * f_0$ and g_0 agree. But these are precisely y_0 and z_0. ■

Proof of Proposition 14.13. We see that

$$[X : B] \;\underset{\substack{\uparrow \\ \text{definition of degree}}}{=}\; \#p^{-1}(b_0) \;\underset{\substack{\uparrow \\ \text{Covering–Bijection Lemma 14.14}}}{=}\; \#p_*(\pi_1(X, x_0))\backslash\pi_1(B, b_0) \;\underset{\substack{\uparrow \\ \text{definition of index}}}{=}\; [\pi_1(B, b_0) : p_*(\pi_1(X, x_0))].$$

■

14.6. **Pullbacks of coverings.**

Lemma 14.15. (Pullback-Covering Lemma) Let $p\colon X \to B$ be a continuous map of topological spaces, let C be a topological space and let $f\colon C \to B$ be a continuous map. As in the Topological-Pullback Proposition 5.7 we consider the pullback

$$f^*X \;:=\; \{(c, x) \in C \times X \mid f(c) = p(x)\}$$

which gives us the following commutative diagram:

$$\begin{tikzcd} f^*X \arrow[r, "\tilde{f}"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\ C \arrow[r, "f"'] & B. \end{tikzcd}$$

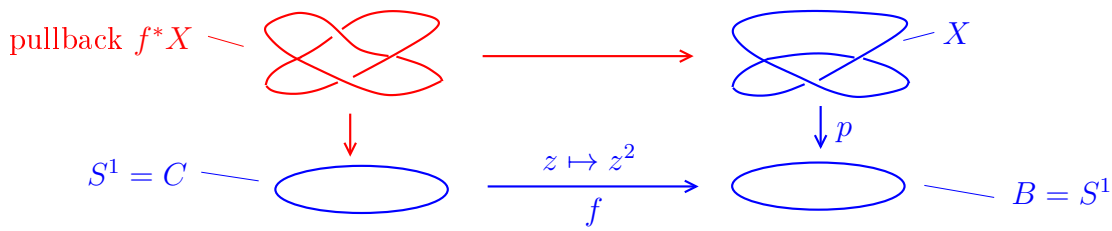
If $p\colon X \to B$ is in fact a covering, then the following statements hold:
(1) The map $q\colon f^*X \to C$ is also a covering.
(2) If B and C are connected, then

$$\deg(q\colon f^*X \to C) \;=\; \deg(p\colon X \to B).$$

(3) Let $c \in C$. We write $b = f(c)$ and we pick $x \in X$ such that $p(x) = b$. The following equality holds:

$$q_*(\pi_1(f^*X, (c, x))) \;=\; f_*^{-1}(p_*(\pi_1(X, x))) \;\subset\; \pi_1(C, c).$$

Example. Similar to the discussion on page 75 we consider the covering $p\colon X \to B = S^1$ that is shown in the figure below and we consider the map $f\colon C = S^1 \to S^1 = B$ that is given by $f(z) = z^2$. The pullback covering $f^*X \to C = S^1$ is a product, i.e. there exists a homeomorphism $f^*X \cong S^1 \times \{-1, 1\}$ that commutes with the projections to S^1.



Proof. Let $p\colon X \to B$ be a covering. We write $Z := f^*X$ and $z := (c, x)$. Finally we denote by $q\colon f^*X \to C$ the map given by $(c, x) \mapsto c$.

(1) First we show that $q\colon f^*X \to C$ is surjective. Thus let $c \in C$. Since $p\colon X \to B$ is in particular surjective we see that there exists an $x \in X$ with $p(x) = f(c)$. Note that $(c, x) \in f^*X$. Since $q(c, x) = c$ we have shown that q is surjective.

Next let $c \in C$. We need to show that c admits a uniformly covered neighborhood. Since p is a covering the point $f(c) \in B$ admits a uniformly covered neighborhood U. This means that $p^{-1}(U) = \bigsqcup_{i \in I} A_i$ such that each A_i is open in X and such that each $p|_{A_i}\colon A_i \to U$ is a homeomorphism.

We set $V := f^{-1}(U)$. We claim that V is uniformly covered. Given $i \in I$ we consider $B_i := \{(c, x) \in f^*X \mid c \in V, x \in U_i\}$. Since $B_i = i^*X \cap (V \times X_i)$ we see that B_i is an open subset of i^*X. Finally one can easily verify that each restriction $q|_{B_i}\colon B_i \to V$ is a homeomorphism.

(2) Since B and C are connected it makes sense to speak of the degrees of p and q. For any $c \in C$ we have a bijection

$$q^{-1}(c) \;=\; \{(c, x) \in C \times X \mid f(c) = p(x)\} \xrightarrow{\text{bijection}} p^{-1}(f(c)) \;=\; \deg(p).$$

This shows that the degree of q equals the degree of p.

(3) We need to show that $q_*(\pi_1(Z, z)) = f_*^{-1}(\pi_1(X, x))$. We show the two inclusions "$\subset$" and "$\supset$" separately.

"$\subset$" Let $\widetilde{f}\colon Z \to X$ be the map given by $(c, x) \to x$. By construction we have $f \circ q = \widetilde{f} \circ p$. But this implies immediately that $f_*(q_*(\pi_1(Z, z)))$ is contained in $p_*(\pi_1(X, x))$.

"$\supset$" Let $\gamma\colon [0, 1] \to C$ be a loop in (C, c) with $[f \circ \gamma] = f_*([\gamma]) \in p_*(\pi_1(X, x))$. We write $g := f \circ \gamma$. Let $\widetilde{g}\colon [0, 1] \to X$ be the lift with $\widetilde{g}(0) = x$. Since $[g] \in p_*(\pi_1(X, x))$ we obtain from the Loop–Lifting Proposition 14.12 that $\widetilde{g}$ is actually a loop. Thus the map $\alpha\colon [0, 1] \to C \times X$ that is given by $t \mapsto (\gamma(t), \widetilde{g}(t))$ is a loop. By construction the loop takes values in Z and evidently we have $q_*([\alpha]) = [\gamma]$. This shows that $[\gamma] \in q_*(\pi_1(Z, z))$. ■

15. Group actions and fundamental groups

15.1. The main theorem on group actions and fundamental groups.

Theorem 15.1. (Action–π_1-Theorem) Let X be a topological space and let G be a group that acts continuously and discretely on X. We choose an $x_0 \in X$. We denote by $p\colon X \to X/G$ the natural projection.

(1) If X is simply connected, then the map

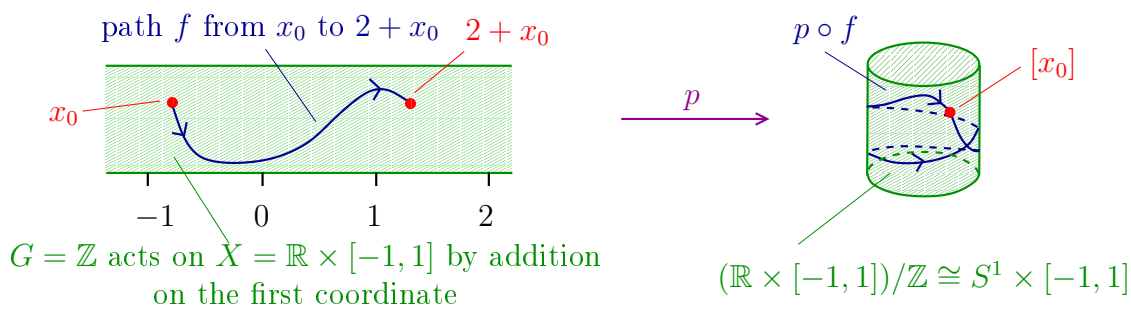
$$\begin{array}{rcl} G & \to & \pi_1(X/G, [x_0]) \\ g & \mapsto & [p \circ (\text{path in } X \text{ from } x_0 \text{ to } g \cdot x_0)] \end{array}$$

is a well-defined isomorphism of groups. In particular we have $\pi_1(X/G) \cong G$.

(2) If X is path-connected (but not necessarily simply connected), then $p_*(\pi_1(X, x_0))$ is a normal subgroup of $\pi_1(X/G, [x_0])$ and the map

$$\begin{array}{rcl} G & \to & \pi_1(X/G, [x_0])/p_*(\pi_1(X, x_0)) \\ g & \mapsto & [p \circ (\text{path in } X \text{ from } x_0 \text{ to } g \cdot x_0)] \end{array}$$

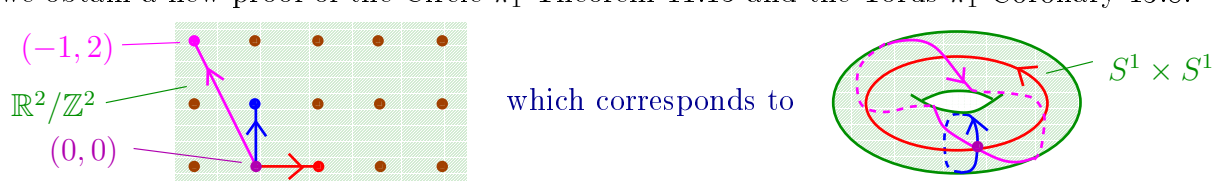
is a well-defined isomorphism of groups.



Example. Let $n \in \mathbb{N}$. It follows from the discussion on page 118 that $\mathbb{R}^n$ is simply connected. We saw on page 133 that the usual continuous action of $(\mathbb{Z}^n, +)$ on $\mathbb{R}^n$ is discrete. It follows from the Action–π_1-Theorem 15.1 (1) that the map

$$\begin{array}{rcl} \mathbb{Z}^n & \to & \pi_1(\mathbb{R}^n/\mathbb{Z}^n, [(0)]) \\ v & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & \mathbb{R}^n/\mathbb{Z}^n \\ t & \mapsto & [t \cdot v] \end{array} \right] \end{array}$$

is an isomorphism. Using the homeomorphism $(S^1)^n \to \mathbb{R}^n/\mathbb{Z}^n$ from the Torus-as-Quotient Lemma 6.6 this also gives a new calculation of the fundamental group of $(S^1)^n$. In particular we obtain a new proof of the Circle-π_1-Theorem 11.13 and the Torus-π_1-Corollary 13.8.

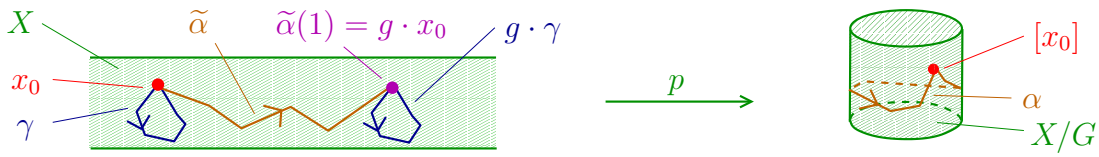


Proof of the Action–π_1-Theorem 15.1. Note that Statement (1) is just a user-friendly special case of Statement (2). Thus it suffices to prove Statement (2). Let X be a path-connected topological space and let G be a group that acts continuously and discretely on X. Note that it follows from the Action–Covering Proposition 14.5 that the natural

projection $p\colon X \to X/G$ is a covering. Finally let $x_0 \in X$. We start out with the following claim.

Claim 1. The image $p_*(\pi_1(X, x_0)) \subset \pi_1(X/G, [x_0])$ is a normal subgroup.

Proof. Let $[\gamma] \in \pi_1(X, x_0)$ and let $[\alpha] \in \pi_1(X/G, [x_0])$. We denote by $\widetilde{\alpha}\colon [0,1] \to X$ the lift of α with the starting point x_0. Note that $p(\widetilde{\alpha}(1)) = \alpha(1) = [x_0]$. This shows that there exists a $g \in G$ with $\widetilde{\alpha}(1) = g \cdot x_0$. We denote by $g \cdot \gamma$ the loop in X that is given by $t \mapsto g \cdot \gamma(t)$. Note that we can form the loop $\widetilde{\alpha} * g \cdot \gamma * \overline{\widetilde{\alpha}}$ since the endpoints always match the starting point.



Now we see that

$$\begin{aligned}
[\alpha] \cdot p_*([\gamma]) \cdot [\alpha]^{-1} &= [p \circ \widetilde{\alpha}] \cdot [p \circ \gamma] \cdot [p \circ \overline{\widetilde{\alpha}}] \overset{\substack{\text{since } p(g \cdot x) = p(x) \in X/G \text{ for all } x \in X \\ \downarrow}}{=} [p \circ \widetilde{\alpha}] \cdot [p \circ (g \cdot \gamma)] \cdot [p \circ \overline{\widetilde{\alpha}}] \\
&= [p \circ (\underbrace{\widetilde{\alpha} * g \cdot \gamma * \overline{\widetilde{\alpha}}}_{\substack{\text{defined since the endpoints} \\ \text{always match the starting points}}})] = p_*([\underbrace{\widetilde{\alpha} * g \cdot \gamma * \overline{\widetilde{\alpha}}}_{\text{loop in } (X, x_0)}]) \in p_*(\pi_1(X, x_0)).
\end{aligned}$$

Thus we have shown that $p_*(\pi_1(X, x_0))$ is a normal subgroup of $\pi_1(X/G, [x_0])$. ⊞

Similar to the Covering–Bijection Lemma 14.14 we consider the map

$$\begin{aligned}
\Phi\colon G &\to \pi_1(X/G, [x_0])/p_*(\pi_1(X, x_0)) \\
g &\mapsto [p \circ (\text{path in } X \text{ from } x_0 \text{ to } g \cdot x_0)].
\end{aligned}$$

Claim 2. The map Φ is well-defined and it is a bijection.

Proof. We consider the maps

$$\begin{array}{ccccc}
G & \to & p^{-1}([x_0]) & \to & \pi_1(X/G, [x_0])/p_*(\pi_1(X, x_0)) \\
g & \mapsto & g \cdot x_0 & & \\
 & & y_0 & \mapsto & [p \circ (\text{path in } X \text{ from } x_0 \text{ to } y_0)].
\end{array}$$

The map to the left is clearly a bijection. The claim now follows from the observation that we showed in the Covering–Bijection Lemma 14.14 that the map to the right is well-defined and that it is a bijection. ⊞

It remains to prove the following claim.

Claim 3. The map $\Phi\colon G \to \pi_1(X/G, [x_0])/p_*(\pi_1(X, x_0))$ is a group homomorphism.

Proof. Let $g, h \in G$. We pick a path α from x_0 to $g \cdot x_0$ and we pick a path β from x_0 to $h \cdot x_0$. Then $g \cdot \beta$ is a path from $g \cdot x_0$ to $(g \cdot (h \cdot x_0)) = gh \cdot x_0$ and $\alpha * g \cdot \beta$ is a path from x_0 to $gh \cdot x_0$. It follows that

$$\begin{aligned}
\Phi(g \cdot h) \underset{\substack{\uparrow \\ \text{since } \alpha * g \cdot \beta \text{ is a path from } x_0 \text{ to } gh \cdot x_0}}{=} [p \circ (\alpha * g \cdot \beta)] &\overset{\substack{\text{since } p \circ (\gamma * \delta) = p \circ \gamma * p \circ \delta \\ \downarrow}}{=} [(p \circ \alpha) * (p \circ g \cdot \beta)] \overset{\substack{\text{since } p(g \cdot x) = p(x) \in X/G \text{ for all } x \in X \\ \downarrow}}{=} [(p \circ \alpha) * (p \circ \beta)] \\
&\underset{\substack{\uparrow \\ \text{definition of } \cdot \text{ in } \pi_1(X/G, [x_0])}}{=} [p \circ \alpha] \cdot [p \circ \beta] = \Phi(g) \cdot \Phi(h).
\end{aligned}$$

■

15.2. Applications of the Action–π_1-Theorem 15.1.

Proposition 15.2. (Lens Space-π_1-Proposition)

(1) Let $p \in \mathbb{Z} \setminus \{0\}$ and let $q \in \mathbb{Z}$ with $\gcd(p,q) = 1$. If $L(p,q) = S^3/\mathbb{Z}_p$ is a lens space as defined on page 135, then
$$\pi_1(L(p,q)) \;\cong\; \mathbb{Z}_p.$$

(2) Let $p, \widetilde{p} \in \mathbb{Z} \setminus \{0\}$ and let $q, \widetilde{q} \in \mathbb{Z}$ with $\gcd(p,q) = 1$ and $\gcd(\widetilde{p}, \widetilde{q}) = 1$. If $L(p,q)$ is homotopy equivalent (e.g. homeomorphic) to $L(\widetilde{p}, \widetilde{q})$, then $p = \widetilde{p}$.

Proof.

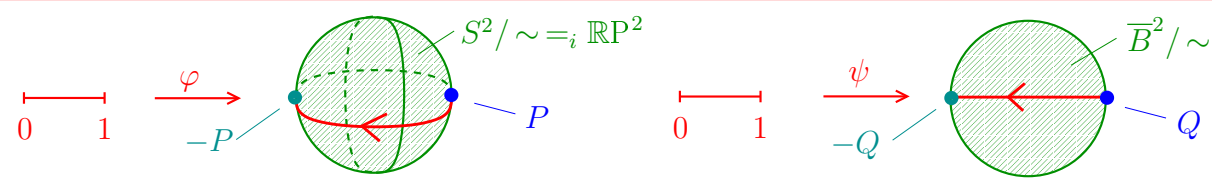
(1) First we note that by the Sphere-π_1-Proposition 11.14 we know that S^3 is simply connected. Next note that in the Lens Space-Discrete Action Lemma 14.2 we showed that the action of $\mathbb{Z}_p$ on S^3 is discrete. Thus the statement follows from the Action–π_1-Theorem 15.1 (1).

(2) This statement follows from (1) and the Homotopy-π_1-Proposition 13.4 (2a). ■

Proposition 15.3. (Real Projective Space–π_1-Proposition) Let $n \in \mathbb{N}_{\geq 2}$.

(1) We consider the real projective space $\mathbb{R}\mathrm{P}^n =_i S^n/\{\pm 1\}$.
  (a) $\pi_1(\text{real projective space } \mathbb{R}\mathrm{P}^n) \;=_i\; \pi_1(S^n/\{\pm 1\}) \;\cong\; \mathbb{Z}_2$.
  (b) We set $P := (1, 0, \ldots, 0) \in S^n$. The unique non-trivial element of the fundamental group $\pi_1(\mathbb{R}\mathrm{P}^n, [P]) =_i \pi_1(S^n/\{\pm 1\}, [P])$ is represented by the loop
$$\begin{array}{rcl} \varphi\colon [0,1] & \to & \mathbb{R}\mathrm{P}^n =_i S^n/\{\pm 1\} \\ t & \mapsto & [(\cos(\pi t), \sin(\pi t), 0, \ldots, 0)]. \end{array}$$

(2) Let "$\sim$" be the equivalence relation on $\overline{B}^n$ that is generated by $x \sim -x$ for $x \in S^{n-1}$.
  (a) $\pi_1(\overline{B}^n/\!\sim) \cong \mathbb{Z}_2$.
  (b) With $Q = (1, 0, \ldots, 0) \in \overline{B}^n$ the unique non-trivial element of $\pi_1(\overline{B}^n/\!\sim), [Q])$ is represented by the loop
$$\begin{array}{rcl} \psi\colon [0,1] & \to & \overline{B}^n/\!\sim \\ t & \mapsto & [2t-1, 0, \ldots, 0)]. \end{array}$$



Proof.

(1) These two statements follows almost immediately from the following observations:
  (i) We consider the usual continuous action of $\{\pm 1\}$ on S^n. On page 133 we saw that the action of $\{\pm 1\}$ on S^n is discrete. We denote by $p\colon S^n \to S^n/\{\pm 1\}$ the projection.
  (ii) Since $n \geq 2$ we obtain from the Sphere-π_1-Proposition 11.14 that the sphere S^n is simply connected.
  (iii) Note that it follows from (i) and (ii) together with the Action–π_1-Theorem 15.1 (1) that the map
$$\begin{array}{rcl} \{\pm 1\} & \to & \pi_1(S^n/\!\sim) \\ \epsilon & \mapsto & [p \circ (\text{path in } S^n \text{ from } P \text{ to } \epsilon \cdot P)] \end{array}$$

is an isomorphism. Since $t \mapsto (\cos(\pi t), \sin(\pi t), 0, \dots, 0)$ is precisely a path from P to $-P$ we see that φ represents the non-trivial element of $\pi_1(S^n/\{\pm 1\}, [P])$.

(iv) Recall that in the Projective Space-Sphere Quotient Lemma 6.9 we saw that the inclusion $S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ induces a homeomorphism $S^n/\{\pm 1\} \to \mathbb{R}\mathrm{P}^n$.

(2) In the Projective Space-Avatar Lemma 6.11 (2) we gave an explicit homeomorphism $f\colon \overline{B}^n/\sim \to S^n/\{\pm 1\}$ with $f([Q]) = [P]$. It follows from this observation together with the results in (1) that $\pi_1(\overline{B}^n/\sim) \cong \mathbb{Z}_2$ and that the unique non-trivial of $\pi_1(\overline{B}^n/\sim, [Q])$ is presented by the loop $f \circ \varphi$, i.e. by the loop $t \mapsto [(\cos(\pi t), \sin(\pi t), 0, \dots, 0)]$. It is straightforward to see that this loop is path-homotopic to the loop ψ. ■

Proposition 15.4. **(SO(3)–π_1-Proposition)**

(1) $\pi_1(\mathrm{SO}(3), \mathrm{id}) \cong \mathbb{Z}_2$.

(2) The unique non-trivial element is represented by the loop

$$\begin{array}{rcl} \gamma\colon [0,1] & \to & \mathrm{SO}(3) \\ t & \mapsto & \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\pi t) & \sin(2\pi t) \\ 0 & -\sin(2\pi t) & \cos(2\pi t) \end{pmatrix}}_{\text{rotation around the } x\text{-axis}} \end{array}$$

at the base point $\mathrm{id} \in \mathrm{SO}(3)$.

Proof.

(1) (a) Let $\sim$ be the equivalence relation on $\overline{B}^3$ that is generated by $x \sim -x$ for $x \in S^2$.

(b) We set $Q := (1,0,0) \in \overline{B}^3$.

(c) In the Projective Space-Avatar Lemma 6.11 (3) we saw that the map

$$\begin{array}{rcl} f\colon \overline{B}^3/\sim & \to & \mathrm{SO}(3) \\ [\underbrace{r \cdot v}_{r \in [0,1] \text{ and } v \in S^2}] & \mapsto & \text{matrix in SO(3) which represents rotation by the angle } \pi \cdot r \\ & & \text{around the axis } \mathbb{R}_{\geq 0} \cdot v \text{ where the rotation} \\ & & \text{direction is given by the right-hand rule} \end{array}$$

is a homeomorphism. Note that $f(0) = \mathrm{id}$.

(d) In the Real Projective Space–π_1-Proposition 15.3 (2) we showed that $\pi_1(\overline{B}^3/\sim, [Q]) \cong \mathbb{Z}_2$.

(e) It follows from (c) and (d) that $\pi_1(\mathrm{SO}(3), f([Q])) \cong \mathbb{Z}_2$.

(2) We still need to show that γ represents the unique non-trivial element of $\pi_1(\mathrm{SO}(3), \mathrm{id})$.

(f) Given $\epsilon \in \{-1, 1\}$ let $\alpha_\epsilon\colon [0,1] \to \overline{B}^3$ be the path that is given by $t \mapsto [t \cdot Q \cdot \epsilon]$.

(g) In the Real Projective Space–π_1-Proposition 15.3 (2) we showed that the unique non-trivial element of $\pi_1(\overline{B}^3/\sim, [Q])$ is represented by the loop $\overline{\alpha}_+ * \alpha_-$.

(h) Note that α_+ is a path from 0 to Q. It follows from (d) and the Change-of-Base Point Proposition 11.6 that the unique non-trivial element of $\pi_1(\overline{B}^3/\sim, 0)$ is represented by $\alpha_+ * (\overline{\alpha}_+ * \alpha_-) * \overline{\alpha}_+ \simeq_{\mathrm{p}} \alpha_- * \overline{\alpha}_+$.

(i) It follows from (c) and (h) and the fact that $f(0) = \mathrm{id}$ that the unique non-trivial element of $\pi_1(\mathrm{SO}(3), \mathrm{id})$ is represented by $\varphi := f \circ (\overline{\alpha}_+ * \alpha_-)$.

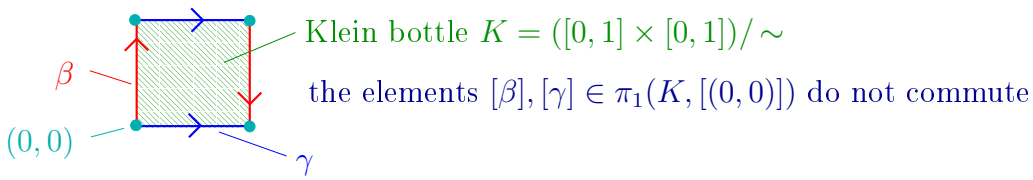
(j) The loop that we gave in the statement of the proposition is precisely $\overline{\varphi}$. But if φ represents a non-trivial element of $\pi_1(\mathrm{SO}(3), \mathrm{id})$, then so does $\overline{\varphi}$. ■

Proposition 15.5. (Klein Bottle–π_1-Proposition) As in the Klein Bottle-as-Quotient Lemma 14.1 we consider the group G that is given by all self-homeomorphisms of $\mathbb{R}^2$ that can be written as concatenations of the two self-homeomorphisms

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x, y+1) \end{array}$$

and their inverses.

(1) The fundamental group of the Klein bottle $K = ([0,1]\times[0,1])/\sim$ is isomorphic to G.
(2) The fundamental group of the Klein bottle K is non-abelian, in particular the two elements $[\beta]$ and $[\gamma]$ shown in the figure below do not commute.



Proof.

(1) In the Klein Bottle-as-Quotient Lemma 14.1 we showed the following two facts:
  (a) The group G acts discretely on $\mathbb{R}^2$.
  (b) The inclusion induced map $f\colon ([0,1]\times[0,1])/\sim \to \mathbb{R}^2/G$ is a homeomorphism.
  It follows from (a) together with the Action–π_1-Theorem 15.1 (1) that the map

$$\begin{array}{rcl} \Theta\colon G & \to & \pi_1(\mathbb{R}^2/G,[(0,0)]) \\ g & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & \mathbb{R}^2/G \\ t & \mapsto & [t\cdot \underbrace{g(0,0)}_{\in\mathbb{R}^2}] \end{array}\right] \end{array}$$

  is an isomorphism. If we combine this calculation with (b) we see that the fundamental group of the Klein bottle is isomorphic to G.
(2) (a) We set $C := B^{-1}\cdot A$. An easy calculation shows that the two homeomorphisms $B(x,y) = (x, y+1)$ and $C(x,y) = (x+1, -y)$ do not commute.
  (b) It follows from (a) that $(f_*)^{-1}(\Theta(B)), (f_*)^{-1}(\Theta(C)) \in \pi_1(K,[(0,0)])$ do not commute.
  (c) We consider the two loops β,γ in $(K,[(0,0)])$ that are shown in the above figure. Note that it follows easily from the definitions that $(f_*)^{-1}(\Theta(B)) = [\beta]$ and that $(f_*)^{-1}(\Theta(C)) = [\gamma]$.
  (d) It follows from (b) and (c) that $[\beta]$ and $[\gamma]$ do not commute. ■

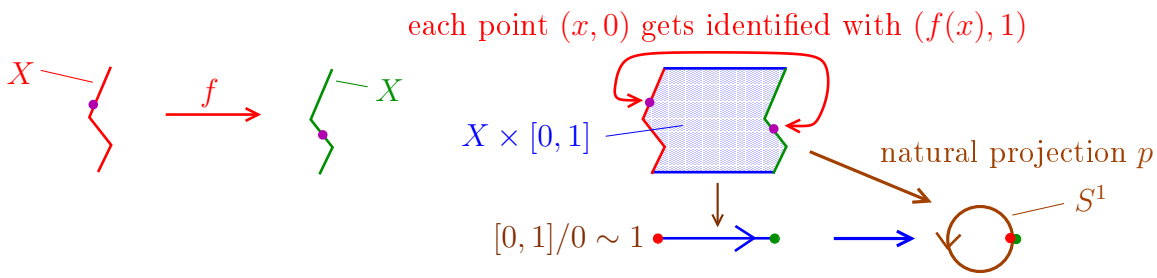
15.3. **The fundamental group of mapping tori.** We start out with the following definition which we will use on several occasions.

Definition. Let X be a topological space and let $f\colon X\to X$ be a homeomorphism. We refer to

$$\mathrm{Tor}(X,f) \;:=\; (X\times[0,1])/\sim \qquad \text{where } (x,0)\sim(f(x),1) \text{ for all } x\in X,$$

as the mapping torus of (X, f). We introduce two maps:

$$\underbrace{\begin{array}{rcl} X & \to & \mathrm{Tor}(X,f) \\ x & \mapsto & [(x,0)] \end{array}}_{\text{the natural injection}} \quad \text{and} \quad \underbrace{\begin{array}{rcl} \mathrm{Tor}(X,f) & \to & S^1 \\ [(x,t)] & \mapsto & \exp(2\pi\,\mathrm{i}t). \end{array}}_{\text{the natural projection}}$$



Examples.

(1) Let X be a topological space. In [Fri23] it is shown that the map

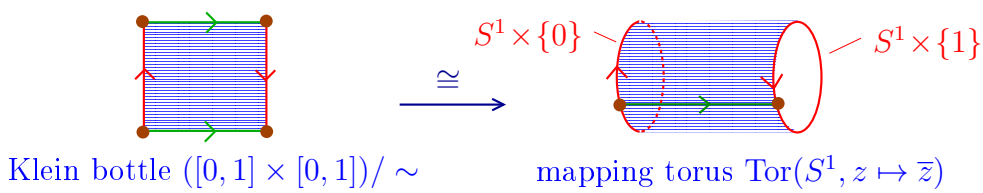
$$\begin{array}{rcl} \Phi\colon \mathrm{Tor}(X,\mathrm{id}) & \to & X \times S^1 \\ [(x,t)] & \mapsto & (x, \exp(2\pi\,\mathrm{i}t)) \end{array}$$

is a homeomorphism.

(2) We consider the homeomorphism $f\colon S^1 \to S^1$ that is given by $z \mapsto \overline{z}$. We also consider the Klein bottle $([0,1]\times[0,1])/\sim$, as defined on page 82. It is not difficult to show that the map

$$\begin{array}{rcl} ([0,1]\times[0,1])/\sim & \to & \mathrm{Tor}(S^1, z \mapsto \overline{z}) \\ [(x,y)] & \mapsto & [(\exp(2\pi\,\mathrm{i}x), y)] \end{array}$$

is a homeomorphism.



Definition. Let N be a group, let $k \in \mathbb{N}_0$ and let $\varphi\colon N \to N$ be an isomorphism such that $\varphi^k = \mathrm{id}$. We define the semidirect product of N and $\mathbb{Z}_k$ with respect to φ as the group $N \rtimes_\varphi \mathbb{Z}_k$ where the underlying set is given by the direct product $N \times \mathbb{Z}_k$, but where the group multiplication is given by

$$(h,m)\cdot(h',m') := (h\cdot\varphi^m(h'), m+m').$$

If φ is understood from the context, then we drop it from the notation, i.e. we write $N \rtimes \mathbb{Z}_k$ instead of $N \rtimes_\varphi \mathbb{Z}_k$.

Examples.

(1) If N is a group and if $\varphi = \mathrm{id}_N$, then $N \rtimes_{\mathrm{id}} \mathbb{Z} = N \times \mathbb{Z}$ is the direct product of N and $\mathbb{Z}$.

(2) Let $N = \mathbb{Z}_n$ for some $n \in \mathbb{N}_0$. We denote by $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ the isomorphism that is given by multiplication by -1. Clearly this isomorphism satisfies $\varphi^2 = \mathrm{id}$, thus we can form

the semidirect product $\mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$, which is called the n-th dihedral group D_n. It has $2n$ elements.

$$\begin{array}{rcl} \Psi\colon \mathbb{Z}_3 \rtimes \mathbb{Z}_2 & \to & S_3 \\ (h,m) & \mapsto & (1\ 2\ 3)^h \cdot (1\ 2)^m. \end{array}$$

Lemma 15.6. (Semidirect Product Lemma) Let G be a group, let $\xi\colon G \to \mathbb{Z}$ be an epimorphism and let $t \in G$ with $\xi(t) = 1$. We write $N := \ker(\xi)$ and we denote by φ the isomorphism

$$\begin{array}{rcl} \varphi\colon N & \to & N \\ h & \mapsto & t \cdot h \cdot t^{-1}. \end{array}$$

Then the map

$$\begin{array}{rcl} \Theta\colon G & \to & N \rtimes_\varphi \mathbb{Z} \\ g & \mapsto & \big(g \cdot t^{-\xi(g)}, \xi(g)\big) \end{array}$$

is an isomorphism of groups.

Proof. It is straightforward to verify that Θ is injective and surjective. It remains to show that Θ is a homomorphism: given $g, h \in G$ we have indeed

$$\begin{aligned} \Theta(g\cdot h) \;=\; (g\cdot h\cdot t^{-\xi(g\cdot h)}, \xi(g\cdot h)) \;&=\; (g\cdot t^{-\xi(g)}\cdot \underbrace{t^{\xi(g)}\cdot h\cdot t^{-\xi(h)}\cdot t^{-\xi(g)}}_{=\varphi^{\xi(g)}(h\cdot t^{-\xi(h)})}, \xi(g)+\xi(h)) \\ &=\; (g\cdot t^{-\xi(g)}, \xi(g))\cdot(h\cdot t^{-\xi(h)}, \xi(h)) \;=\; \Theta(g)\cdot\Theta(h). \end{aligned}$$

■

Proposition 15.7. (Mapping Torus-π_1-Proposition) Let (X, x_0) be a pointed path-connected topological space, let $f\colon X \to X$ be a homeomorphism and let γ be a path from x_0 to $f(x_0)$. We denote by $\gamma_*\colon \pi_1(X, f(x_0)) \to \pi_1(X, x_0)$ the isomorphism given by the Change-of-Base Point Proposition 11.6 and we define

$$\varphi := \gamma_* \circ f_*\colon \pi_1(X, x_0) \xrightarrow[\cong]{f_*} \pi_1(X, f(x_0)) \xrightarrow[\cong]{\gamma_*} \pi_1(X, x_0).$$

Let $i\colon X \to \mathrm{Tor}(X, f)$ be the natural inclusion and let $p\colon \mathrm{Tor}(X, f) \to S^1$ be the natural projection. There exists a natural isomorphism

$$\Theta\colon \pi_1(\mathrm{Tor}(X, f), [(x_0, 0)]) \xrightarrow{\cong} \pi_1(X, x_0) \rtimes_\varphi \mathbb{Z}$$

such that the following diagram commutes

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "i_*"] \arrow[d, "\mathrm{id}"] & \pi_1(\mathrm{Tor}(X, f), [(x_0, 0)]) \arrow[r, "p_*"] \arrow[d, "\Theta", "\cong"'] & \pi_1(S^1, 1) \arrow[d, "\cong"] \\ \pi_1(X, x_0) \arrow[r, "g \mapsto (g,0)"] & \pi_1(X, x_0) \rtimes_\varphi \mathbb{Z} \arrow[r, "(g,n) \mapsto n"] & \mathbb{Z}. \end{tikzcd}$$

Proof. The proof of the proposition can be summarized by the following sequence of isomorphisms:

$$\pi_1(\mathrm{Tor}(X, f)) \underset{\substack{\uparrow \\ \text{Mapping Torus-of-Homeomorphism Lemma ??}}}{=} \pi_1((X \times \mathbb{R})/\mathbb{Z}_f) \overset{\substack{\text{Action–}\pi_1\text{-Theorem 15.1 (2) and} \\ \text{Semidirect Product Lemma 15.6} \\ \downarrow}}{\cong} \pi_1(X \times \mathbb{R}) \rtimes \mathbb{Z} \underset{\substack{\uparrow \\ \text{Homotopy-}\pi_1\text{-Proposition 13.4}}}{\cong} \pi_1(X) \rtimes \mathbb{Z}.$$

Thus we have obtained an isomorphism $\Theta\colon \pi_1(\mathrm{Tor}(X, f)) \xrightarrow{\cong} \pi_1(X) \rtimes \mathbb{Z}$. A more detailed version of the proof and a proof that the given diagram actually commutes is given in [Fri23]. ■

Example. Let $K = \mathrm{Tor}(S^1, f(z) := \overline{z})$ be the Klein bottle. It follows from the discussion on page **??** that the automorphism $f_* =: \varphi$ of $\pi_1(S^1) = \mathbb{Z}$ is multiplication by -1. It follows from the Mapping Torus-π_1-Proposition 15.7 that

$$\pi_1(\text{Klein bottle}) \;\cong\; \mathbb{Z} \rtimes_\varphi \mathbb{Z}.$$

16. Applied topology

16.1. Topological robotics.

Lemma 16.1. The map

$$\begin{array}{rcl} \Phi\colon S^1 \times S^1 \times S^1 & \to & \mathrm{SO}(3) \\ (\alpha, \beta, \gamma) & \mapsto & \begin{array}{c}\text{rotation by } \alpha \\ \text{around the } x\text{-axis}\end{array} \circ \begin{array}{c}\text{rotation by } \beta \\ \text{around the } y\text{-axis}\end{array} \circ \begin{array}{c}\text{rotation by } \gamma \\ \text{around the } z\text{-axis}\end{array} \end{array}$$

is a surjection.

Sketch of proof. Let $A \in \mathrm{SO}(3)$ be matrix with columns v_1, v_2, v_3. After a rotation around the z-axis we can assume that v_1 lies in the xz-plane. After a rotation around the y-axis we can then arrange that v_1 lies in fact on the x-axis, i.e. we can arrange that $v_1 = e_1$. Since $v_1 = e_1, v_2, v_3$ is an orthonormal basis we see that v_2, v_3 lie in the yz-plane. After a rotation around the x-axis we can finally arrange that $v_2 = e_2$ and $v_3 = e_3$. ■

Proposition 16.2. There is no continuous map $\Psi\colon \mathrm{SO}(3) \to S^1 \times S^1 \times S^1$ such that $\Phi \circ \Psi = \mathrm{id}_{\mathrm{SO}(3)}$.

Proof. Let us suppose there exists a continuous map $\Psi\colon \mathrm{SO}(3) \to S^1 \times S^1 \times S^1$ such that $\Phi \circ \Psi = \mathrm{id}_{\mathrm{SO}(3)}$. Then we get the following commutative diagram

$$\begin{tikzcd} & \overbrace{\pi_1(S^1 \times S^1 \times S^1, \Psi(\mathrm{id}))}^{\cong \mathbb{Z}^3} \arrow[dr, "\Phi_*"] & \\ \underbrace{\pi_1(\mathrm{SO}(3), \mathrm{id})}_{\cong \mathbb{Z}_2} \arrow[ur, "\Psi_*"] \arrow[rr, "\mathrm{id}_*"] & & \underbrace{\pi_1(\mathrm{SO}(3), \mathrm{id})}_{\cong \mathbb{Z}_2}. \end{tikzcd}$$

By the SO(3)–π_1-Proposition 15.4 we have that $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$ and by the Torus-π_1-Corollary 13.8 we know that $\pi_1(S^1 \times S^1 \times S^1) \cong \mathbb{Z}^3$. Since $\mathbb{Z}_2$ is a finite group and since $\mathbb{Z}^3$ is torsion-free we see that the homomorphism $\Psi_*\colon \pi_1(\mathrm{SO}(3)) \to \pi_1(S^1 \times S^1 \times S^1)$ is the trivial map. But that is not possible since the horizontal map is an isomorphism of a non-trivial group. ■

17. Wedges and topological graphs

17.1. The wedge of pointed topological spaces.

Definition. Let $\{(X_i, x_i)\}_{i\in I}$ be a non-empty family of pointed topological spaces.

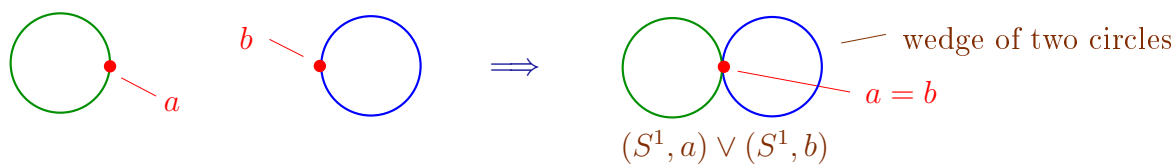
(1) We refer to
$$\bigvee_{i\in I}(X_i, x_i) \;:=\; \Big(\bigsqcup_{i\in I} X_i\Big)\Big/\sim \qquad \text{where } x_i \sim x_j \text{ for all } i, j \in I$$
as the wedge of the pointed topological spaces (X_i, x_i). We refer to the point given by the identifications as the wedge point. If we do not say anything else it will be denoted by $*$. We equip the wedge with this base point to view the wedge as a pointed topological space.

(2) Given $j \in I$ we refer to the map
$$\begin{aligned} \iota_j \colon X_j &\to \bigvee_{i\in I}(X_i, x_i) \\ x &\mapsto [x] \end{aligned}$$
as the natural injection.

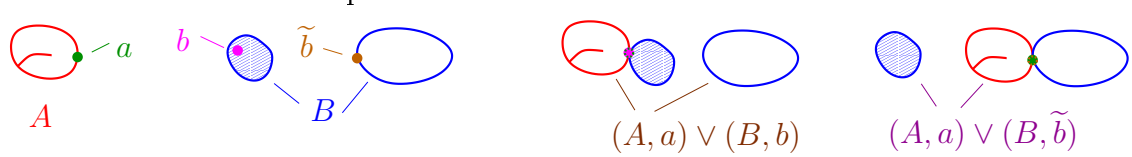
The wedge of the empty family of pointed topological spaces is defined as the one-point topological space $\{*\}$.

Example.

(1) The wedge $(A, a) \vee (B, b)$ of two pointed topological spaces (A, a) and (B, b) should be viewed as the result of gluing A and B along the points a and b.



(2) In general the homeomorphism type of the wedge depends on the choice of base points. This becomes particularly clear if we consider disconnected spaces. For example it is pretty clear that the wedges $(A, a) \vee (B, b)$ and $(A, a) \vee (B, \widetilde{b})$ considered in the figure below are not homeomorphic.



Proposition 17.1. (Wedge Proposition) Let $\{(X_i, x_i)\}_{i\in I}$ be a family of pointed topological spaces.

(0) A subset $U \subset \bigvee_{i\in I}(X_i, x_i)$ is open (closed) if and only if for each $i \in I$ the preimage $\iota_i^{-1}(U)$ is an open (closed) subset of X_i.

(1) (a) Let $j \in J$. The natural injection $\iota_j \colon X_j \to \bigvee_{i\in I}(X_i, x_i)$ is continuous and it is an embedding.

(b) If (Z, z_0) is a pointed topological space and if $\{f_i\colon (X_i, x_i) \to (Z, z_0)\}_{i\in I}$ is a family of continuous maps, then the map

$$\begin{array}{rcl} \bigvee_{i\in I} f_i\colon \bigvee_{i\in I} (X_i, x_i) & \to & Z \\ [(x,i)] & \mapsto & f_i(x) \end{array}$$

is continuous and it sends the wedge point to z_0.

(∗) A map $f\colon \bigvee_{i\in I} (X_i, x_i) \to Z$ is continuous if and only if each $f \circ \iota_i\colon X_i \to Z$ is continuous.

(2) The pointed topological space $\left(\bigvee_{i\in I} (X_i, x_i), *\right)$ together with the natural injections ι_i is a coproduct of $\{(X_i, x_i)\}_{i\in I}$ in the category $\mathcal{PTop}$ of pointed topological spaces, that we defined on page 127.

Proof. We leave the routine verification of this proposition to the reader. The details can be found in [Fri23]. ■

Example.

(1) Let $(X_i, x_i)_{i\in I}$ be a family of pointed topological spaces. Let $J \subset I$. Using the Wedge Proposition 17.1 one can easily show that the natural map $\bigvee_{j\in J} (X_j, x_j) \to \bigvee_{i\in I} (X_i, x_i)$ is an embedding. Furthermore we see that a retraction from $\bigvee_{i\in I} (X_i, x_i)$ to the subspace $\bigvee_{j\in J} (X_j, x_j)$ is given by sending all points in X_i with $i \neq j$ to the wedge point.

(2) Let $G = (V = \{v\}, E, i, t)$ be an abstract graph with a *single* vertex. Using the Abstract Graph-Topology Lemma 7.1 (3) and the Wedge Proposition 17.1 one can easily show that the topological realization $|G|$ is homeomorphic to the wedge of $\#E$ copies of $(S^1, 1)$.

Lemma 17.2. (Wedge–Properties Lemma) Let $\{(X_i, x_i)\}_{i\in I}$ be a family of pointed topological spaces.

(1) If I is finite and if each X_i is compact, then $\bigvee_{i\in I} (X_i, x_i)$ is compact.

(2) If each X_i is (path)-connected, then the wedge $\bigvee_{i\in I} (X_i, x_i)$ is (path)-connected.

(3) The wedge $\bigvee_{i\in I} (X_i, x_i)$ is Hausdorff if and only if each X_i is Hausdorff.

Proof.

(1) This statement follows immediately from the Disjoint Union Topology-Properties Lemma 5.9 (1) together with the Quotient Topology-Properties Proposition 6.2 (1).

(2) This statement follows immediately from the Union Connected Lemma 2.19 together with the Wedge Proposition 17.1 (1a).

(3) This statement can be proved easily using the Wedge Proposition 17.1 (0a). ■

Definition. We say a topological space X is homogeneous if given any two points $x, y \in X$ there exists a homeomorphism $f\colon X \to X$ with $f(x) = y$.

Example. It is clear that the topological space $\mathbb{R}^n$ is homogeneous. One can also easily show "by hand" that the sphere S^n is homogeneous.

Lemma 17.3. (Wedging Point–Irrelevance Lemma) Let $\{X_i\}_{i\in I}$ be a family of topological spaces. If each X_i is homogeneous, then different choices of base points still lead to homeomorphic wedges.

Proof. Suppose that for each $i \in I$ we have base points $x_i, y_i \in X_i$. Since each X_i is homogeneous we know that for each $i \in I$ there exists a homeomorphism $f_i\colon X_i \to X_i$ with $f_i(x_i) = y_i$. It follows from the Wedge Proposition 17.1 (1) that the maps

$$\bigvee_{i\in I}(X_i, x_i) \;\to\; \bigvee_{i\in I}(X_i, y_i) \qquad \text{and} \qquad \bigvee_{i\in I}(X_i, y_i) \;\to\; \bigvee_{i\in I}(X_i, x_i)$$

induced by the maps f_i and by the maps f_i^{-1} are continuous. Clearly they are inverses of one another, so they are homeomorphisms. ■

Definition. Let $\{X_i\}_{i\in I}$ be a family of non-empty homogeneous topological spaces (e.g. a family of spheres). For each $i \in I$ we pick a point $x_i \in X_i$ and we write

$$\bigvee_{i\in I} X_i \;:=\; \bigvee_{i\in I}(X_i, x_i).$$

By the above Lemma 17.3 the homeomorphism type of $\bigvee_{i\in I} X_i$ is well-defined.

17.2. Abstract graphs and wedges of circles.

Proposition 17.4. (Finite Tree–Contractible Proposition) Let T be an abstract graph. If T is a finite tree, then the following hold:

(1) The topological realization $|T|$ admits a deformation retraction to a vertex, in particular it is contractible.
(2) The fundamental group $\pi_1(|T|)$ is trivial.

Proof. Note that Statement (2) is an immediate consequence of Statement (1) together with the Homotopy-π_1-Proposition 13.4 (3). Thus it suffices to prove Statement (1).

We will prove that any tree admits a deformation retraction to a vertex. We will prove this statement by induction on the number of edges. If a tree has zero edges, then it follows from the connectedness of T that T consists of precisely one vertex. So we are done with this special case.

Now suppose that we know that all trees with n edges admit a deformation retraction to a vertex. Let $T = (V, E)$ be a tree with $n + 1$ edges. By the Tree–Valence One Edge Lemma 7.8 we know that T admits a vertex v of valence one. We denote by e the unique edge with v as an endpoint. We set $T' = (V \setminus \{v\}, E \setminus \{e\})$. Note that T' is a non-empty connected abstract subgraph of T. It follows from the Tree–Basics Lemma 7.4 (1) that T' is again a tree.

Claim. $|T'|$ is a deformation retract of $|T|$.

Proof. We denote by u the other vertex of e. Since $u \neq v$ we obtain from the Abstract Graph-Topology Lemma 7.1, see also the discussion on page 95, that there exists a closed

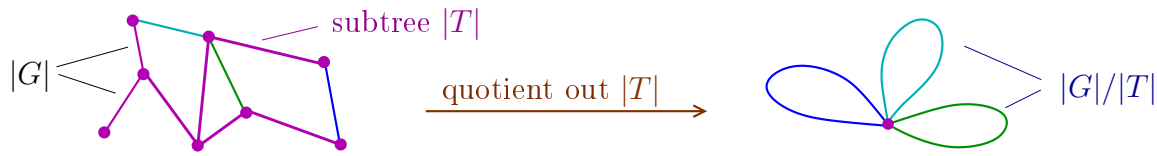
embedding $\Phi\colon [0,1] \to |T|$ with $\Phi(0) = u$ and $\Phi(1) = v$. We consider the map

$$\begin{array}{rcl} H\colon |T| \times [0,1] & \to & |T| \\ (x,t) & \mapsto & \begin{cases} x, & \text{if } x \in |T'|, \\ \Phi(t \cdot y), & \text{if } x = \Phi(y) \text{ for some } y \in [0,1]. \end{cases} \end{array}$$

It follows from the Abstract Graph-Topology Lemma 7.1 that $|T'|$ and $|e| = \Phi([0,1])$ are closed subsets of $|T|$. It thus follows from the Homotopy Combination Lemma 8.2 that the map H is continuous. It is now clear that H is a deformation retraction from $|T|$ to $|T'|$. ⊞

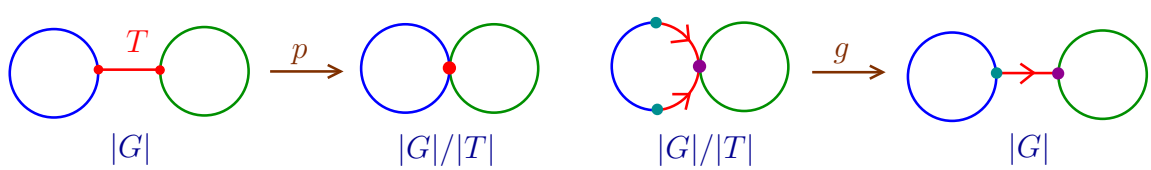
We have thus shown that $|T'|$ is a deformation retract of $|T'|$. By our induction hypothesis $|T'|$ admits a deformation retraction to a vertex $v \in T'$. It follows that $|T|$ also admits a deformation retraction to the vertex v. ■

Proposition 17.5. (Graph-mod-Tree Proposition) Let G be an abstract graph. For every finite subtree T of G the projection $|G| \to |G|/|T|$ is a homotopy equivalence.



Proof. We refer to [Fri23] for the rather technical and uninspiring proof. ■

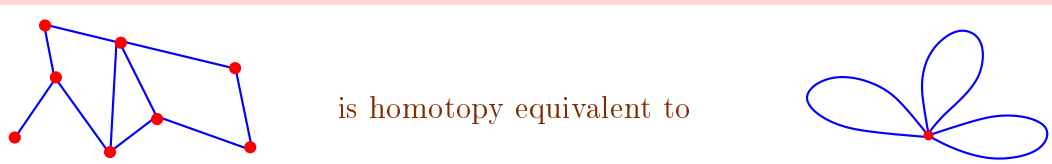
Example. We consider the abstract graph G and the subtree T that are shown in the figure below. It follows from the Graph-mod-Tree Proposition 17.5 that the projection $p\colon |G| \to |G|/|T|$ is a homotopy equivalence. On page 109 we already saw that the map $g\colon |G|/|T| \to |G|$ that is also sketched in the figure is a homotopy inverse to p. The Graph-mod-Tree Proposition 17.5 can be viewed as a notationally slightly heroic generalization of this example.



Corollary 17.6. (Graph-Homotopy Type Corollary)

(1) Let $G = (V, E)$ be a finite abstract graph. If G is connected and non-empty, then the topological realization $|G|$ is homotopy equivalent to the wedge of $1 - \chi(G)$ circles.

(2) Let G, G' be two finite abstract connected non-empty graphs. If $\chi(G) = \chi(G')$, then the topological realizations $|G|$ and $|G'|$ are homotopy equivalent.



topological graph of Euler characteristic $8 - 10 = -2$

wedge of three circles

Proof. Let G be a finite abstract graph.

(1) By the Spanning Tree–Existence Proposition 7.6 (1) we know that G admits a spanning tree T. Next note that by the Graph-mod-Tree Proposition 17.5 we know that the map $p\colon |G| \to |G|/|T|$ is a homotopy equivalence. By the Abstract Graph–Quotient Lemma 7.9 (2) we know that $|G|/|T|$ is homeomorphic to $|G/T|$. Thus it remains to show that $|G/T|$ is homeomorphic to the wedge of $1-\chi(G)$ circles. We make the following observation:

(a) Note that by definition the spanning tree T contains all vertices of G. In other words, the abstract graph G/T contains only a single vertex.

(b) Note that
$$\chi(G/T) \underset{\substack{\uparrow \\ \text{Abstract Graph–Quotient Lemma 7.9 (1)}}}{=} \chi(G)-\chi(T)+1 \underset{\substack{\uparrow \\ \text{Tree Characterization Proposition 7.7}}}{=} \chi(G)-1+1 = \chi(G).$$

(c) By (a) and (b) we know that G/T has $1-\chi(G)$ edges.

(d) It follows from (a) and (c), together with the discussion on page 156 that $|G/T|$ is indeed homeomorphic to the wedge of $1-\chi(G)$ circles.

(2) The last statement is an immediate consequence of (1) and the Homotopy-Equivalence Basics Lemma 9.5. ■

18. Basics of group theory I: Abelian groups

18.1. The direct sum of abelian groups.

Definition. Let $\{X_s\}_{s\in S}$ be a family of abelian groups.

(1) We define the product of the X_s to be the group

$$\prod_{s\in S} X_s \;:=\; \Big\{ f\colon S \to \bigcup_{s\in S} X_s \;\Big|\; \text{for every } s\in S \text{ we have } f(s)\in X_s \Big\},$$

where the group structure is given by the obvious "component wise multiplication".

(2) We define the direct sum of the X_s to be the group

$$\bigoplus_{s\in S} X_s \;:=\; \Big\{ (g_s)_{s\in S} \in \prod_{s\in S} X_s \;\Big|\; \text{all but finitely many } g_s \text{ are trivial} \Big\}.$$

(3) Given $t\in S$ we refer to

$$\begin{array}{rcl} \iota_t\colon X_t & \to & \bigoplus_{s\in S} X_s \\ x & \mapsto & \left(\begin{array}{rcl} S & \to & \bigcup_{s\in S} X_s \\ s & \mapsto & \begin{cases} x, & \text{if } s=t, \\ 0_{X_t}, & \text{if } s\neq t. \end{cases} \end{array} \right) \end{array}$$

as the natural homomorphism.

Lemma 18.1. (Coproduct-of-Abelian Groups Lemma) Let $\{X_s\}_{s\in S}$ be a family of abelian groups.

(1) Let Z be an abelian group and let $\{f_s\colon X_s\to Z\}_{s\in S}$ be a family of homomorphisms. The map

$$\begin{array}{rcl} \bigoplus_{s\in S} f_s\colon \bigoplus_{s\in S} X_s & \to & Z \\ (g_s)_{s\in S} & \mapsto & \sum\limits_{\substack{s\in S \text{ with} \\ g_s\neq e}} f_s(g_s) \end{array}$$

is the unique homomorphism such that for each $s\in S$ we have $f_s = \Phi\circ\iota_s$.

(2) The coproduct in the category $\mathcal{AbGroup}$ of *abelian* groups of $\{X_s\}_{s\in S}$ is given by $\bigoplus_{s\in S} X_s$ together with the above natural homomorphisms $\iota_t\colon X_t\to \bigoplus_{s\in S} X_s$.

Proof. This statement is basically the content of the Coproduct-of-Sets-and-Abelian Groups Lemma 4.4 (2). But of course one can also prove it easily by hand. ■

Convention.

(1) Let $\{X_s\}_{s\in S}$ be a family of abelian groups. For $t\in S$ we use the natural injection $\iota_s\colon X_t\to \bigoplus_{s\in S} X_s$ from the Coproduct-of-Abelian Groups Lemma 18.1 to view X_t as a subgroup of the direct sum $\bigoplus_{s\in S} X_s$.

(2) Let G be an abelian group and let $\{X_s\}_{s\in S}$ be a family of subgroups. The inclusions $X_s\to G$ define a natural homomorphism

$$\bigoplus_{s\in S} X_s \;\to\; G.$$

If this natural homomorphism is a monomorphism, then we refer to the image also as the direct sum of the X_s.

Theorem 18.2. (Chinese Remainder Theorem) If $r, s \in \mathbb{N}$ are coprime, then the map

$$\begin{array}{rcl} \Phi\colon \mathbb{Z}_{r\cdot s} & \to & \mathbb{Z}_r \times \mathbb{Z}_s = \mathbb{Z}_r \oplus \mathbb{Z}_s \\ [k] & \mapsto & ([k],[k]) \end{array}$$

is an isomorphism.

Proof. We leave the elementary algebra proof to the reader. Alternatively see [Fri23]. ■

18.2. Free abelian groups.

Definition. We say that a group π is a free abelian group on a subset $S \subset \pi$, if the following universal property holds: for any *map* $f\colon S \to G$ of the set S to an abelian group G there exists a unique *homomorphism* $\psi\colon \pi \to G$ that makes the following diagram commute

$$\begin{tikzcd} S \arrow[r, hook] \arrow[dr, "f"'] & \pi \arrow[d, "\exists!\,\psi"] \\ & G. \end{tikzcd}$$

We call S a basis of the free abelian group π.

Lemma 18.3. (Free Abelian Group–Uniqueness Lemma) Let G be a free abelian group on a subset $S \subset G$ and let H be a free abelian group on a subset $T \subset H$. Given any bijection $f\colon S \to T$ there exists a unique isomorphism $\psi\colon G \to H$ such that $\psi(s) = f(s)$ for all $s \in S$.

Example.

(1) Let $m \in \mathbb{N}_0$. One can easily verify that $\mathbb{Z}^m$ is a free abelian group on the subset $e_1, \dots, e_m$ given by the m standard basis vectors of $\mathbb{Z}^m$. It follows from the Free Abelian Group–Uniqueness Lemma 18.3 that if G is a free abelian group on a finite set S, then G is isomorphic to $\mathbb{Z}^{\#S}$.

(2) Let π be a free abelian group with basis S. If $\Phi\colon \pi \to \pi$ is a homomorphism such that $\Phi(s) = s$ for all $s \in S$, then it follows immediately from the uniqueness statement of the above definition, that $\Phi = \mathrm{id}$.

Proof of Lemma 18.3. The proof of the uniqueness follows along the same lines as all uniqueness proofs of this kind. We refer to [Fri23] for details. ■

Definition. Let S be a set.

(1) We define

$$\mathbb{Z}^{(S)} := \bigoplus_{s\in S} \mathbb{Z} = \{\text{all maps } S \to \mathbb{Z} \text{ that are zero for all but finitely many } s \in S\}.$$

(2) Let $s \in S$. The map

$$\begin{array}{rcl} S & \to & \mathbb{Z} \\ t & \mapsto & \begin{cases} 1, & \text{if } t = s, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

lies in $\mathbb{Z}^{(S)}$ and it is also called s. Thus we can view S as a subset of $\mathbb{Z}^{(S)}$.

Lemma 18.4. (Free Abelian Group–Existence Lemma) Let S be a set. The group $\mathbb{Z}^{(S)}$ is a free abelian group on the subset $S \subset \mathbb{Z}^{(S)}$.

Definition. Let S be a set. We refer to $\mathbb{Z}^{(S)}$ as *the* free abelian group generated by S.

Proof of Lemma 18.4. Let $f\colon S \to G$ be a map to an abelian group. We consider the map

$$\begin{array}{rcl} \psi\colon \mathbb{Z}^{(S)} & \to & G \\ \sum_{i=1}^{k} n_i \cdot s_i & \mapsto & \sum_{i=1}^{k} n_i \cdot f(s_i). \end{array}$$

It is straightforward to verify that ψ is in fact a homomorphism, that ψ has the desired property and that ψ is the unique such homomorphism. ■

Lemma 18.5. (Free Abelian Group–Splitting Lemma) Let $\varphi\colon G \to A$ be an epimorphism between abelian groups.

(1) If A is a free abelian group, then there exists a right inverse $\sigma\colon A \to G$ of φ.
(2) If $\sigma\colon A \to G$ is a right inverse of φ, then $G = \ker(\varphi) \oplus \sigma(A)$.

Proof. Let $\varphi\colon G \to A$ be an epimorphism between abelian groups.

(1) We assume that A is a free abelian group. We pick a basis B for the free abelian group A. Since $\varphi\colon G \to A$ is an epimorphism we can choose for each $b \in B$ an element $\widetilde{b} \in G$ with $\varphi(\widetilde{b}) = b$.

Since A is a free abelian group on B there exists a unique homomorphism $\sigma\colon A \to G$ with $\sigma(b) = \widetilde{b}$ for all $b \in B$. For each $b \in \widetilde{B}$ we have $(\varphi \circ \sigma)(b) = \varphi(\widetilde{b}) = b$. It follows from the discussion on page 161 that $\varphi \circ \sigma = \mathrm{id}_A$.

(2) Let $\sigma\colon A \to G$ be a right inverse of φ. We start out with the following claim:

Claim. We have $\ker(\varphi) \cap \sigma(A) = \{0\}$.

Proof. If $g = \sigma(a) \in \ker(\varphi) \cap \sigma(A)$, then

$$g = \sigma(a) = (\sigma \circ \mathrm{id}_A)(a) \underset{\substack{\uparrow \\ \text{since } \varphi\circ\sigma = \mathrm{id}_A}}{=} \sigma(\varphi(\sigma(a))) \underset{\substack{\uparrow \\ \text{since } g = \sigma(a)}}{=} \sigma(\varphi(g)) \underset{\substack{\uparrow \\ \text{since } g \in \ker(\varphi)}}{=} \sigma(0) = 0.$$

⊞

As we discussed on page **??**, it follows from the above claim that the natural homomorphism

$$\begin{array}{rcl} \Theta\colon \ker(\varphi) \oplus \sigma(A) & \to & G \\ g \oplus h & \mapsto & g + h \end{array}$$

is a monomorphism. It remains to show that Θ is an epimorphism. Let $g \in G$. Note that

$$\varphi(g - \sigma(\varphi(g))) = \varphi(g) - \varphi(\sigma(\varphi(g))) \underset{\substack{\uparrow \\ \text{since } \varphi\sigma = \mathrm{id}_A}}{=} \varphi(g) - \varphi(g) = 0.$$

This shows that $g - \sigma(\varphi(g)) \in \ker(\varphi)$. It follows that $g \in \mathrm{im}(\sigma) \oplus \ker(\varphi)$. ■

Theorem 18.6. (Basis Cardinality Theorem) Let F be an abelian group. If F is free on two subsets S and T, i.e. if S and T are bases for F, then $\#S = \#T$.

Proof. The proof of this statement is basically the same as the proof that in a vector space any two bases have the same cardinality. This proof is given in detail in [Loe14, Chapter 16.7]. ■

Definition. If F is a free abelian group on a subset S, then we define

$$\operatorname{rank}(F) \;:=\; \#S \quad \in \mathbb{N}_0 \cup \{\infty\}.$$

Theorem 18.7. (Free Abelian Subgroup Theorem) Let F be a free abelian group. Every subgroup H of F is a free abelian subgroup with $\operatorname{rank}(H) \leq \operatorname{rank}(F)$.

Proof. We outsource the proof of this purely algebraic fact to [Fri23]. Alternatively see [Rot88, Theorem 9.3]. ■

18.3. Finitely generated abelian groups.

Definition.

(1) Given an abelian group G we say that $S \subset G$ is a generating set if the natural homomorphism $\Xi_S \colon \mathbb{Z}^{(S)} \to G$ is an epimorphism.
(2) We call an abelian group finitely generated if it admits a finite generating set.

Lemma 18.8. (Finite Generation Lemma) Let $\varphi \colon A \to B$ be a homomorphism of abelian groups.

(1) If φ is an epimorphism and if A is finitely generated, then so is B.
(2) If φ is a monomorphism and if B is finitely generated, then so is A.

Proof.

(1) Let $\varphi \colon A \to B$ be an epimorphism of abelian groups. If S is a finite generating set for A, then clearly the finite set $\varphi(S)$ is a generating set for B.
(2) We only need to show that if B is a finitely generated group, then so is every subgroup A of B. Let T be a finite generating set for B. We denote by $\Xi_T \colon \mathbb{Z}^{(T)} \to B$ the natural epimorphism. Note that $\Xi_T^{-1}(A)$ is a subgroup of $\mathbb{Z}^{(T)}$. By the Free Abelian Subgroup Theorem 18.7 we know that $\Xi_T^{-1}(A)$ is a free abelian group of rank $\leq \#T$. Let S be a basis for $\Xi_T^{-1}(A)$. It follows immediately from the definitions that $\Xi_T(S)$ is a generating set for A. Since $\#\Xi_T(S) \leq \#S \leq \#T$ we see that A is finitely generated. ■

Lemma 18.9. (Matrix Quotient Lemma)

(1) Let H be an abelian group with finite generating set $S = \{s_1, \ldots, s_m\}$. There exists an $n \in \mathbb{N}_0$ and an $m \times n$–matrix A over $\mathbb{Z}$ such that the map

$$\begin{array}{ccc} \mathbb{Z}^m / A \cdot \mathbb{Z}^n & \to & H \\ \left[\begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}\right] & \mapsto & \sum\limits_{i=1}^{m} v_i \cdot s_i \end{array}$$

is a well-defined isomorphism.
(2) If $A \in \mathrm{M}(n \times n, \mathbb{Z})$ is a diagonal matrix with entries $d_1, \ldots, d_n \in \mathbb{Z}$, then

$$\mathbb{Z}^n / A \cdot \mathbb{Z}^n \;\cong\; \bigoplus_{i=1}^{n} \mathbb{Z}_{d_i}.$$

(3) Let A be an $m \times n$–matrix over $\mathbb{Z}$. For matrices $P \in \mathrm{GL}(m, \mathbb{Z})$ and $Q \in \mathrm{GL}(n, \mathbb{Z})$ the map

$$\begin{array}{ccc} \mathbb{Z}^m / A \cdot \mathbb{Z}^n & \cong & \mathbb{Z}^m / PAQ \cdot \mathbb{Z}^n \\ [v] & \mapsto & [P \cdot v] \end{array}$$

is well-defined and it is an isomorphism.

Proof.

(1) Let H be an abelian group with finite generating set $S = \{s_1, \dots, s_m\}$. By definition the natural homomorphism $\Xi_S \colon \mathbb{Z}^{(S)} \to H$ is an epimorphism. Let $\Theta \colon \mathbb{Z}^m \to \mathbb{Z}^{(S)}$ be the natural isomorphism given by $\Theta(e_i) = s_i$. Note that $\Xi_S \circ \Theta \colon \mathbb{Z}^m \to H$ is also an epimorphism. By the Free Abelian Subgroup Theorem 18.7 we know that $\ker(\Xi_s) \subset \mathbb{Z}^m$ is a finitely generated free abelian subgroup. Let $a_1, \dots, a_n \in \mathbb{Z}^m$ be a basis for this free abelian group. We set $A := (a_1 \ \dots \ a_n) \in \mathrm{M}(m \times n)$. One can easily verify that this matrix has the desired properties.

(2) This statement follows easily from the definitions.

(3) First note that for $Q \in \mathrm{GL}(n, \mathbb{Z})$ we have $Q \cdot \mathbb{Z}^n = \mathbb{Z}^n$. It follows that the given map is well-defined. An inverse is given by $[w] \mapsto [P^{-1} \cdot w]$. This shows that the given map is an isomorphism. ■

Theorem 18.10. (Smith Normal Form Theorem) Let A be an $m \times n$–matrix over $\mathbb{Z}$. There exist two matrices $P \in \mathrm{GL}(m, \mathbb{Z})$ and $Q \in \mathrm{GL}(n, \mathbb{Z})$ and $d_1, \dots, d_t \in \mathbb{N}$ such that

$$P \cdot A \cdot Q \ = \ \begin{pmatrix} d_1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & d_t & 0 \\ 0 & 0 & 0 & 0_{(m-t)\times(n-t)} \end{pmatrix} \ \in \ \mathrm{M}(m \times n, \mathbb{Z}).$$

Proof. We refer to [New72, Chapter II.15] and [Nor12, Chapter 1] for a proof. ■

Theorem 18.11. (Abelian Group-Classification Theorem) Let H be a finitely generated abelian group. There exist prime powers $q_1, \dots, q_k$ such that

$$H \ \cong \ \mathbb{Z}^r \oplus \bigoplus_{i=1}^{k} \mathbb{Z}_{q_i}.$$

Furthermore r is uniquely determined and the prime powers $q_1, \dots, q_k$ are uniquely determined up to permutation.

Sketch of proof. Let H be a finitely generated abelian group. We first prove the existence of the promised isomorphism. Note that it follows from the Matrix Quotient Lemma 18.9 (1), the Smith Normal Form Theorem 18.10 and the Matrix Quotient Lemma 18.9 (2) and (3) that there exist $r \in \mathbb{N}$ and $d_1, \dots, d_t \in \mathbb{N}$ such that

$$H \ \cong \ \mathbb{Z}^r \oplus \bigoplus_{i=1}^{t} \mathbb{Z}_{d_i}.$$

The existence of the promised decomposition follows from from the Chinese Remainder Theorem 18.2 which tells us, that each summand $\mathbb{Z}_{d_i}$ is isomorphic to the direct sum of quotients of $\mathbb{Z}$ by prime powers.

It remains to prove the uniqueness of the prime power q_i up to permutation. One can do so by introducing the following invariant of abelian groups: Given a prime p, given $k \in \mathbb{N}$ and given an abelian group V we consider

$$T_{p,k}(V) \ := \ \#\{v \in V \mid p^k \cdot v = 0\}.$$

These numbers are clearly isomorphism invariants of abelian groups. By studying these invariants one can easily prove that the q_i are unique up to permutation. ■

19. Basics of group theory II: The free and amalgamated product of groups

19.1. The free product of groups.

Definition. Let $\{X_s\}_{s\in S}$ be a family of groups. (After possibly replacing the family $\{X_s\}_{s\in S}$ by the family $\{X_s \times \{s\}\}_{s\in S}$ we can and will assume that the groups X_s are pairwise distinct.)

(1) A reduced word in $\{X_s\}_{s\in S}$ is a finite sequence $(x_1, \dots, x_m)$ such that the following conditions are satisfied:
   (a) each x_i lies in one of the groups X_s,
   (b) no x_i is the neutral element of any X_s,
   (c) any two consecutive x_j's lie in two different groups.
   Note that we also allow the "empty sequence" ().

(2) We denote by $\mathop{\ast}\limits_{s\in S} X_s$ the set of all reduced words.

(3) Let $(x_1, \dots, x_m)$ and $(y_1, \dots, y_n)$ be two elements in $\mathop{\ast}\limits_{s\in S} X_s$. We define the product of these two elements as follows:
   (a) First suppose that x_m and y_1 lie in different groups. Then we define
   $$(x_1, \dots, x_m) \cdot (y_1, \dots, y_n) \;:=\; (x_1, \dots, x_m, y_1, \dots, y_n).$$
   (b) Now suppose that x_m and y_1 lie in the same group. Let $j \in \{1, \dots, \min\{m, n\}\}$ be the maximal element such that x_{m+1-i} and y_i lie in the same group and such that $x_{m+1-i} = y_i^{-1}$ for $i = 1, \dots, j$. If $j < \min\{m, n\}$, then we set
   $$(x_1, \dots, x_m) \cdot (y_1, \dots, y_n) \;:=\; (x_1, \dots, \underbrace{x_{m-j} \cdot y_{j+1}}_{\neq e}, \dots, y_n).$$
   If $j = m$, we set $$(x_1, \dots, x_m) \cdot (y_1, \dots, y_n) \;:=\; (y_{m+1}, \dots, y_n).$$
   We proceed similarly if $j = n$.

(4) We refer to $\mathop{\ast}\limits_{s\in S} X_s$ together with this product structure as the free product of the X_s. For a finite family $X_1, \dots, X_k$ of groups we denote the free product by $X_1 * \cdots * X_k$.

Definition. Let t be a symbol. We define
$$\langle t \rangle \;:=\; \{\dots, t^{-2}, t^{-1}, e, t, t^2, \dots\}.$$
The set $\langle t \rangle$ admits an obvious group structure that is given by $t^i \cdot t^j := t^{i+j}$. This group $\langle t \rangle$ is called the infinite cyclic group generated by t. The map $\mathbb{Z} \to \langle t \rangle$, $n \mapsto t^n$ is evidently an isomorphism.

Example. We consider the free product $\langle s \rangle * \langle t \rangle$. Examples of elements in this free product are (s^3, t^{-5}, s^2) and $(s^{-2}, t^4, s^2, t^{-1})$. The product of these two elements is given by
$$(s^3, t^{-5}, s^2) \cdot (s^{-2}, t^4, s^2, t^{-1}) \;=\; (s^3, \underbrace{t^{-5} \cdot t^4}_{=t^{-1}}, s^2, t^{-1}).$$
Note that $(s) \cdot (t) = (s, t) \neq (t, s) = (t) \cdot (s)$. Thus we see that the free product $\langle s \rangle * \langle t \rangle$ is very much non-abelian.

Lemma 19.1. (Free Product Lemma) Let $\{X_s\}_{s\in S}$ be a family of groups. The free product $\mathop{\ast}\limits_{s\in S} X_s$ with the above product is again a group.

Proof. It is clear that the empty sequence is a neutral element. It follows immediately from the definition of the product that the inverse of an element $(x_1,\dots,x_m)\in \mathop{\ast}\limits_{s\in S} X_s$ is given by $(x_m^{-1},\dots,x_1^{-1})$. It remains to show that the multiplication satisfies associativity.

In principle one can show associativity naively "by hand", but there are surprisingly many cases one needs to distinguish, so that the proof becomes rather painful. We refer to [Fri23] for an elegant way to prove associativity. ■

Lemma 19.2. (Coproduct-of-Groups Lemma) Let $\{X_s\}_{s\in S}$ be a family of groups.

(1) (a) Let $t\in S$. The map
$$\begin{aligned} \iota_t\colon X_t &\to \mathop{\ast}\limits_{s\in S} X_s \\ x &\mapsto \begin{cases} (x), & \text{if } x\neq e, \\ (), & \text{if } x=e \end{cases} \end{aligned}$$
is a monomorphism.

(b) Let Z be a group and let $\{f_s\colon X_s\to Z\}_{s\in S}$ be a family of homomorphisms. The map
$$\begin{aligned} \Phi\colon \mathop{\ast}\limits_{s\in S} X_s &\to Z \\ (x_1,\dots,x_m) &\mapsto \underbrace{f_{s_1}(x_1)\cdot\ldots\cdot f_{s_m}(x_m)}_{\text{where } x_i\in X_{s_i}} \end{aligned}$$
is a homomorphism and it is the unique homomorphism such that for each $s\in S$ we have $f_s=\Phi\circ\iota_s$.

(2) The coproduct in the category *Group* of groups of $\{X_s\}_{s\in S}$ is given by $\mathop{\ast}\limits_{s\in S} X_s$ together with the homomorphisms $\iota_t\colon X_t\to \mathop{\ast}\limits_{s\in S} X_s$ from (1a).

Proof. All statements follow easily from the definitions. We refer to [Fri23] for full details. ■

Convention. Let $\{X_s\}_{s\in S}$ be a family of groups.

(1) Let $t\in T$. We refer to
$$\begin{aligned} \iota_t\colon X_t &\to \mathop{\ast}\limits_{s\in S} X_s \\ x &\mapsto \begin{cases} (x), & \text{if } x\neq e, \\ (), & \text{if } x=e \end{cases} \end{aligned}$$
as the natural monomorphism $\iota_t\colon X_t\to \mathop{\ast}\limits_{s\in S} X_s$.

(2) For $t\in S$ we use the natural monomorphism $\iota_s\colon X_t\to \mathop{\ast}\limits_{s\in S} X_s$ to view X_t as a subgroup of the free product $\mathop{\ast}\limits_{s\in S} X_s$.

(3) Let $(x_1,\dots,x_m)\in \mathop{\ast}\limits_{s\in S} X_s$. Since $(x_1,\dots,x_m)=(x_1)\cdot\dots\cdot(x_m)$ we can use (2) to denote this element by $x_1\cdot\ldots\cdot x_m$.

Example. We consider again the free product $\langle s\rangle * \langle t\rangle$. Using the above convention we now write $s^3\cdot t^{-5}\cdot s^2=(s^3,t^{-5},s^2)$ and $s^{-2}\cdot t^4\cdot s^2\cdot t^{-1}=(s^{-2},t^4,s^2,t^{-1})$. The product of

these two elements is given by very reasonable looking

$$(s^3 \cdot t^{-5} \cdot s^2) \cdot (s^{-2} \cdot t^4 \cdot s^2 \cdot t^{-1}) \;=\; s^3 \cdot \underbrace{t^{-5} \cdot t^4}_{=t^{-1}} \cdot s^2 \cdot t^{-1}.$$

Also note that the discussion on page 166 shows that $s \cdot t \neq t \cdot s$, in particular we see that the free product $\langle s \rangle * \langle t \rangle$ is non-abelian.

19.2. **Free groups.**

Definition. We say that a group π is a free group on a subset $S \subset \pi$, if the following universal property holds: for any *map* $f\colon S \to G$ of the set S to a group G there exists a unique *homomorphism* $\psi\colon \pi \to G$ that makes the following diagram commute

$$\begin{tikzcd} S \arrow[r, hook] \arrow[dr, "f"'] & \pi \arrow[d, "\exists!\,\psi"] \\ & G. \end{tikzcd}$$

We call S a basis of the free group π.

Lemma 19.3. (Free Group–Uniqueness Lemma) Let G be a free group on a subset $S \subset G$ and let H be a free group on a subset $T \subset H$. Given any bijection $f\colon S \to T$ there exists a unique isomorphism $\psi\colon G \to H$ such that $\psi(s) = f(s)$ for all $s \in S$.

Proof. The proof of this lemma is basically identical to the proof of the Free Abelian Group–Uniqueness Lemma 18.3. ■

Definition. For a set S we write $\quad \langle S \rangle \;:=\; \mathop{\ast}\limits_{s \in S} \langle s \rangle.$

The lemma below allows us to call $\langle S \rangle$ the free group on the set S. The map $S \to \langle S \rangle$ that is given by $s \mapsto (s)$ is an injection and we use it to view S as a subset of $\langle S \rangle$.

Lemma 19.4. (Free Group–Existence Lemma) Let S be a set. The group $\langle S \rangle$ is a free group on the subset $S \subset \langle S \rangle$.

Proof. Let G be a group and let $f\colon S \to G$ be a map. For $s \in S$ the map

$$\begin{array}{rcl} \psi_s\colon \langle s \rangle & \to & G \\ s^n & \mapsto & f(s)^n \end{array}$$

is evidently the unique homomorphism $\psi_s\colon \langle s \rangle \to G$ with $\psi_s(s) = f(s)$. It now follows from the Coproduct-of-Groups Lemma 19.2 (2) that the map

$$\begin{array}{rcl} \psi\colon \langle S \rangle = \mathop{\ast}\limits_{s \in S} \langle s \rangle & \to & G \\ s_{i_1}^{n_1} \cdot \ldots \cdot s_{i_k}^{n_k} & \mapsto & f(s_{i_1})^{n_1} \cdot \ldots \cdot f(s_{i_k})^{n_k} \end{array}$$

is the unique homomorphism $\langle S \rangle \to G$ with $\psi(s) = \psi_s(s) = f(s)$ for $s \in S$. ■

Definition.

(1) For a set S we refer to $\langle S \rangle = \mathop{\ast}\limits_{s \in S} \langle s \rangle$ as *the* free group on the set S.

(2) Let $s_1, \ldots, s_k$ be distinct elements of a set. We refer to

$$\langle s_1, \ldots, s_k \rangle \;:=\; \langle s_1 \rangle * \cdots * \langle s_k \rangle$$

as the free group on the generators $s_1, \ldots, s_k$.

19.3. The abelianization of a group.

Definition. Let G be a group.

(1) Given $x, y \in G$ we refer to $[x, y] := x \cdot y \cdot x^{-1} \cdot y^{-1}$ as the commutator of x and y.

(2) We refer to
$$[G, G] \;:=\; \left\{ \prod_{i=1}^{n} [x_i, y_i] \,\middle|\, x_1, y_1, \ldots, x_n, y_n \in G \right\}$$
as the commutator subgroup of G.

Lemma 19.5. (Commutator Subgroup Lemma) Let G be a group.

(1) The commutator subgroup $[G, G]$ is a normal subgroup of G.

(2) The group $G_{\mathrm{ab}} := G/[G, G]$ is abelian.

(3) Let $\alpha\colon G \to H$ be a homomorphism to an abelian group. There exists a unique homomorphism $\beta\colon G_{\mathrm{ab}} = G/[G, G] \to H$ such that the following diagram commutes:
$$\begin{tikzcd} G \arrow[d, "g \mapsto [g]"'] \arrow[dr, "\alpha"] & \\ G_{\mathrm{ab}} = G/[G, G] \arrow[r, "\exists!\,\beta"'] & H \text{ abelian}. \end{tikzcd}$$

Proof. Let G be a group.

(1) We leave the straightforward verification of this statement to the reader, alternatively see [Fri23].

(2) For any two elements $x, y \in G$ we have
$$xy\,[G, G] \;=\; yx \cdot x^{-1}y^{-1}xy\,[G, G] \;=\; yx \cdot \underbrace{[x^{-1}, y^{-1}]}_{\in [G,G]}\,[G, G] \;=\; yx\,[G, G].$$
This shows that $G_{\mathrm{ab}} = G/[G, G]$ is indeed abelian.

(3) Let G be a group and let $\alpha\colon G \to H$ be a homomorphism to an abelian group. Using the fact that H is abelian one can easily verify that $[G, G]$ lies in the kernel of α. The desired statement now follows immediately from the universal property of quotient groups. ■

Definition. Let G be a group.

(1) We call
$$G_{\mathrm{ab}} \;:=\; G/[G, G]$$
the abelianization of G.

(2) We denote by
$$\begin{aligned} \Phi_G\colon G &\to G_{\mathrm{ab}} = G/[G, G] \\ g &\mapsto [g] \end{aligned}$$
the natural projection.

(3) Let $\alpha\colon G \to H$ be a homomorphism. By the Commutator Subgroup Lemma 19.5 (3) there exists a unique homomorphism $\alpha_*\colon G_{\mathrm{ab}} \to H_{\mathrm{ab}}$ such that the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, two heads, "\Phi_G"'] & H \arrow[d, two heads, "\Phi_H"] \\ G_{\mathrm{ab}} \arrow[r, "\alpha_*"] & H_{\mathrm{ab}}. \end{tikzcd}$$

We refer to α_* as the induced homomorphism.

Proposition 19.6. (Abelianization Functor Proposition)

(1) The maps
$$\begin{array}{rcl} G & \mapsto & G_{\mathrm{ab}} = G/[G,G] \\ (\alpha\colon G \to H) & \mapsto & (\alpha_*\colon G_{\mathrm{ab}} \to H_{\mathrm{ab}}) \end{array}$$
define a covariant functor from the category *Group* of groups to the category *AbGroup* of abelian groups.

(2) If $\varphi\colon G \to H$ is an isomorphism of groups, then the induced map $\varphi_*\colon G_{\mathrm{ab}} \to H_{\mathrm{ab}}$ is also an isomorphism.

Proof.

(1) This statement follows easily from the definitions.

(2) This statement follows from (1) and the fact that functors turn isomorphisms into isomorphisms. ■

Lemma 19.7. (Abelianization-Epimorphism Lemma) If $\alpha\colon G \to H$ is an epimorphism, then the induced map $\alpha_*\colon G_{\mathrm{ab}} \to H_{\mathrm{ab}}$ is also an epimorphism.

Proof. By definition of α_* we have the following commutative diagram:

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, two heads, "\Phi_G"'] & H \arrow[d, two heads, "\Phi_H"] \\ G_{\mathrm{ab}} \arrow[r, "\alpha_*"] & H_{\mathrm{ab}}. \end{tikzcd}$$

By hypothesis $\alpha\colon G \to H$ is an epimorphism. It follows that $\Phi_H \circ \alpha\colon G \to H_{\mathrm{ab}}$ is an epimorphism, which implies that α_* is an epimorphism. ■

Proposition 19.8. (Abelianization Calculation Proposition)

(1) If π is abelian, then the natural homomorphism $\Phi_\pi\colon \pi \to \pi_{\mathrm{ab}} = \pi/[\pi,\pi]$ is an isomorphism.

(2) Let $\{X_s\}_{s\in S}$ be a family of groups. We have natural homomorphisms
$$\begin{array}{rcl} \left(\mathop{\ast}_{s\in S} X_s\right)_{\mathrm{ab}} & \to & \bigoplus_{s\in S} (X_s)_{\mathrm{ab}} \\ [(x_1,\ldots,x_k)] & \mapsto & \sum_{i=1}^{k} [x_i] \end{array} \quad \text{and} \quad \begin{array}{rcl} \bigoplus_{s\in S} (X_s)_{\mathrm{ab}} & \to & \left(\mathop{\ast}_{s\in S} X_s\right)_{\mathrm{ab}} \\ \sum_{i=1}^{k} [x_i] & \mapsto & [x_1 \cdot \ldots \cdot x_k]. \end{array}$$
These are inverses of one another, in particular they are isomorphisms.

(3) Let S be a set. The map
$$\begin{array}{rcl} \langle S\rangle_{\mathrm{ab}} & \to & \mathbb{Z}^{(S)} \\ [s_{i_1}^{n_1} \cdot \ldots \cdot s_{i_k}^{n_k}] & \mapsto & \sum_{i=1}^{k} n_i \cdot s_i \end{array}$$
is a natural isomorphism.

Proof of the Abelianization Calculation Proposition 19.8.

(1) If π is abelian, then every commutator is trivial, thus $[\pi, \pi] = \{e\}$. This implies that the natural homomorphism $\Phi_\pi \colon \pi \to \pi/[\pi, \pi]$ is an isomorphism.
(2) One can easily verify that the maps are well-defined homomorphisms and inverses of one another.
(3) This statement follows immediately from the statements (1) and (2) together with the observation that by definition we have $\langle S \rangle = \mathop{\ast}_{s \in S} \langle s \rangle$ and $\mathbb{Z}^{(S)} = \bigoplus_{s \in \mathbb{Z}} \mathbb{Z}$. ■

Proposition 19.9. (Basis Cardinality Proposition) Let G be a group that is free on a subset S and let H be a group that is free on a subset $T \subset H$. If G and H are isomorphic, then $\#S = \#T$.

Proof. This proposition follows easily from the Abelianization Functor Proposition 19.6 (2) and the Basis Cardinality Theorem 18.6. ■

19.4. The amalgamated product of groups.

Definition. Let G be a group and let $A \subset G$ be a subset. We define

$$\text{subgroup of } G \text{ normally generated by } A \;:=\; \langle\langle A \rangle\rangle \;:=\; \bigcap_{\substack{A \subset H \lhd G \\ H \text{ normal subgroup}}} H,$$

i.e. $\langle\langle A \rangle\rangle$ is the intersection of all normal subgroups H of G that contain A. This is again a normal subgroup of G.

Lemma 19.10. (Normal Closure–Factorization Lemma) Let G be a group and let $A \subset G$ be a subset. Let $\alpha\colon G \to \Gamma$ be a group homomorphism such that $\alpha(a) = e \in \Gamma$ for all $a \in A$. Then there exists a unique group homomorphism $\beta\colon G/\langle\langle A \rangle\rangle \to \Gamma$ such that the following diagram commutes:

$$\begin{tikzcd} G \arrow[d] \arrow[dr, "\alpha"] & \\ G/\langle\langle A \rangle\rangle \arrow[r, "\exists!\,\beta"'] & \Gamma. \end{tikzcd}$$

Proof. Let G be a group, let $A \subset G$ be a subset and let $\alpha\colon G \to \Gamma$ be a group homomorphism such that $\alpha(a) = e \in \Gamma$ for all $a \in A$. This means that $A \subset \ker(\alpha)$. Since $\ker(\alpha)$ is a normal subgroup of G it follows immediately from the definition of $\langle\langle A \rangle\rangle$ that $\langle\langle A \rangle\rangle \subset \ker(\alpha)$. The lemma follows from the universal property of the quotient group. ■

Proposition 19.11. (Group Pushout Proposition) Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. We consider the group

$$A *_G B \;:=\; (A * B)/\langle\langle \{\alpha(g) \cdot \beta(g)^{-1}\}_{g \in G} \rangle\rangle$$

and the maps

$$\begin{array}{rcl} \mu\colon A & \to & A *_G B \\ a & \mapsto & [(a)] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \nu\colon B & \to & A *_G B \\ b & \mapsto & [(b)] \end{array}$$

The following statements hold:

(1) (a) The maps $\mu\colon A \to A *_G B$ and $\nu\colon B \to A *_G B$ are homomorphisms.

(b) Suppose we are given two homomorphisms $\mu'\colon A \to W$ and $\nu'\colon B \to W$ to some group W such that $\mu' \circ \alpha = \nu' \circ \beta\colon G \to W$. By the Coproduct-of-Groups Lemma 19.2 the homomorphisms μ' and ν' induce a homomorphism $A * B \to W$. This homomorphism descends to a homomorphism

$$\Phi\colon A *_G B \;\to\; W.$$

This homomorphism is the unique homomorphism $\Phi\colon A *_G B \to W$ with $\Phi \circ \mu = \mu'$ and $\Phi \circ \nu = \nu'$. In particular we obtain the following commutative diagram:

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, "\beta"] & A \arrow[d, "\mu"', "a \mapsto [(a)]"] \arrow[ddr, bend left, "\mu'"] & \\ B \arrow[r, "\nu"', "b \mapsto [(b)]"] \arrow[drr, bend right, "\nu'"'] & A *_G B \arrow[dr, dashed, "\exists!\,\Phi"] & \\ & & W. \end{tikzcd}$$

(2) The group $A*_G B$ with the natural homomorphisms $\alpha\colon A \to A*_G B$ and $\beta\colon B \to A*_G B$ is the pushout in the category *Group* of groups.

Definition. Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. We write

$$A *_G B \;:=\; (A * B)/\langle\langle \{\alpha(g)\cdot\beta(g)^{-1}\}_{g\in G}\rangle\rangle$$

and we refer to $A *_G B$ as the amalgamated product of A and B with amalgam G. We refer to the obvious homomorphisms $A \to A *_G B$ and $B \to A *_G B$ as the natural homomorphisms.

Proof.

(1) (a) The statement follows from the Coproduct-of-Groups Lemma 19.2 (1a).
(b) This statement follows easily from the Coproduct-of-Groups Lemma 19.2 (1b). together with the Normal Closure–Factorization Lemma 19.10.

(2) This statement follows easily from the definitions and Statement (1). ■

Example. We consider the following pushout diagram

$$\begin{tikzcd}
\langle a,b\rangle \arrow[r, "a\mapsto a", "b\mapsto e"'] \arrow[d, "b\mapsto b"', "a\mapsto e"] & \langle a\rangle \arrow[d] \\
\langle b\rangle \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \langle a\rangle *_{\langle a,b\rangle} \langle b\rangle.
\end{tikzcd}$$

It follows easily from the universal property of a pushout or from the explicit description of the amalgamated product that $\langle a\rangle *_{\langle a,b\rangle} \langle b\rangle = \{e\}$.

Lemma 19.12. (Amalgamated Product Lemma)

(1) For two groups A and B and the trivial group $G = \{e\}$, the corresponding amalgamated product is the usual free product, i.e.

$$A *_{\{e\}} B \;= A * B.$$

(2) Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be group homomorphisms. We consider the diagram

$$\begin{tikzcd}
G \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d] \\
B \arrow[r] \arrow[ur, phantom, "\lrcorner"] & A *_G B.
\end{tikzcd}$$

The following two statements hold:
(a) If the left vertical map β is an isomorphism, then the right vertical map is also an isomorphism and the inverse $A *_G B \to A$ is given by the identity on A and $\beta^{-1}\colon B \to G$.
(b) If the left vertical map β is an epimorphism, then the right vertical map is also an epimorphism.
The analogous statements of course hold with the roles of "A" and "B" swapped.

Proof. As we will now see, all statements follow from formal properties of pushouts:

(1) This statement follows from the fact that $\{e\}$ is the initial object of the category of groups and together with the Pushout Lemma 4.8.

(2) These two statements follows from the Pushout Lemma 4.8 (2a) and (2b). ■

Lemma 19.13. (Amalgamated Product–Quotient Lemma) If $\alpha\colon G \to A$ is a group homomorphism and if $\beta\colon G \to \{e\}$ is the trivial homomorphism, then the projection $A \to A/\langle\langle \alpha(G) \rangle\rangle$ induces a natural isomorphism

$$A *_G \{e\} \xrightarrow{\cong} A/\langle\langle \alpha(G) \rangle\rangle.$$

Proof. The statement follows immediately from the definition of the amalgamated product and the natural isomorphism $A * \{e\} \xrightarrow{\cong} A$. ■

20. The Seifert–van Kampen Theorem

20.1. The formulation of the general Seifert–van Kampen Theorem.

Theorem 20.1. (Seifert–van Kampen Theorem) Let X be a topological space and let $X = U \cup V$ be a decomposition of X in two *open* subsets U and V such that $U \cap V$ is non-empty and *path-connected*. Let $x_0 \in U \cap V$. We denote by $i\colon U \cap V \to U$ and $j\colon U \cap V \to V$ the inclusions.

(1) (a) The inclusions $k\colon U \to X$ and $l\colon V \to X$ induce an isomorphism

$$\Phi\colon \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \xrightarrow{\cong} \pi_1(X, x_0)$$

which has the property that the following diagram commutes:

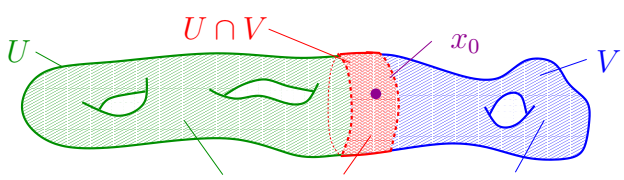
$$\begin{tikzcd}
\pi_1(U\cap V, x_0) \arrow[r, "i_*"] \arrow[d, "j_*"'] & \pi_1(U, x_0) \arrow[d] \arrow[ddr, bend left, "k_*"] & \\
\pi_1(V, x_0) \arrow[r] \arrow[drr, bend right, "l_*"'] & \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \arrow[dr, "\Phi", "\cong"'] & \\
& & \pi_1(X, x_0).
\end{tikzcd}$$

Here the "undecorated" maps are natural homomorphism to the amalgamated product.

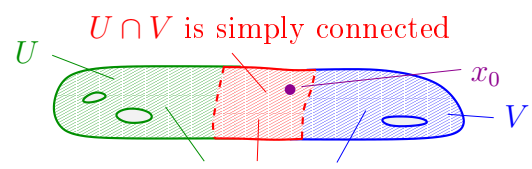
(b) The group $\pi_1(X, x_0)$ is a pushout.

(2) If $U \cap V$ is simply connected, then the inclusions $U \to X$ and $V \to X$ induce an isomorphism

$$\Phi\colon \pi_1(U, x_0) * \pi_1(V, x_0) \xrightarrow{\cong} \pi_1(X, x_0)$$



$\pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \xrightarrow{\cong} \pi_1(X, x_0)$



$\pi_1(U, x_0) * \pi_1(V, x_0) \xrightarrow{\cong} \pi_1(X, x_0)$

Summary of proof. First note that Statement (2) is a special case of Statement (1a). Furthermore note that Statement (1b) follows from Statement (1a) and the Group Pushout Proposition 6.12 (2),

Thus it remains to prove Statement (1a). By the Group Pushout Proposition 6.12 (1b) there exists a unique homomorphism

$$\Phi\colon \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \to \pi_1(X, x_0)$$

such that the following diagram commutes:

$$\begin{tikzcd}
\pi_1(U\cap V,x_0) \arrow[r, "i_*"] \arrow[d, "j_*"'] & \pi_1(U,x_0) \arrow[d] \arrow[ddr, bend left, "k_*"] & \\
\pi_1(V,x_0) \arrow[r] \arrow[drr, bend right, "l_*"'] & \pi_1(U,x_0) *_{\pi_1(U\cap V,x_0)} \pi_1(V,x_0) \arrow[dr, "\exists!\,\Phi"] & \\
& & \pi_1(X,x_0).
\end{tikzcd}$$

It remains to show that Φ is surjective and that Φ is injective. The plan is as follows:

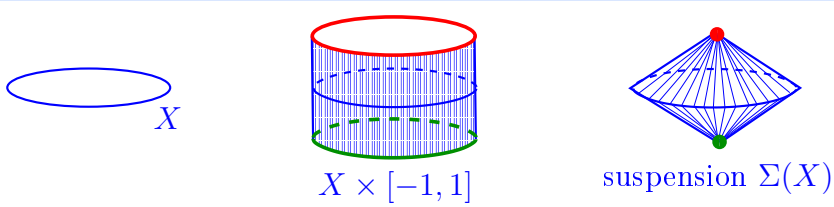
(a) In Section 20.2 we will see that Φ is surjective.

(b) In Section 20.4 we will show that Φ is injective by showing that there exists a map $\Psi\colon \pi_1(X,x_0) \to \pi_1(U,x_0) *_{\pi_1(U\cap V,x_0)} \pi_1(V,x_0)$ with $\Psi\circ\Phi = \mathrm{id}$. To construct the map Ψ it is convenient to work with an alternative description of free and amalgamated products. We will explain this alternative point of view in Section 20.3. ■

Definition. Let X be a non-empty topological space. We define the suspension of X to be the topological space

$$\Sigma(X) \;:=\; (X\times[-1,1])/\sim$$

where we identify all points in $X\times\{-1\}$ to a single point and we identify all points in $X\times\{1\}$ to a single point.



Lemma 20.2. (Suspension-π_1-Lemma) If X is a path-connected non-empty topological space, then its suspension $\Sigma(X) = (X\times[-1,1])/\sim$ is simply connected.

Example. One can easily show that the suspension $\Sigma(S^n)$ of the n-dimensional sphere is homeomorphic to the $(n+1)$-dimensional sphere S^{n+1}. By the Ball-Sphere-Connected Lemma 2.17 we know that for $n\geq 1$ the sphere S^n is path-connected. Thus it follows from the Suspension-π_1-Lemma 20.2 that for $n\geq 1$ we have $\pi_1(S^{n+1}) = 0$. In other words, we have obtained a new proof of the Sphere-π_1-Proposition 11.14.

Proof. Let X be a path-connected non-empty topological space. In the following we denote by $p\colon X\times[-1,1]\to\Sigma(X)$ the natural projection. We consider the two subsets $U = p(X\times[-1,\frac{1}{2}))$ and $V = p(X\times(-\frac{1}{2},1])$. It follows from the definition of the quotient topology on $\Sigma(X)$ that U and V are open subsets of $\Sigma(X)$. We pick a base point x on $X = X\times\{0\}\subset\Sigma(X)$.
We calculate that

$$\pi_1(\Sigma(X),x) \;\underset{\uparrow}{\cong}\; \pi_1(U,x) *_{\pi_1(U\cap V,x)} \pi_1(V,x) \;\underset{\uparrow}{=}\; \begin{array}{c}\text{amalgamated product}\\ \text{of trivial groups}\end{array} \;=\; \text{trivial group.}$$

by the Seifert–van Kampen -Theorem 20.1 which we can apply since $U\cap V\cong X\times(-\frac{1}{2},\frac{1}{2})$ is path-connected

one can easily show that U and V are contractible, hence their fundamental groups are trivial by the Homotopy-π_1-Proposition 13.4 (3)

We have thus shown that $\Sigma(X)$ is indeed simply connected. ■

20.2. Proof of the Seifert–van Kampen-Theorem 20.1: Surjectivity.

Proof of the Seifert–van Kampen-Theorem 20.1: Surjectivity. Let X be a topological space and let $U, V \subset X$ be two open subsets with $X = U \cup V$ and such that $U \cap V$ is path-connected and non-empty. We choose a base point $x_0 \in U \cap V$. We need to show that the map

$$\Phi\colon \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \;\to\; \pi_1(X, x_0)$$

from page **??** is surjective. We consider the following commutative diagram of homomorphisms:

$$\begin{tikzcd}
\pi_1(U,x_0) * \pi_1(V,x_0) \arrow[d] \arrow[drr, "\Xi"] & & \\
\pi_1(U,x_0) *_{\pi_1(U\cap V,x_0)} \pi_1(V,x_0) \arrow[rr, "\Phi"'] & & \pi_1(X,x_0)
\end{tikzcd}$$

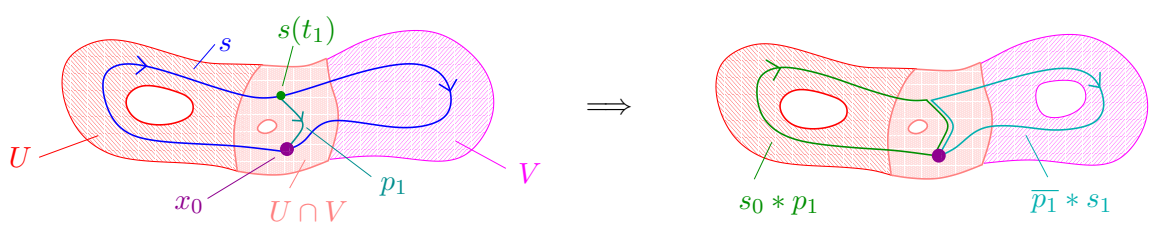
where the horizontal homomorphism Ξ and the left diagonal homomorphism are given by the Coproduct-of-Groups Lemma 19.2 (1b). It remains to prove that the horizontal homomorphism $\Xi\colon \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$ is surjective. Clearly it suffices to prove the following claim.

Claim. Every loop $s\colon [0,1] \to X = U \cup V$ in x_0 is path-homotopic to a loop which is the concatenation of finitely many loops, each of which lies in (U, x_0) or in (V, x_0).

Proof. Let $s\colon [0,1] \to X = U \cup V$ be a loop in x_0. Since U and V are *open* we can use the Lebesgue Lemma 11.9 to find a subdivision

$$0 \;=\; t_0 \;<\; t_1 \;<\; t_2 \;<\; \ldots \;<\; t_k \;<\; t_{k+1} \;=\; 1$$

of the interval $[0,1]$ such that the image of each $s_i := s|_{[t_i, t_{i+1}]}$ lies in U or in V. The same argument as in the proof of the Sphere-π_1-Proposition 11.14 shows that we can assume that each $s(t_i)$ lies in $U \cap V$. The problem is that even though the paths s_i lie in U respectively in V, they are not necessarily loops in x_0. But since $U \cap V$ is path-connected we can find for each $i \in \{1, \ldots, k\}$ a path p_i in $U \cap V$ that connects $s(t_i) \in U \cap V$ to the base point $x_0 \in U \cap V$.



Now we have the following path-homotopies:

by the Reparametrization–Homotopy Lemma 11.1, since s is a reparametrization of $s_0 * \cdots * s_k$

$$s \;\simeq_{\mathrm{p}}\; s_0 * \cdots * s_k \;\simeq_{\mathrm{p}}\; s_0 * p_1 * \overline{p_1} * s_1 * p_2 * \overline{p_2} * s_2 * p_3 * \overline{p_3} * \ldots * \overline{p_k} * s_k$$

since by the Path-Homotopy Product Proposition 11.4 (3) we know that each $p_i * \overline{p_i}$ is null-homotopic

$$\simeq_{\mathrm{p}} (\underbrace{s_0 * p_1}_{\substack{\text{loop in } (U,x_0)\\ \text{or in } (V,x_0)}}) * (\underbrace{\overline{p_1} * s_1 * p_2}_{\substack{\text{loop in } (U,x_0)\\ \text{or in } (V,x_0)}}) * (\underbrace{\overline{p_2} * s_2 * p_3}_{\substack{\text{loop in } (U,x_0)\\ \text{or in } (V,x_0)}}) * (\overline{p_3} * \ldots * (\overline{p_k} * s_k).$$

Thus we have shown that s is path-homotopic to a concatenation of loops, each of which lies in (U, x_0) or in (V, x_0). ■

20.3. Interlude: An alternative definition of the free product of groups.

Definition. Let A and B be two groups.

(1) We set
$$S(A,B) \;:=\; \{\text{all finite sequences } (g_1,\dots,g_k) \text{ with each } g_i \text{ in } A \text{ or } B\}.$$
We equip $S(A,B)$ with the product structure given by
$$(x_1,\dots,x_k)\cdot(y_1,\dots,y_l) \;:=\; (x_1,\dots,x_k,y_1,\dots,y_l).$$
This turns $S(A,B)$ into a monoid.

(2) Let $\alpha\colon G\to A$ and $\beta\colon G\to B$ be two group homomorphisms. We denote by $\sim_G$ the equivalence relation on $S(A,B)$ that is generated by the relations

(a) $$(x_1,\dots,x_l,e,x_{l+2},\dots,x_k) \;\sim_G\; (x_1,\dots,x_l,x_{l+2},\dots,x_k)$$
(b) $$(x_1,\dots,x_l,x_{l+1},\dots,x_k) \;\underset{\substack{\uparrow\\ \text{if both } x_l \text{ and } x_{l+1} \text{ lie in } A \text{ or both lie in } B}}{\sim_G}\; (x_1,\dots,x_l\cdot x_{l+1},\dots,x_k).$$
(c) $$(x_1,\dots,x_l,\alpha(g),x_{l+2},\dots,x_k) \;\underset{\substack{\uparrow\\ \text{for any } g\in G}}{\sim_G}\; (x_1,\dots,x_l,\beta(g),x_{l+2},\dots,x_k).$$

If $G=\{e\}$, then we just write $\sim$ instead of $\sim_G$.

(3) The monoid structure on $S(A,B)$ descends to a well-defined map
$$\begin{array}{rcl} S(A,B)/\!\sim_G \;\times\; S(A,B)/\!\sim_G & \to & S(A,B)/\!\sim_G \\ ([(g_1,\dots,g_k)],[h_1,\dots,h_l]) & \mapsto & [(g_1,\dots,g_k,h_1,\dots,h_l)]. \end{array}$$
One can easily verify that this defines a group structure on $S(A,B)/\!\sim_G$.

Proposition 20.3. (Alternative–Amalgamated Product Proposition) Let A and B be two groups and let $\alpha\colon G\to A$ and $\beta\colon G\to B$ be two group homomorphisms. The map
$$\begin{array}{rcl} \Phi\colon S(A,B)/\!\sim_G & \to & A *_G B = (A*B)/\langle\!\langle\{\alpha(g)\cdot\beta(g)^{-1}\}_{g\in G}\rangle\!\rangle \\ [\underbrace{(x_1,\dots,x_k)}_{\in S(A,B)}] & \mapsto & [\underbrace{x_1\cdot\ldots\cdot x_k}_{\in A*B}] \end{array}$$
is well-defined and it is a group isomorphism.

Auxiliary Lemma 20.4. Let $\Phi\colon G\to H$ and $\Psi\colon H\to G$ be two group homomorphisms such that $\Phi\circ\Psi=\mathrm{id}_H$ and such that Ψ is an epimorphism. Then Φ is an isomorphism.

Proof of Lemma 20.4. It follows from $\Phi\circ\Psi=\mathrm{id}_H$ that Φ is an epimorphism. It remains to show that Φ is a monomorphism. So let $g\in\ker(\Phi)$. Since Ψ is an epimorphism there exists an $h\in H$ with $\Psi(h)=g$. Since $\Phi\circ\Psi=\mathrm{id}_H$ we see that $h=\Phi(\Psi(h))=\Phi(g)=e$. Thus $h=e$ and thus $g=\Psi(h)=\Psi(e)=e$. ■

Proof of Proposition 20.3. We start out with the following claim.

Claim 1. The map
$$\begin{array}{rcl} \Phi\colon S(A,B)/\!\sim_G & \to & A *_G B = (A*B)/\langle\!\langle\{\alpha(g)\cdot\beta(g)^{-1}\}_{g\in G}\rangle\!\rangle \\ [(x_1,\dots,x_k)] & \mapsto & [x_1\cdot\ldots\cdot x_k] \end{array}$$
is well-defined, it is a homomorphism and it is surjective.

Proof. We consider the map
$$\begin{array}{rcl} \widetilde{\Phi}\colon S(A,B) & \to & A *_G B \\ (x_1,\dots,x_k) & \mapsto & [x_1\cdot\ldots\cdot x_k]. \end{array}$$
As above we view $S(A,B)$ as a monoid. With this structure it is clear that $\widetilde{\Phi}$ is a monoid homomorphism. Since $A*B$ is a subset of $S(A,B)$ we see that $\widetilde{\Phi}$ is surjective.

It remains to show that $\widetilde{\Phi}$ descends to a well-defined map on $S(A,B)/\sim_G$. The first two relations in the definition of $\sim_G$ do not pose a problem. But let us consider relation (c). Let $w=(x_1,\dots,x_l)$, $\widetilde{w}=(x_{l+2},\dots,x_k)$ in $S(A,B)$ and let $g\in G$. Note that we have the following equality in $A*_G B$:
$$\begin{aligned} \widetilde{\Phi}(\underbrace{(w,\alpha(g),\widetilde{w})}_{\in S(A,B)}) &\overset{\substack{\text{since } \widetilde{\Phi} \text{ is a monoid homomorphism}\\ \downarrow}}{=} \widetilde{\Phi}(w)\cdot\alpha(g)\cdot\widetilde{\Phi}(\widetilde{w}) &&= \widetilde{\Phi}(w)\cdot\overbrace{\alpha(g)\cdot\beta(g)^{-1}}^{=e\in A*_G B}\cdot\beta(g)\cdot\widetilde{\Phi}(\widetilde{w}) \\ &&&= \widetilde{\Phi}(w)\cdot\beta(g)\cdot\widetilde{\Phi}(\widetilde{w}) = \widetilde{\Phi}(\underbrace{(w,\beta(g),\widetilde{w})}_{\in S(A,B)}). \end{aligned}$$
We have thus verified that $\widetilde{\Phi}$ factors through the relation given by (c). ⊞

Claim 2. The map
$$\begin{array}{rcl} \Psi\colon A*_G B & \to & S(A,B)/\sim_G \\ [\underbrace{(x_1,\dots,x_k)}_{\in A*B}] & \mapsto & [\underbrace{(x_1,\dots,x_k)}_{\in S(A,b)}] \end{array}$$
is well-defined and it is a homomorphism.

Proof. First we consider the map
$$\begin{array}{rcl} \widetilde{\Psi}\colon A*B & \to & S(A,B)/\sim_G \\ (x_1,\dots,x_k) & \mapsto & [(x_1,\dots,x_k)]. \end{array}$$
One can easily verify that this is in fact a group homomorphism. It follows immediately from the definitions that for any $g\in G$ we have
$$\widetilde{\Psi}(\alpha(g)\cdot\beta(g)^{-1}) = [(\alpha(g),\beta(g)^{-1})] \underset{\substack{\uparrow\\ \text{follows from part (c), (b) and (a) of the definition of } \sim_G}}{=} [(\beta(g),\beta(g)^{-1})] \underset{\uparrow}{=} [(\beta(g)\cdot\beta(g)^{-1})] \underset{\uparrow}{=} [()] = e.$$
Thus it follows from the Normal Closure–Factorization Lemma 19.10 that $\widetilde{\Psi}$ descends to a well-defined homomorphism $\Psi\colon A*_G B\to S(A,B)/\sim_G$. ⊞

It is clear that $\Phi\circ\Psi=\mathrm{id}_{A*_G B}$. It is not so obvious that $\Psi\circ\Phi=\mathrm{id}_{S(A,B)/\sim_G}$. In fact we will proceed in a slightly different fashion: to show that Φ is an isomorphism it suffices, according to Lemma 20.4, to show that $\Psi\colon A*_G B\to S(A,B)/\sim_G$ is surjective. This in turn follows from the following claim.

Claim 3. Any element in $S(A,B)$ is equivalent, via $\sim_G$, to an element in $A*B$.

Proof. We prove the claim by induction on the length of a sequence in $S(A,B)$. Clearly the claim holds for the unique sequence in $S(A,B)$ of length 0. Suppose we already know the claim for all sequences of length $\le k-1$. Let $(x_1,\dots,x_k)\in S(A,B)$ be a sequence of length k. If $(x_1,\dots,x_k)$ is a reduced word then by definition it lies in $A*B$ and we are done. Now we suppose that $(x_1,\dots,x_k)$ is not reduced. Then, by the definition on page 166, one of the following occurs:

(1) there exists an $i\in\{1,\dots,k\}$ with $x_i=e$, or
(2) there exists an $i\in\{1,\dots,k-1\}$ with $x_i,x_{i+1}\in A$, or

(3) there exists an $i \in \{1, \dots, k-1\}$ with $x_i, x_{i+1} \in B$.

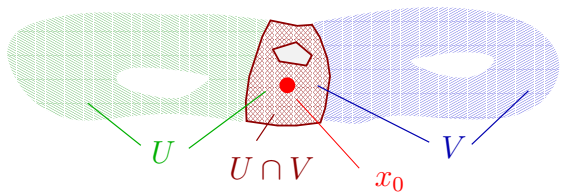
In the first case the sequence is equivalent to $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k)$. In the second and third case the sequence is equivalent to $(x_1, \dots, x_{i-1} \cdot x_{i+1}, \dots, x_k)$. In all three cases we have thus shown that our original sequence is equivalent to a sequence of length $k-1$. So we are done by our induction hypothesis. ■

20.4. Proof of the Seifert–van Kampen-Theorem 20.1: Injectivity.

Proof of the Seifert–van Kampen-Theorem 20.1: Injectivity. We want to show that the map

$$\Phi\colon A *_G B \;=\; \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0) \;\to\; \pi_1(X, x_0)$$

which is induced by the inclusions $i\colon U \to X$ and $j\colon V \to X$ is injective.



To prove injectivity of Φ it is enough to show that there exists a map $\Psi\colon \pi_1(X, x_0) \to A *_G B$ such that $\Psi \circ \Phi = \mathrm{id}_{A *_G B}$. We will prove the existence of such a map Ψ in three steps:

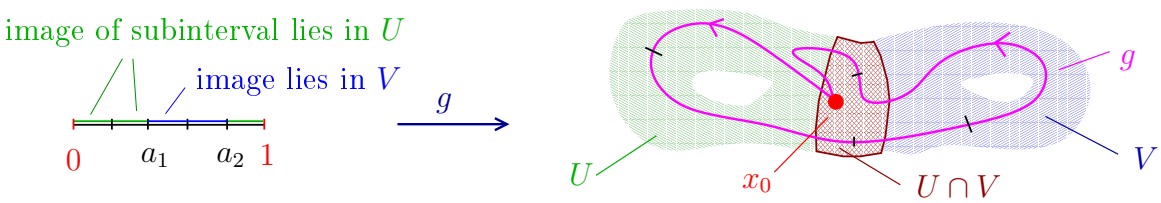
(1) First we construct a map $\overline{\Psi}\colon \{\text{loops in } (X, x_0)\} \to A *_G B$.
(2) Then we show that $\overline{\Psi}$ descends to a well-defined map

$$\begin{aligned} \Psi\colon \pi_1(X, x_0) &\;\to\; A *_G B \\ [g] &\;\mapsto\; \overline{\Psi}(g). \end{aligned}$$

(3) Finally we show that $\Psi \circ \Phi = \mathrm{id}_{A *_G B}$.

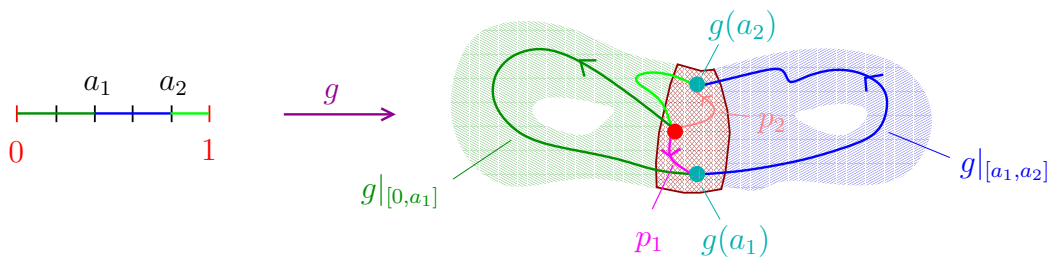
For the remainder of the proof we will identify $A *_G B$ with $S(A, B)/\sim_G$ using the isomorphism given by the Alternative–Amalgamated Product Proposition 20.3.
Step (1). Let $g\colon [0,1] \to X$ be a loop in (X, x_0). We apply the Lebesgue Lemma 11.9 to the map $g\colon [0,1] \to X$ and the open cover $X = U \cup V$. We obtain an N such that for every $i \in \{0, \dots, N-1\}$ the image of the interval $[\frac{i}{N}, \frac{i+1}{N}]$ under the map g lies in U or in V. We introduce a few definitions.

(a) We say a subinterval of $[0,1]$ can be *colored green* if the image of the subinterval under the map g lies in U and we say an interval can be *colored blue* if the image of the subinterval lies in V. Note that the image of a subinterval lies in $U \cap V$ if and only if the subinterval can be colored green as well as blue.
(b) A *coloring* is a map $\alpha\colon \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\} \to \{\text{blue}, \text{green}\}$ such that each subinterval $[i, i + \frac{1}{N}]$ can be colored by $\alpha(i)$.
(c) Given a coloring α we denote by $0 < a_1 < \cdots < a_{k-1} < 1$ the numbers where the coloring of the subintervals changes from one color to another. Note that for each a_i the intervals $[a_i - \frac{1}{N}, a_i]$ and $[a_i, a_i + \frac{1}{N}]$ can be colored in two different colors, which implies that $g(a_i) \in U \cap V$.
(d) A *basing* for a coloring α is a collection of paths $p_1, \dots, p_{k-1}$ in $U \cap V$ such that each path p_i goes from x_0 to $g(a_i)$.



Given a coloring α and a basing $p_1, \ldots, p_{k-1}$ we set

$$\overline{\Psi}(g) \;:=\; \Big[\big[\underbrace{g|_{[0,a_1]} * \overline{p}_1}_{\substack{\text{loop in } (U,x_0) \text{ or} \\ \text{loop in } (V,x_0)}}\big], \big[\underbrace{p_1 * g|_{[a_1,a_2]} * \overline{p}_2}_{\substack{\text{loop in } (U,x_0) \text{ or} \\ \text{loop in } (V,x_0)}}\big], \ldots, \big[\underbrace{p_{k-1} * g|_{[a_{k-1},1]}}_{\substack{\text{loop in } (U,x_0) \text{ or} \\ \text{loop in } (V,x_0)}}\big]\Big] \;\in\; S(A,B)/\sim_G = A *_G B.$$



Our next goal is to prove the following claim.

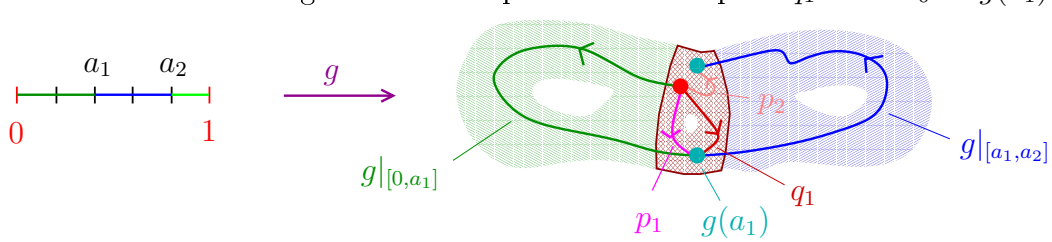
Claim 1. The definition of $\overline{\Psi}(g) \in S(A,B)/\sim_G$ does not depend on any of the choices made.

Proof. We need to prove the following three statements:

(a) Given N and a coloring the definition does not depend on the choice of the basings.
(b) Given N the definition does not depend on the choice of the coloring.
(c) The definition does not depend on the choice of N.

We now prove these three statements:

(a) We first fix a coloring α and we show that the definition of $\overline{\Psi}(g)$ does not depend on the choice of the basing. Thus let us pick a different path q_1 from x_0 to $g(a_1)$.



In this case we have the following equality in $S(A,B)/\sim_G$:

$$\begin{aligned}
&\text{definition of } \overline{\Psi}(g) \text{ using the basing } p_1, p_2, \dots, p_{k-1} \;= \\
&\quad = \big[[g|_{[0,a_1]} * \overline{p}_1],\ [p_1 * g|_{[a_1,a_2]} * \overline{p}_2], \dots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\big] \\
&\quad = \big[[g|_{[0,a_1]} * \overline{p}_1 * \overbrace{(p_1 * \overline{q}_1) * (\underbrace{q_1 * \overline{p}_1}_{\text{loop in } U\cap V})}^{\text{null-homotopic loop}}],\ [p_1 * g|_{[a_1,a_2]} * \overline{p}_2], \dots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\big] \\
&\quad \underset{\uparrow}{=} \big[[g|_{[0,a_1]} * \overline{p}_1 * (p_1 * \overline{q}_1)],\ [(q_1 * \overline{p}_1) * p_1 * g|_{[a_1,a_2]} * \overline{p}_2], \dots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\big]
\end{aligned}$$

note that $q_1 * \overline{p}_1$ is a loop in $U \cap V$, thus it defines elements in $G = \pi_1(U \cap V, x_0)$, thus we get this equality in $S(A,B)/\sim_G$

$$\begin{aligned}
&\quad = \big[[g|_{[0,a_1]} * (\underbrace{\overline{p}_1 * p_1}_{\text{null-homotopic}}) * \overline{q}_1)],\ [q_1 * (\underbrace{\overline{p}_1 * p_1}_{\text{null-homotopic}}) * g|_{[a_1,a_2]} * \overline{p}_2], \dots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\big] \\
&\quad = \big[[g|_{[0,a_1]} * \overline{q}_1],\ [q_1 * g|_{[a_1,a_2]} * \overline{p}_2], \dots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\big] \\
&\quad = \text{definition of } \overline{\Psi}(g) \text{ using the basing } q_1, p_2, \dots, p_{k-1}.
\end{aligned}$$

Basically the same argument (with slightly more messy notation) shows that the choices of the other paths p_i are also irrelevant.

(b) Now we show that the definition does not depend on the choice of the coloring. So suppose we have two colorings α and β. We can turn one coloring into the other by a finite sequence of color swaps of a single subinterval. Therefore it suffices to consider the case that there exists a single $i \in \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\}$ where the values for $\alpha(i)$ and $\beta(i)$ differ.

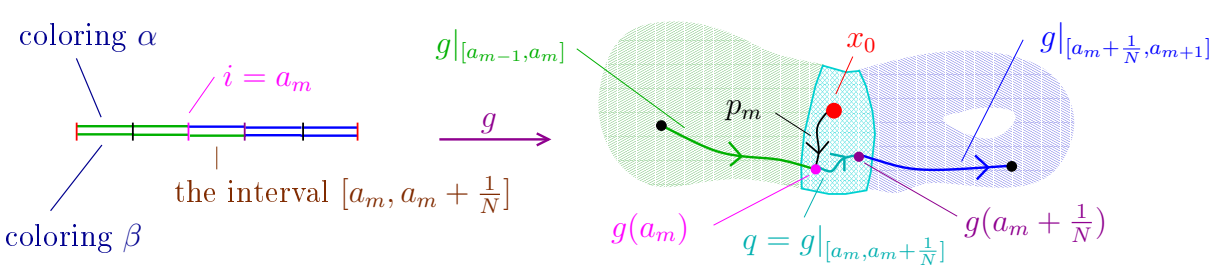
We consider the case, sketched in the figure below, that $\alpha(i) =$ blue, $\beta(i) =$ green and such that the common coloring to the left is green and the common coloring to the right is blue. All other cases are dealt with in a similar way.

Let $0 = a_0 < a_1 < \cdots < a_k = 1$ be the decomposition corresponding to the coloring α. We pick a basing $p_1, \dots, p_{k-1}$ for α. There exists a unique m with $a_m = i$. The corresponding decomposition for the coloring β is given by $0 = a_0 < a_1 < \cdots < a_m + \frac{1}{N} < a_{m+1} < a_k = 1$. We write $q = g_{[a_m, a_m + \frac{1}{N}]}$. Since $[a_m, a_m + \frac{1}{N}]$ can be colored green and blue we see that q lies actually in $U \cap V$. We have

$$\begin{aligned}
&\text{definition of } \overline{\Psi}(g) \text{ using the coloring } \alpha \text{ and the basing } p_1, \dots, p_{k-1} \;= \\
&\quad = \big[[g|_{[0,a_1]} * \overline{p}_1], \dots, [p_{m-1} * g|_{[a_{m-1},a_m]} * \overline{p}_m], [p_m * g|_{[a_m,a_{m+1}]} * \overline{p}_{m+1}], \dots\big] \\
&\quad \underset{\uparrow}{=} \big[[g|_{[0,a_1]} * \overline{p}_1], \dots, [p_{m-1} * g|_{[a_{m-1},a_m+\frac{1}{N}]} * \overline{q} * \overline{p}_m], [p_m * q * g|_{[a_m+\frac{1}{N},a_{m+1}]} * \overline{p}_{m+1}], \dots\big]
\end{aligned}$$

since $g_{[a_{m-1},a_m]} \simeq_{\mathrm{p}} g_{[a_{m-1},a_m+\frac{1}{N}]} * \overline{q}$ and $g_{[a_m,a_m+\frac{1}{N}]} \simeq_{\mathrm{p}} q * g_{[a_m+\frac{1}{N},a_{m+1}]}$

$$= \text{definition of } \overline{\Psi}(g) \text{ using the coloring } \beta \text{ and basing } p_1, \dots, p_{m-1}, p_m * q, p_{m+1}, \dots, p_{k-1}.$$



(c) First suppose that instead of dividing $[0,1]$ into N intervals we divide it into $m \cdot N$ intervals for some $m \in \mathbb{N}$. A coloring for the subdivision into N subintervals then also gives rise to a coloring into $m \cdot N$ subintervals, and it follows immediately from the definitions that we end up with the same element of $A *_G B$. Now if we had chosen a different N' instead, then we can replace N and N' by $N \cdot N'$ and by the argument we just made we obtain the same element of $A *_G B$. This concludes the proof of (c). ⊞

Step (2). We need to prove the following claim.

Claim 2. If $g, g' \colon [0,1] \to X$ are two path-homotopic loops in (X, x_0), then $\overline{\Psi}(g) = \overline{\Psi}(g')$.

Proof. Let $g, g' \colon [0,1] \to X$ be two path-homotopic loops in (X, x_0). We pick a path-homotopy

$$\begin{array}{rcl} H \colon [0,1] \times [0,1] & \to & X \\ (t,s) & \mapsto & H(t,s) \end{array}$$

from the loop g to the loop g'. This time we apply the Lebesgue Lemma 11.9 to the map $H \colon [0,1] \times [0,1] \to X$ and the open cover $X = U \cup V$. We obtain an $N \in \mathbb{N}$ such that for every choice of $i, j \in \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\}$ the image of the square $[i, i + \frac{1}{N}] \times [j, j + \frac{1}{N}]$ under the map H lies in U or in V. Given $a \in [0,1]$ we introduce the two paths

$$\underbrace{\begin{array}{rcl} H_a \colon [0,1] & \to & X \\ t & \mapsto & (t,a) \end{array}}_{\text{“horizontal path”}} \qquad \text{and} \qquad \underbrace{\begin{array}{rcl} V_a \colon [0,1] & \to & X \\ t & \mapsto & (a,t). \end{array}}_{\text{“vertical path”}}$$

Note that each H_a is a loop in x_0 and that $H_0 = g$ and $H_1 = g'$. Therefore it suffices to show that for any $j \in \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\}$ we have $\overline{\Psi}(H_j) = \overline{\Psi}(H_{j+\frac{1}{N}})$. So let $j \in \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\}$.

As above we say that a subset of $[0,1] \times [0,1]$ can be *colored green* if the image of the subset under the map H lies in U and we say the subset can be *colored blue* if the image lies in V. We pick a map $\alpha \colon \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\} \to \{\text{blue}, \text{green}\}$ such that for each i the value $\alpha(i)$ is a color with which the square $[i, i + \frac{1}{N}] \times [\frac{j}{N}, j + \frac{1}{N}]$ can be colored. Note that α also defines a coloring for the loops H_j and $H_{j+\frac{1}{N}}$.

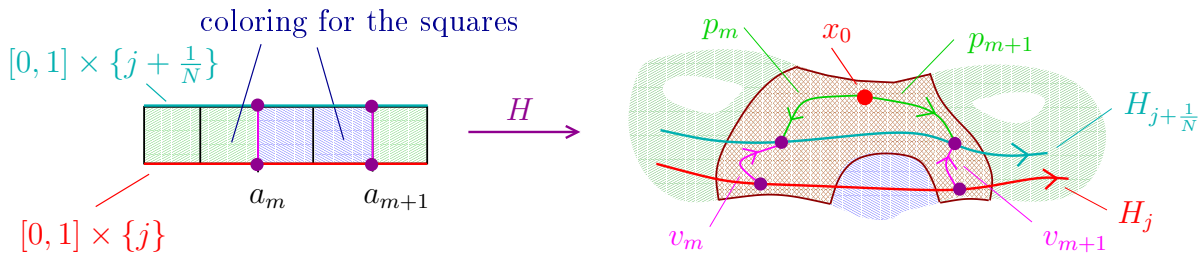
We define $0 = a_0 < a_1 < \cdots < \dots a_k = 1$ as above for the coloring α. We pick a basing p_i, $i = 1, \dots, k-1$ for the loop H_j. For $i = 0, \dots, k$ we write $v_i = V_{a_i}|_{[j, j+\frac{1}{N}]}$. Put differently, v_j connects the point $H_j(a_i)$ to the corresponding point $H_{j+\frac{1}{N}}(a_i)$. Now we have

$$\begin{array}{rcl} \overline{\Psi}(H_j) & = & \overline{\Psi}(H_j) \text{ defined using the coloring } \alpha \text{ and the basing } p_1, \dots, p_{k-1} \\ & = & \big[\dots\big] * \big[p_m * H_j|_{[a_m, a_{m+1}]} * \overline{p}_{m+1}\big] * \big[\dots\big] \\ & \underset{\uparrow}{=} & \big[\dots\big] * \big[p_m * v_m * \overline{v}_m * H_j|_{[a_m, a_{m+1}]} * v_{m+1} * \overline{v}_{m+1} * \overline{p}_{m+1}\big] * \big[\dots\big] \\ & & \text{since each } v_m * \overline{v}_m \text{ is null-homotopic} \\ & \underset{\uparrow}{=} & \big[\dots\big] * \big[p_m * v_m * H_{j+\frac{1}{N}}|_{[a_m, a_{m+1}]} * \overline{v}_{m+1} * \overline{p}_{m+1}\big] * \big[\dots\big] \\ & & \text{since } H \text{ gives rise to path-homotopies from } \overline{v}_m * H_j|_{[a_m, a_{m+1}]} * v_{m+1} \text{ to } H_{j+\frac{1}{N}}|_{[a_m, a_{m+1}]} \text{ for } m = 0, \dots, k-1 \\ & = & \overline{\Psi}(H_{j+\frac{1}{N}}) \text{ defined using } \alpha \text{ and the basing } p_1 * v_1, \dots, p_{k-1} * v_{k-1}. \end{array}$$

This concludes the proof of the claim and thus of part (2) of our three step program. ⊞

Summarizing we now showed that the map $\overline{\Psi}$ descends to a well-defined map

$$\begin{array}{rcl} \Psi \colon \pi_1(X, x_0) & \to & A *_G B \\ [g] & \mapsto & \big[\overline{\Psi}(g)\big]. \end{array}$$



Step (3). It remains to prove the following claim.

Claim 3. We have $\Psi\circ\Phi=\mathrm{id}_{A*_G B}$.

Proof. So let $[y_1]\cdot\ldots\cdot[y_k]\in A*_G B$ where $y_1,\ldots,y_k$ are loops in (U,x_0) and (V,x_0). Then

$$\begin{aligned}(\Psi\circ\Phi)([y_1]\cdot\ldots\cdot[y_k]) &\overset{\substack{\text{definition of } \Phi\colon A*_G B=\pi_1(U,x_0)*\pi_1(V,x_0)\to\pi_1(X,x_0)\\ \downarrow}}{=} \Psi([y_1*\cdots*y_k]) = \left[\overline{\Psi}(y_1*\cdots*y_k)\right] \underset{\substack{\uparrow\\ \text{by definition of } \overline{\Psi}\text{, where we use } N=k \text{ and where}\\ \text{we can use the constant paths as basings since}\\ y_1,\ldots,y_k \text{ are already loops in } (U,x_0) \text{ and } (V,x_0)}}{=} [y_1*\cdots*y_k]\\ &= [y_1]\cdot\ldots\cdot[y_k].\end{aligned}$$

This concludes the proof of the claim and thus also the proof of the Seifert–van Kampen-Theorem 20.1 ■

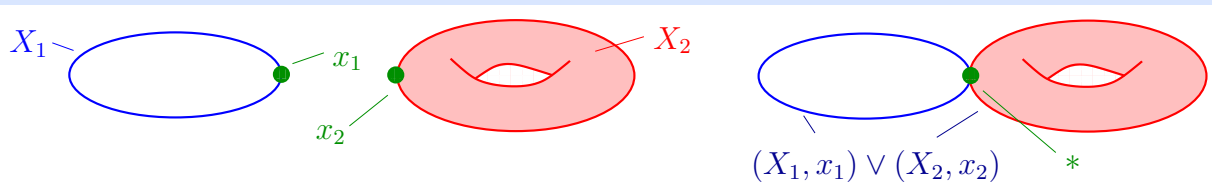
20.5. **Fundamental groups of wedges.**

Definition. Let $\{(X_i,x_i)\}_{i\in I}$ be a non-empty family of pointed topological spaces.

(1) We refer to
$$\bigvee_{i\in I}(X_i,x_i) := \Big(\bigsqcup_{i\in I}X_i\Big)/\sim \quad \text{where } x_i\sim x_j \text{ for all } i,j\in I$$
as the wedge of the pointed topological spaces (X_i,x_i). We refer to the point given by the identifications as the wedge point and we denote it by $*$.

(2) Let $j\in I$. We refer to
$$\begin{aligned}\iota_j\colon X_j &\to \bigvee_{i\in I}(X_i,x_i)\\ x &\mapsto [x]\end{aligned}$$
as the natural inclusion. Note that in the Wedge Proposition 17.1 we showed that each natural inclusion is an embedding, thus we can view each X_j as a subspace of the wedge.

The wedge of the empty family of pointed topological spaces is defined as the one-point topological space $\{*\}$.



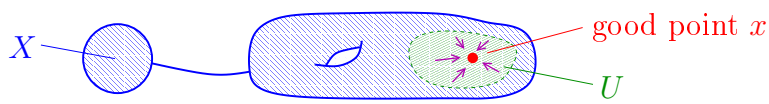
Let $(X_i,x_i)_{i\in I}$ be a family of pointed topological spaces. The natural inclusions ι_{i*} induce a homomorphism

$$\underbrace{\mathop{\ast}_{i\in I} \pi_1(X_i, x_i)}_{\substack{\text{coproduct of the}\\ \text{fundamental groups}}} \;\to\; \pi_1\Big(\underbrace{\bigvee_{i\in I} (X_i, x_i)}_{\substack{\text{coproduct of}\\ \text{the } (X_i, x_i)}} , *\Big).$$

It seems reasonable to expect that this homomorphism is an isomorphism. For simplicity let us restrict ourself for a second to the case that we are dealing with just two topological spaces X_1 and X_2. It is now very tempting to apply the Seifert–van Kampen-Theorem 20.1 to the decomposition $X_1 \vee X_2 = X_1 \cup X_2$. Unfortunately this approach hits a snag: in the Seifert–van Kampen Theorem we have to deal with decompositions of a topological space X into *open* subsets. But in most cases X_1 and X_2 will certainly not be open subspaces of $X_1 \vee X_2$.

Definition. We say a point x in a topological space X is good, if the following two conditions are satisfied:

(1) $\{x\}$ is a closed subset of X .
(2) There exists an open neighborhood U of x such that x is a deformation retract of U.



Lemma 20.5. (Good Point-Lemma)

(1) Every point in a topological manifold (e.g. on a sphere) is good.
(2) Let (X_i, x_i) be a family of pointed topological spaces. If each $x_i \in X_i$ is good, then the wedge point $*$ is a good point in $\bigvee_{i\in I} (X_i, x_i)$.

Proof.

(1) This follows easily from the fact that topological manifolds are by definition Hausdorff and that they are locally homeomorphic to open (half-) balls.
(2) Let (X_i, x_i) be a family of pointed topological spaces such that each $x_i \in X_i$ is good. It follows easily from the Wedge Proposition 17.1 (0) that the wedge point $*$ defines a closed subset of the wedge. For each $i \in I$ let U_i be an open neighborhood of $x_i \in X_i$ such that x_i is a deformation retract of U_i. We can think of $U := \bigvee_{i\in I} U_i$ as an open neighborhood of the wedge point $*$. One can easily show that U admits a deformation retraction to $*$. ■

Proposition 20.6. (Wedge-π_1-Proposition) Let $\{(X_i, x_i)\}_{i\in I}$ be a family of pointed topological spaces. We suppose that each x_i is a good point in X_i. Given $j \in I$ we denote by $\iota_j \colon X_j \to \bigvee_{i\in I} X_i$ the natural inclusion. We denote by $*$ the wedge point of $\bigvee_{i\in I} (X_i, x_i)$. The map

$$\mathop{\ast}_{i\in I} \pi_1(X_i, x_i) \;\to\; \pi_1\Big(\bigvee_{i\in I} (X_i, x_i), *\Big),$$

which is induced by the inclusions ι_i together with the Coproduct-of-Groups Lemma 19.2 (1b), is an isomorphism.

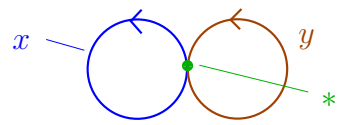
Proof. We postpone the proof of the Wedge-π_1-Proposition 20.6 to the next section. ■

Lemma 20.7. (Wedge-of-Circles–π_1-Lemma) Let S be a set. The map

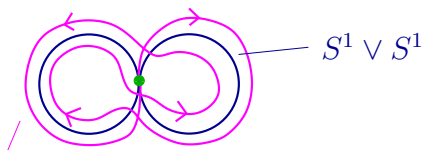
$$\begin{array}{rcl} \langle S\rangle & \to & \pi_1\Big(\bigvee\limits_{s\in S}(S^1,1),*\Big) \\ s & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & \bigvee\limits_{s\in S}(S^1,1) \\ t & \mapsto & [\exp(2\pi\,\mathrm{i}t),s] \end{array}\right] \end{array}$$

that is determined by

is an isomorphism.



$\langle x,y\rangle \xrightarrow{\cong} \pi_1(S^1\vee S^1,*)$



$y^{-1}x^{-1}yx$ is a non-trivial element in $\pi_1(S^1\vee S^1,*)=\langle x,y\rangle$

Proof. We consider the following maps:

$$\langle S\rangle \underset{\text{(a)}}{=} \mathop{\ast}_{s\in S}\langle s\rangle \xrightarrow[\text{(b)}]{\cong} \mathop{\ast}_{s\in S}\pi_1(S^1,1) \xrightarrow[\text{(c)}]{\cong} \pi_1\Big(\bigvee_{s\in S}(S^1,1),*\Big)$$

We now clarify what we mean by the above maps:

(a) By the definition on page 168 we have $\langle S\rangle = \mathop{\ast}\limits_{s\in S}\langle s\rangle$ where $\langle s\rangle = \{\dots,s^{-1},e,s,s^2,\dots\}$.

(b) It follows from the Circle-π_1-Theorem 11.13 that for each $s\in S$ we have an isomorphism from the infinite cyclic group $\langle s\rangle$ to $\pi_1(S^1,1)$ by sending the generator s of $\langle s\rangle$ to the path-homotopy class of the loop given by $t\mapsto\exp(2\pi\,\mathrm{i}t)$.

(c) By the Good Point-Lemma 20.5 (or alternatively by a completely elementary argument) we know that $1\in S^1$ is a good point. The isomorphism is thus given by the Wedge-π_1-Proposition 20.6.

We have now seen that all of the above maps are indeed isomorphisms. The composition is precisely the map that we claimed was an isomorphism. ■

Example. We consider the loop γ from page **??** in $\mathbb{C}\setminus\{\pm1\}$ that is also illustrated in the figure below. One can easily show that there exists a deformation retraction

$$f\colon \mathbb{C}\setminus\{\pm1\} \to S^1\vee S^1 \overset{\text{we use the homeomorphism from page ?? to make this identification}}{=} \underbrace{\{z\in\mathbb{C}\mid |z+1|=1\}\cup\{z\in\mathbb{C}\mid |z-1|=1\}}_{\text{two circles that meet at the origin}}.$$

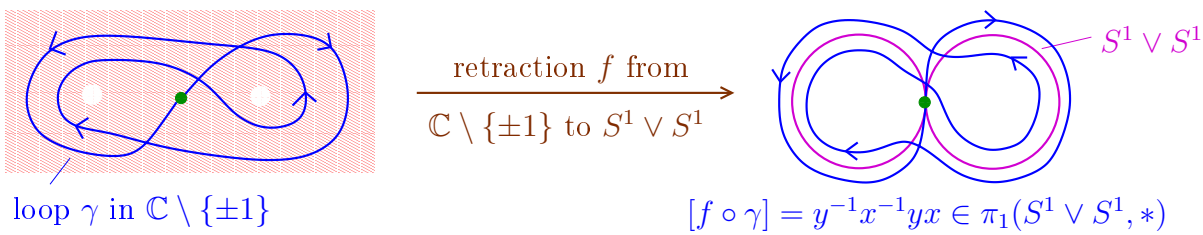
Note that

$$f_*([\gamma]) = [f\circ\gamma] = y^{-1}x^{-1}yx \in \langle x,y\rangle \underset{\text{we use the Wedge-of-Circles–}\pi_1\text{-Lemma 20.7 to make this identification}}{=} \pi_1(S^1\vee S^1,*),$$

which is non-trivial. Thus we see that $f\circ\gamma$ is not null-homotopic in $S^1\vee S^1$. This implies that γ is not null-homotopic in $\mathbb{C}\setminus\{\pm1\}$.

20.6. **Proof of the Wedge-π_1-Proposition 20.6.** In this section we provide the proof of the Wedge-π_1-Proposition 20.6. We proceed as follows:

(1) In the Finite Wedge-π_1-Lemma 20.8 we first deal with the case of the wedge of two pointed topological spaces. By applying this lemma iteratively we get the desired statement also for the wedge of finitely many pointed topological spaces.



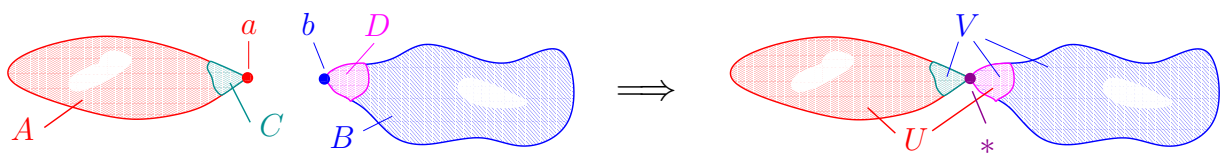
(2) We reduce the general case of the Wedge-π_1-Proposition 20.6 via a compactness argument to the case of the wedge of finitely many pointed topological spaces.

Lemma 20.8. (Finite Wedge-π_1-Lemma) Let A and B be two path-connected topological spaces and let $a \in A$ and $b \in B$ be good points. The inclusion maps induce an isomorphism

$$\pi_1(A, a) * \pi_1(B, b) \xrightarrow{\cong} \pi_1((A, a) \vee (B, b), *).$$

Proof. Let A and B be two path-connected topological spaces and let $a \in A$ and $b \in B$ be good points. All wedges below will be with respect to these two base points, thus, to simplify the notation we will drop them from the notation. In this spirit we write $X = A \vee B$. As usual we denote by $*$ the wedge point in $A \vee B$. By the Wedge Proposition 17.1 (1b) we can view A and B as subspaces of $X = A \vee B$ in the obvious way.

Since a is a good point we can pick an open neighborhood C in A of a that deformation retracts to a. We can also pick an open neighborhood D of b in B that deformation retracts to b. We consider $U := A \vee D$ and $V := B \vee C$.



Next we consider the following diagram

$$\begin{tikzcd}
\pi_1(A, a) * \pi_1(B, b) \arrow[r] \arrow[d] & \pi_1(X, *) \arrow[d, "\mathrm{id}"] \\
\pi_1(U, *) * \pi_1(V, *) \arrow[r] & \pi_1(X, *).
\end{tikzcd}$$

We make the following observations:

(1) The vertical and horizontal maps are all induced by the various inclusions of topological spaces. In particular the diagram commutes.
(2) We can view $U = A \vee D$ and $V = B \vee C$ as open subsets of $X = A \vee B$.
(3) One can easily show that A is a deformation retract of $U = A \vee D$.
(4) As in (3) we see that B is a deformation retract of $V = B \vee C$.
(5) As in (3), we see that $A \cap B = \{*\}$ is a deformation retract of $U \cap V = C \vee D$. It follows from the Homotopy-π_1-Proposition 13.4 (2b) that $U \cap V$ is simply connected.
(6) By (2) and (5) we can apply the Seifert–van Kampen-Theorem 20.1 to the decomposition $A \vee B = U \cup V$ and we obtain that the bottom horizontal map of the above diagram is an isomorphism.

(7) It follows from (3), (4) and the Deformation Retract–Homotopy Equivalence Lemma 9.3 that the inclusions $A \to U = A \cup D$ and $B \to V = B \cup C$ are homotopy equivalences.
(8) It follows from (7) together with the Homotopy-π_1-Proposition 13.4 (2) that the two inclusion induced maps $\pi_1(A, a) \to \pi_1(U, *)$ and $\pi_1(B, b) \to \pi_1(V, *)$ are isomorphisms. This implies that the left vertical map in the above commutative diagram is an isomorphism.
(9) We obtain from (1), (6) and (8) that the top horizontal map is an isomorphism. ■

Proof of the Wedge-π_1-Proposition 20.6. Let $\{(X_i, x_i)\}_{i \in I}$ be a family of pointed topological spaces such that each x_i is a good point in X_i. We use these points to form the wedge $\bigvee_{i \in I} X_i$. Recall that the fact that x_i is good means that

(1) $\{x_i\}$ is a closed subset of X_i and
(2) that there exists an open neighborhood U_i of x_i in X_i such that x_i is a deformation retract of U_i.

We need to show that the homomorphism

$$\Phi\colon \mathop{\ast}_{i \in I} \pi_1(X_i, x_i) \;\to\; \pi_1\Big(\bigvee_{i \in I} X_i, *\Big)$$

that is induced by the natural inclusions ι_i is an isomorphism. We will show that the homomorphism Φ is surjective and injective.

Claim 1. The map

$$\Phi\colon \mathop{\ast}_{i \in I} \pi_1(X_i, x_i) \;\to\; \pi_1\Big(\bigvee_{i \in I} X_i\Big)$$

is injective.

Proof. Given a finite subset $J \subset I$ we consider the following diagram:

$$\begin{tikzcd}
\mathop{\ast}_{i \in J} \pi_1(X_i, x_i) \arrow[r, "\cong"] \arrow[d] & \pi_1\Big(\bigvee_{i \in J} X_i, *\Big) \arrow[d, hook, "\iota_*"] \\
\mathop{\ast}_{i \in I} \pi_1(X_i, x_i) \arrow[r] & \pi_1\Big(\bigvee_{i \in I} X_i, *\Big).
\end{tikzcd}$$

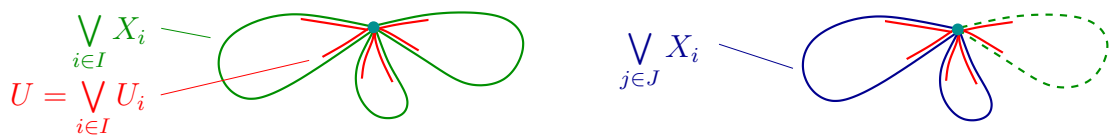
We make the following clarifications and observations:

(1) The maps are induced by the various obvious inclusion. It follows easily from the definitions that the diagram commutes.
(2) Since J is finite we know by the Finite Wedge-π_1-Lemma 20.8, applied finitely many times, that the top horizontal homomorphism is an isomorphism.
(3) Note that as on page 156 we see that $\bigvee_{i \in J} X_i \subset \bigvee_{i \in I} X_i$ is a retract. It follows from the functoriality of fundamental groups that the right vertical map is a monomorphism.
(4) Note that it follows easily from the explicit description of a free product, that given any $g = (g_1, \ldots, g_m) \in \mathop{\ast}_{i \in I} \pi_1(X_i, x_i)$ there exists a finite subset $J \subset I$ such that g lies in the image of the left vertical map.
(5) The claim follows from the above observations. ⊞

The proof that Φ is also surjective requires more of an effort.

Claim 2. We set $U := \bigcup_{i\in I} U_i$. Given any compact subset $K \subset \bigvee_{i\in I} X_i$ there exists a finite subset $J \subset I$ with $K \subset U \cup \bigvee_{i\in J} X_i$.

Proof. Given $i \in I$ we set $V_i := X_i \setminus \{x_i\}$. Since x_i is a good point we see that V_i is an open subset of X_i. Using the Wedge Proposition 17.1 (0) one can now easily see that V_i is in fact an open subset of $\bigvee_{i\in I} X_i$. Evidently the V_i together with U form an open cover of $\bigvee_{i\in I} X_i$. Since K is compact we see that there exists a finite subset $J \subset I$ with $K \subset U \cup \bigcup_{j\in J} V_j$. But this set is precisely $U \cup \bigvee_{i\in J} X_i$. ⊞



Claim 3. The map

$$\Phi\colon \mathop{\ast}_{i\in I} \pi_1(X_i, x_i) \;\to\; \pi_1\Big(\bigvee_{i\in I} X_i, *\Big)$$

is surjective.

Proof. Given a finite subset $J \subset I$ we consider the following commutative diagram

$$\begin{tikzcd}
\mathop{\ast}_{j\in J} \pi_1(X_j, *) \arrow[r, "\cong", "\Phi_J"'] \arrow[d, hook] & \pi_1\Big(\bigvee_{i\in J} X_i, *\Big) \arrow[dr, "\cong"] \arrow[d] & \\
 & & \pi_1\Big(U \cup \bigvee_{i\in J} X_i, *\Big) \arrow[dl] \\
\mathop{\ast}_{i\in I} \pi_1(X_i, x_i) \arrow[r, "\Phi"] & \pi_1\Big(\bigvee_{i\in I} X_i, *\Big). &
\end{tikzcd}$$

We make the following observations and comments:

(1) The maps are either the obvious algebraic inclusion maps or they are the obvious maps induced by inclusions of topological spaces. It follows almost immediately from the definitions that the diagram commutes.
(2) Since J is finite we obtain, from applying the Finite Wedge-π_1-Lemma 20.8 finitely many times, that the top horizontal map is an isomorphism.
(3) Note that $\bigvee_{i\in J} X_i$ is a deformation retract of $U \cup \bigvee_{i\in J} X_i$. Thus it follows from the Homotopy-π_1-Proposition 13.4 (2b) that the top diagonal map is an isomorphism.

Now we turn to the actual proof that the above map Φ is surjective. So suppose we are given some $[\gamma] \in \pi_1\Big(\bigvee_{i\in J} X_i, *\Big)$. It follows from the Compact Image Lemma 2.8 together with Claim 2 that there exists a finite subset $J \subset I$ such that the image of γ is contained in $U \cup \bigvee_{i\in J} X_i$. But considering the above diagram, and using the fact that the top horizontal map is an isomorphism we see that $[\gamma]$ actually lies in the image of the bottom horizontal map Φ. ■

20.7. **Fundamental groups of topological graphs.**

Proposition 20.9. (Graph-π_1-Proposition)

(1) Let $G = (V, E, i, t)$ be a finite connected non-empty abstract graph and let T be a spanning tree of G. Let $e_1, \ldots, e_n$ be the edges not contained in T. We pick some $v \in V$. For $j = 1, \ldots, n$ we pick a path $\alpha_j \colon [0,1] \to |T|$ from v to $i(e_j)$, we denote by $\beta_j \colon [0,1] \to |e_j| \subset |G|$ the obvious path from $i(e_j)$ to $t(e_j)$, and we pick a path $\gamma_j \colon [0,1] \to |T|$ from $t(e_j)$ to v. The map

$$\begin{aligned} \Theta_G \colon \langle x_1, \ldots, x_n \rangle &\to \pi_1(|G|, v) \\ x_j &\mapsto [\alpha_j * \beta_j * \gamma_j] \end{aligned}$$

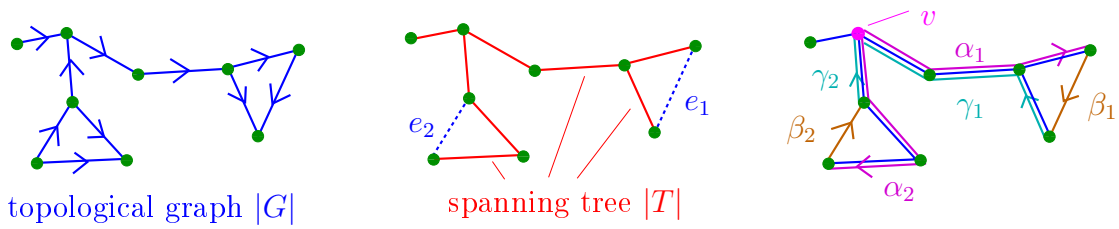
that is given by

is an isomorphism.

(2) Let X be a finite connected non-empty topological graph with v vertices and e edges. Then

$$\pi_1(X) \;\cong\; \text{free group on } 1 - \chi(X) = 1 + e - v \text{ generators.}$$

(3) Two finite connected topological graphs are homotopy equivalent if and only if they have the same Euler characteristic.



Proof.

(1) We consider the maps

$$\langle x_1, \ldots, x_n \rangle \xrightarrow{\Theta_G} \pi_1(|G|, [v]) \underset{\uparrow}{\to} \pi_1(|G|/|T|, v) \overset{\downarrow}{\underset{\cong}{\leftarrow}} \pi_1\Big(\bigvee_{j=1}^{n} S^1_j, *\Big) \underset{\uparrow}{\overset{\cong}{\leftarrow}} \langle y_1, \ldots, y_n \rangle.$$

isomorphism induced by the homeomorphism on page 156

isomorphism by the Graph-Homotopy Type Corollary 17.6 (1) and the Homotopy-π_1-Proposition 13.4 (2)

isomorphism given by the Wedge-of-Circles–π_1-Lemma 20.7

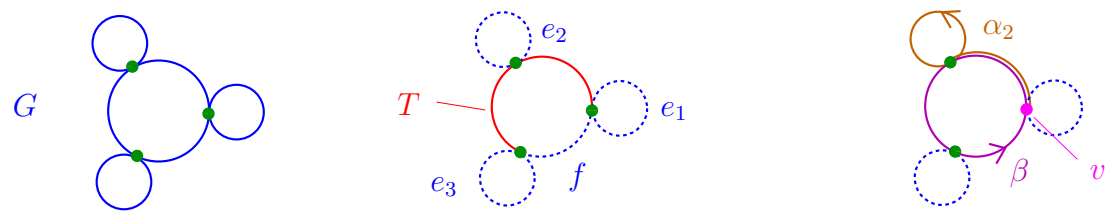
It follows fairly easily from the explicit descriptions of the various maps that for each $j \in \{1, \ldots, n\}$ the images of x_j and y_j in $\pi_1(|G|/|T|, v)$ are represented by path-homotopic loops, in particular the images of x_j and y_j in $\pi_1(|G|/|T|, [v])$ are the same. It follows from the above and the fact that all other maps are isomorphisms that Θ_G itself is indeed an isomorphism.

(2) Basically by definition we only need to prove the statement if $X = |G|$ is the topological realization of a finite connected non-empty abstract graph $G = (V, E, i, t)$. By the Spanning Tree–Existence Proposition 7.6 we know that G admits a spanning tree $T = (W, F)$. By definition T contains all vertices of V. Furthermore, since T is a tree we know by the Tree Characterization Proposition 7.7 that $\#W - \#F = 1$. Thus it follows that there are precisely $\#E - \#F = e - (\#W - 1) = e - v + 1$ edges of G that are not contained in T. The desired statement now follows from (1).

(3) Let X and X' be two finite connected topological graphs. If $\chi(X) = \chi(X')$, then we already saw in the Graph-Homotopy Type Corollary 17.6 (2) that X and X' are homotopy equivalent. On the other hand, if $\chi(X) \neq \chi(X')$, then it follows from (1)

together with the Basis Cardinality Proposition 19.9 that X and X' are not homotopy equivalent. ■

Example. We look at the graph G shown in the figure below. We also show a spanning tree T and a "base vertex v". We make the following observations:

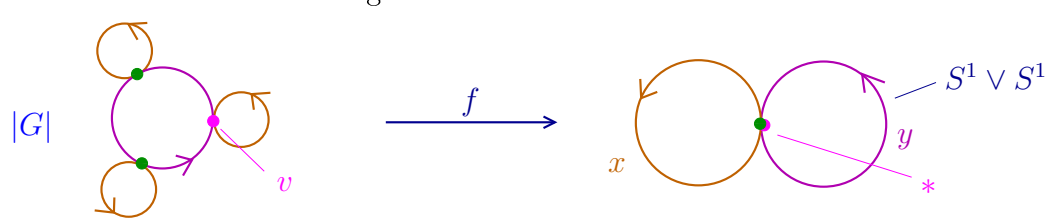


(1) The three small loops e_1, e_2, e_3 and the edge f are not contained in T.
(2) Each loop e_i gives rise to a loop α_i with base point v, by traveling along T, going once around e_i and going back to v along T.
(3) The edge f gives rise to a loop β by going once around the central circle.

It follows from the Graph-π_1-Proposition 20.9 that the map

$$\begin{array}{rcl} \langle a_1, a_2, a_3, b\rangle & \to & \pi_1(|G|, v) \\ a_i & \mapsto & [\alpha_i] \\ b & \mapsto & [\beta] \end{array}$$

is an isomorphism. Next we do something much more interesting: We consider the map $f\colon |G| \to S^1 \vee S^1$ shown in the figure below.



One can easily verify that f is a covering of degree 3. We use the Wedge-of-Circles–π_1-Lemma 20.7 to make the usual identification $\pi_1(S^1 \vee S^1, *) = \langle x, y\rangle$. By working directly with the definition of f_* one can see that f_* is given by

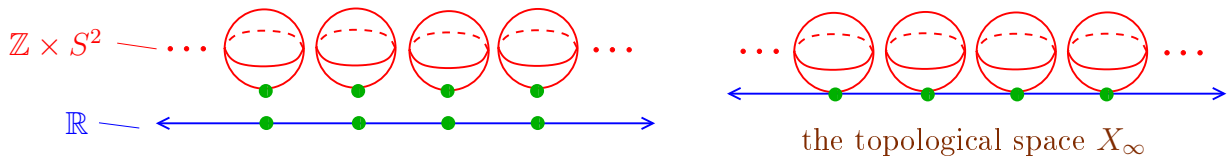
$$\begin{array}{rcl} f_*\colon \langle a_1, a_2, a_3, b\rangle = \pi_1(|G|, v) & \to & \pi_1(S^1 \vee S^1, *) = \langle x, y\rangle \\ a_i & \mapsto & y^{i-1} \cdot x \cdot y^{-(i-1)} \\ b & \mapsto & y^3. \end{array}$$

Since $f\colon |G| \to S^1 \vee S^1$ is a covering of degree 3 we obtain from the Covering-π_1-Monomorphism Proposition 14.11 together with the Covering-Degree Proposition 14.13 that the induced map f_* is a monomorphism and that the image of f_* is a subgroup of index three. In summary, we have now shown that the free group $\langle x, y\rangle$ on two generators contains the subgroup $f_*(\langle a_1, a_2, a_2, b\rangle) = \langle x, yxy^{-1}, y^2xy^{-2}, y\rangle$ which is a free group on four generators. Thus we see that the free group of rank two can contain "larger" free groups. This is in complete contrast to the situation in free abelian groups which we studied in the Free Abelian Subgroup Theorem 18.7.

20.8. **Exhaustions of topological spaces.** In many situations one is dealing with non-compact topological spaces where one would be tempted to apply the Seifert–van Kampen-Theorem 20.1 or the Wedge-π_1-Proposition 20.6 "infinitely many times", which does not really make sense. For example let us consider the topological space

$$X_\infty \;=\; \big(\mathbb{R} \sqcup (\mathbb{Z} \times S^2)\big) \,/\, n \sim (n, (0,0,-1)).$$

The topological space X consists of an infinite line where to each $n \in \mathbb{Z}$ we attach a 2-dimensional sphere. We would like to show that X_∞ is simply connected. The following lemma lets us get around the issue of "infinite operations".



Proposition 20.10. (Exhaustion–Simply Connected Proposition) Let X be a topological space and let

$$X_0 \;\subset\; X_1 \;\subset\; X_2 \;\subset\; X_3 \;\subset\; \ldots$$

be a sequence of subspaces of X such the following three conditions are satisfied:

(1) We have $\bigcup\limits_{i \in \mathbb{N}_0} X_i = X$.

(2) For each $i \in \mathbb{N}$ we have $X_i \subset \mathring{X}_{i+1}$.

(3) Each X_i is simply connected.

Then for each $x_0 \in X$ we have $\pi_1(X, x_0) = 0$.

Lemma 20.11. (Compact Subspace–Exhaustion Lemma) Let X be a topological space and let

$$X_0 \;\subset\; X_1 \;\subset\; X_2 \;\subset\; X_3 \;\subset\; \ldots$$

be a sequence of subspaces of X such the following two conditions are satisfied:

(1) We have $\bigcup\limits_{i \in \mathbb{N}_0} X_i = X$.

(2) For each $i \in \mathbb{N}_0$ we have $X_i \subset \mathring{X}_{i+1}$.

Given any compact subset K of X there exists an $i \in \mathbb{N}_0$ with $K \subset X_i$.

Proof of the Compact Subspace–Exhaustion Lemma 20.11. We start out with the following little claim.

Claim. We have $$\bigcup_{i \in \mathbb{N}} \mathring{X}_i \;=\; X.$$

Proof. We have the following equalities and inclusions

$$X \;\underset{\substack{\uparrow \\ \text{by hypothesis (1)}}}{=}\; \bigcup_{i \in \mathbb{N}} X_i \;\underset{\substack{\uparrow \\ \text{by hypothesis (2)}}}{=}\; \bigcup_{i \in \mathbb{N}} \mathring{X}_i \;\subset\; X.$$

We see that the inclusion to the right is in fact an equality. ⊞

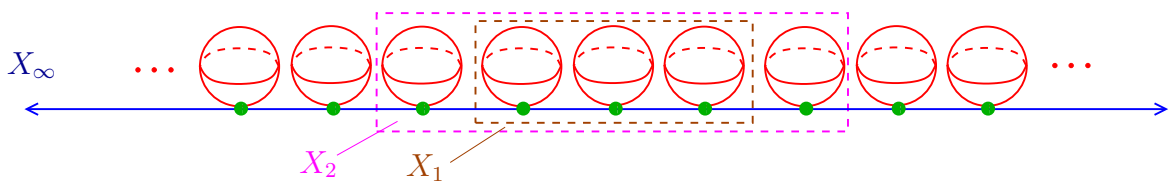
Now let $K \subset X$ be a compact subspace. Recall that by the Interior-Closure Lemma 1.6 (1) we know that interiors are always open subsets. It follows from the compactness of K and the claim that K is contained in the union of finitely many of the $\mathring{X}_i$. It follows from our hypothesis on the X_i that K is in fact contained in a single X_i. ■

Proof of the Exhaustion–Simply Connected Proposition 20.10. Let $x_0 \in X$ and let $f\colon [0,1] \to X$ be a loop in x_0. By the Compact Image Lemma 2.8 we know that $f([0,1])$ is a compact subset of X. By the Compact Subspace–Exhaustion Lemma 20.11 we know that $f([0,1])$ is contained in a single X_i. It follows from Hypothesis (3) that f is null-homotopic in X_i. It is clear that this implies that f is also null-homotopic in X. ■

Example. We return to the topological space X_∞ from the beginning of the section. Given $k \in \mathbb{N}$ we set

$$X_k \;=\; \big((-k-1,k+1) \sqcup (\{-k,\dots,k\} \times S^2)\big) \,/\, n \sim (n,(0,0,-1)).$$

Using the Wedge-π_1-Proposition 20.6 one can easily show that each X_k is simply connected. It follows from the Exhaustion–Simply Connected Proposition 20.10 that X_∞ is simply connected.



Proposition 20.12. (Tree–π_1-Proposition) Let $T = (V,E,\varphi)$ be a non-empty connected undirected abstract graph. The following statements are equivalent:

(1) T is a tree.
(2) $\pi_1(|T|)$ is the trivial group.

Proof. Let $T = (V,E,\varphi)$ be a non-empty connected undirected abstract graph.

We start out with a proof of the (1)⇒(2)-direction. Since T is non-empty we can pick a vertex $v \in V$. We consider the subgraph given by $T_0 := \{v\}$. Iteratively we build a sequence of subgraphs $T_0, T_1, T_2, \dots$ of T as follows: given T_i we add all edges (and the corresponding vertices) that have at least one vertex in common with T_i. We make the following observations:

(a) One can easily verify that each T_i is connected.
(b) It follows from (a) and the Tree–Basics Lemma 7.4 (1) that each T_i is a tree.
(c) It follows from (b) and Finite Tree–Contractible Proposition 17.4 that each T_i is simply connected.
(d) One can easily verify that for each $i \in \mathbb{N}_0$ we have $T_i \subset \mathring{T}_{i+1}$.
(e) We have $T = \bigcup\limits_{i\in\mathbb{N}_0} T_i$.

It follows from the above together with the Exhaustion–Simply Connected Proposition 20.10 that T is simply connected.

We turn to the proof of the (2)⇒(1)-direction. By the Spanning Tree–Existence Proposition 7.6 we know that T admits a spanning tree S. It remains to show that if $S \neq T$, then $\pi_1(|T|)$ is non-trivial. Thus suppose that there exists an edge $e \in T \setminus S$. We denote by $|e| \subset |T|$ the corresponding subspace. By definition there exist $v, w \in V$ with $\varphi(e) = \{v,w\}$. By definition of the topological realization there exists a continuous map $\alpha\colon [0,1] \to |e|$ with $\alpha(0) = v$ and $\alpha(1) = w$ such that α restricts to a homeomorphism $(0,1) \to |e| \setminus \{v,w\}$. Since S is a spanning tree we know that $v, w \in S$. Also note that trees are connected. Thus it follows from the Abstract Graph-Topology Lemma 7.1 that

$|S|$ is connected. Therefore there exists a continuous map $\beta\colon [0,1] \to |S|$ with $\beta(0) = v$ and $\beta(1) = w$. Next we consider the map

$$\begin{array}{rcl} f\colon |T| & \to & S^1 \\ x & \mapsto & \begin{cases} 1, & \text{if } x \notin \alpha(0), \\ \exp(2\pi\,\mathrm{i}t), & \text{if } x = \alpha(t). \end{cases} \end{array}$$

It follows from the Topological-Pushout Proposition 6.12 that f is continuous.

Note that $\alpha * \beta$ is a loop in v and note that $f \circ (\alpha * \beta)\colon [0,1] \to S^1$ is path-homotopic to the loop $t \mapsto \exp(2\pi\,\mathrm{i}t)$. It follows from these observations together with the Circle-π_1-Theorem 11.13 that $f_*([\beta * \alpha]) \in \pi_1(S^1, 1)$ is non-trivial. But this shows that $\pi_1(|T|, v)$ is non-trivial. ■

21. Presentations of groups

21.1. Generating sets of groups.

Definition. Let G be a group and let $S \subset G$ be a subset.

(1) A word in S is a finite sequence $(s_1^{\epsilon_1}, \dots, s_k^{\epsilon_k})$ with $s_1, \dots, s_k \in S$ and $\epsilon_1, \dots, \epsilon_k \in \{-1, 1\}$. We refer to k as the length of the word.
(2) A word $(s_1^{\epsilon_1}, \dots, s_k^{\epsilon_k})$ defines an element $s_1^{\epsilon_1} \cdot \ldots \cdot s_k^{\epsilon_k}$ in G. In casual language we also refer to an expression of the form $s_1^{\epsilon_1} \cdot \ldots \cdot s_k^{\epsilon_k}$ as a word.

Definition.

(1) Let G be a group and let $S \subset G$ be a subset. We denote by $\Xi_S \colon \langle S \rangle \to G$ the natural homomorphism (given by the Free Group–Existence Lemma 19.4) from the free group $\langle S \rangle$ to G that is given by $s \mapsto s$.
(2) Given a group G we say that $S \subset G$ is a generating set if the natural homomorphism $\Xi_S \colon \langle S \rangle \to G$ is an epimorphism.
(3) We say that a group G is finitely generated if it admits a finite generating set. Otherwise we say that the group is infinitely generated.

Lemma 21.1. (Generating Set–Characterization Lemma) Let G be a group and let $S \subset G$ be a subset. The following statements are equivalent:

(1) S is a generating set for G.
(2) Every element of G can be represented by a word in S.
(3) If H is a subgroup of G that contains S, then $H = G$.

Proof. The lemma follows easily from the definitions. ■

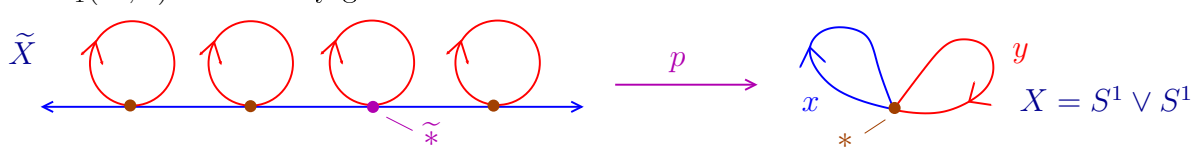
Lemma 21.2. (Generating Set–Lemma)

(1) If G is a free group on a subset S, then S is a generating set for G.
(2) If $\varphi \colon G \to H$ is an epimorphism between two groups and if S is a generating set for G, then $\varphi(S)$ is a generating set of H.

Proof. Both statements follow easily from the definitions. ■

Example. First recall that it follows from the Wedge-of-Circles–π_1-Lemma 20.7 that we can make the identification $\pi_1(S^1 \vee S^1, *) = \langle x, y \rangle$. In the figure below we show a covering $p \colon \widetilde{X} \to S^1 \vee S^1$. It follows from the Covering-π_1-Monomorphism Proposition 14.11 that $\pi_1(\widetilde{X}, \widetilde{*})$ is isomorphic to a subgroup of $\pi_1(S^1 \vee S^1, *) = \langle x, y \rangle$.

Using the Graph-π_1-Proposition 20.9 one can show that $\pi_1(\widetilde{X}, \widetilde{*})$ admits epimorphisms onto free groups of any rank $n \in \mathbb{N}$. Thus it follows from the Generating Set–Lemma 21.2 that $\pi_1(\widetilde{X}, \widetilde{*})$ is infinitely generated.



We have thus shown that there exists a finitely generated group that admits an infinitely generated subgroup.

Lemma 21.3. (Generating Set Properties–Lemma) Let G be a group and let S be a generating set for G.

(1) If $\varphi\colon G \to G$ is a homomorphism such that $\varphi(s) = s$ for all $s \in S$, then $\varphi = \mathrm{id}$.
(2) If for all $s, t \in S$ we have $s \cdot t = t \cdot s$, then G is abelian.
(3) If $\varphi\colon H \to G$ is a group homomorphism with $S \subset \varphi(H)$, then φ is an epimorphism.

Proof.

(1) Let $\varphi\colon G \to G$ be a homomorphism such that $\varphi(s) = s$ for every $s \in S$. We consider $H := \{g \in G \,|\, \varphi(g) = g\}$. It is straightforward to verify that H is a subgroup of G. Since H contains S it follows from the Generating Set–Characterization Lemma 21.1 (1)⇒(3) that $H = G$, i.e. $\varphi = \mathrm{id}$.

(2) We assume that for all $s, t \in S$ we have $s \cdot t = t \cdot s$. We want to show that G is abelian. We start out with the following claim.

Claim. For all $s, t \in S$ and all $\mu, \nu \in \{-1, 1\}$ we have $s^\mu \cdot t^\nu = t^\nu \cdot s^\mu$.

Proof. First we consider the case $\mu = 1$ and $\nu = -1$:

$$s \cdot t^{-1} \;=\; \underbrace{t^{-1} \cdot t}_{=e} \cdot s \cdot t^{-1} \;\underset{\substack{\uparrow \\ \text{since } s \cdot t = t \cdot s}}{=}\; t^{-1} \cdot s \cdot t \cdot t^{-1} \;=\; t^{-1} \cdot s.$$

By swapping the roles of s and t we see that the above also takes care of the case $\mu = -1$ and $\nu = 1$. Thus it remains to deal with the case $\mu = -1$ and $\nu = -1$. By hypothesis we have $t \cdot s = s \cdot t$. Taking inverses we see that $(t \cdot s)^{-1} = (s \cdot t)^{-1}$. But this means precisely that $s^{-1} \cdot t^{-1} = t^{-1} \cdot s^{-1}$. ⊞

Now let $g, h \in G$. By the Generating Set–Characterization Lemma 21.1 (1)⇒(2) we can write $g = s_1^{\epsilon_1} \cdot \ldots \cdot s_k^{\epsilon_k}$ and $h = t_1^{\tau_1} \cdot \ldots \cdot t_l^{\tau_l}$ for some $s_1, \ldots, s_k, t_1, \ldots, t_l \in S$ and $\epsilon_1, \ldots, \epsilon_k, \tau_1, \ldots, \tau_l \in \{-1, 1\}$. By applying the above claim altogether $k \cdot l$ times we see that $g \cdot h = h \cdot g$.

(3) Let $\varphi\colon H \to G$ be a group homomorphism with $S \subset \varphi(G)$. Note that $\varphi(H)$ is a subgroup of G. Since $S \subset \varphi(G)$ it thus follows from the Generating Set–Characterization Lemma 21.1 (1)⇒(3) that $\varphi(H) = G$. ■

21.2. Presentations of groups.

Definition. Let X be a set and let R be a subset of the free group $\langle X \rangle$ generated by the set X. We write

$$\langle X \mid R \rangle \;:=\; \langle X \rangle \,/\, \langle\langle R \rangle\rangle.$$

If $X = \{x_1, \ldots, x_k\}$ and $R = \{r_1, \ldots, r_l\}$, then we use the convenient shorthand

$$\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle \;:=\; \langle \{x_1, \ldots, x_k\} \,|\, \{r_1, \ldots, r_l\} \rangle.$$

Examples.

(1) For $k \in \mathbb{N}_0$ we have

$$\langle g \mid g^k \rangle \;=\; \langle g \rangle / \langle\langle g^k \rangle\rangle \;\underset{\substack{\uparrow \\ \text{this follows easily from the observation that } \langle g \rangle \text{ is abelian}}}{=}\; \langle g \rangle / \{g^{k \cdot n} \,|\, n \in \mathbb{Z}\} \;\cong \mathbb{Z}_k.$$

(2) The group $\langle x_1, \ldots, x_k \,|\rangle$ with $R = \varnothing$ is just the free group on $x_1, \ldots, x_k$.

Convention. Given a group $\langle X \,|\, R\rangle$ and given $g \in \langle X\rangle$ we denote the images of the g in the quotient $\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l\rangle$ again by g.

Definition. Let $r \in \langle x_1, \ldots, x_k\rangle$. Furthermore let G be a group and let $g_1, \ldots, g_k$ be elements of G. We say that $g_1, \ldots, g_k$ satisfy the relation r if the homomorphism $\langle x_1, \ldots, x_k\rangle \to G$ that is given by $x_i \mapsto g_i$, $i = 1, \ldots, k$ sends r to the trivial element.

Example. We consider the word $r = [x_1, x_2] = x_1x_2x_1^{-1}x_2^{-1} \in \langle x_1, x_2\rangle$. If a group G is abelian, then any two elements $g_1, g_2 \in G$ satisfy the relation $x_1x_2x_1^{-1}x_2^{-1}$.

Lemma 21.4. (Free Abelian Group-Presentation Lemma) Let $m \in \mathbb{N}$. There exists a unique isomorphism

$$\alpha\colon \langle x_1, \ldots, x_m \mid [x_i, x_j] = e \text{ for all } 1 \le i < j \le m\rangle \xrightarrow{\cong} \mathbb{Z}^m$$

which sends each x_i to the i-th standard basis vector e_i of $\mathbb{Z}^m$.

Proof. We set $X = \{x_1, \ldots, x_m\}$ and $R = \{[x_i, x_j] \,|\, 1 \le i < j \le m\}$. Since $\mathbb{Z}^m$ is abelian we obtain from the Free Group–Existence Lemma 19.4 together with the Normal Closure–Factorization Lemma 19.10 that there exists a unique homomorphism $\alpha\colon \langle X \,|\, R\rangle \to \mathbb{Z}^m$ with $\alpha(x_i) = e_i$. Since the image of α contains the generating set $\{e_1, \ldots, e_m\}$ of $\mathbb{Z}^m$ we obtain from the Generating Set Properties–Lemma 21.3 (3) that α is an epimorphism. Thus it remains to show that α is a monomorphism.

Claim. The group $\langle X \,|\, R\rangle$ is abelian.

Proof. Recall that it follows from the Generating Set–Lemma 21.2 that $X = \{x_1, \ldots, x_m\}$ is a generating set for $\langle X \,|\, R\rangle$. Let $i < j$. We have the following equality in $\langle X \,|\, R\rangle$:

$$x_j \cdot x_i \;=\; \underbrace{x_i \cdot x_j \cdot x_i^{-1} \cdot x_j^{-1}}_{=[x_i,x_j]=e\in\langle X \,|\, R\rangle} \cdot x_j \cdot x_i \;=\; x_i \cdot x_j.$$

This shows that the elements of the generating set commute. Thus it follows from the Generating Set Properties–Lemma 21.3 (2) that the group $\langle X \,|\, R\rangle$ is abelian. ⊞

As we mentioned on page 161 (2), the group $\mathbb{Z}^m$ is free abelian on the set $e_1, \ldots, e_m$. We just learned in the above claim that $\langle X \,|\, R\rangle$ is abelian. This implies in particular that there exists a unique homomorphism $\beta\colon \mathbb{Z}^m \to \langle X \,|\, R\rangle$ with $\beta(e_i) = x_i$. Note that for each $i \in \{1, \ldots, m\}$ we have $(\beta \circ \alpha)(x_i) = \beta(\alpha(x_i)) = \beta(e_i) = x_i$. Thus it follows from the Generating Set Properties–Lemma 21.3 (1) that $\beta \circ \alpha = \mathrm{id}$. We have thus shown that $\alpha\colon \langle X \,|\, R\rangle \to \mathbb{Z}^m$ is a monomorphism. ■

Definition.

(1) A presentation is a pair (X, R) consisting of a set X together with a subset $R \subset \langle X\rangle$. We call the elements of X the generators and we call the elements of R the relators of the presentation.

(2) Let π be a group. A presentation for π is a presentation (X, R) together with an isomorphism

$$\langle X \mid R\rangle \xrightarrow{\cong} \pi.$$

Very often one just says $\langle X \mid R\rangle$ is a presentation for π and the explicit isomorphism is suppressed from the notation.

(3) We say a group π is finitely presented, if π admits a presentation with finitely many generators and finitely many relators.

Example. We had just seen in the Free Abelian Group-Presentation Lemma 21.4 that

$$\mathbb{Z}^n \;\cong\; \langle x_1, \ldots, x_n \mid [x_i, x_j] = e \text{ with } 1 \le i < j \le n \rangle.$$

This shows that $\mathbb{Z}^n$ is a finitely presented group.

Proposition 21.5. (Abelianization-of-Presentation Proposition) Suppose we are given a finitely presented group $\pi = \langle x_1, \ldots, x_m \mid r_1, \ldots, r_n \rangle$. We denote by

$$\Phi\colon \langle x_1, \ldots, x_m \rangle \;\to\; \mathbb{Z}^m$$

the homomorphism that is given by $\Phi(x_i) = e_i$. There exists unique isomorphism

$$\Psi\colon \underbrace{\pi/[\pi,\pi]}_{=\pi_{\mathrm{ab}}} \xrightarrow{\cong} \mathbb{Z}^m/\operatorname{span}(\Phi(r_1), \ldots, \Phi(r_n)).$$

Example. We consider the group $\quad \pi \;=\; \langle x, y \mid x^2y^{-2}x, y^{-1}x^3yx \rangle.$

Let $\Phi\colon \langle x, y\rangle \to \mathbb{Z}^2$ be the epimorphism given by $\Phi(x) = (1,0)$ and $\Phi(y) = (0,1)$. It follows from the Abelianization-of-Presentation Proposition 21.5 that the abelianization of π is given by

$$\pi_{\mathrm{ab}} \;=\; \mathbb{Z}^2/\operatorname{span}(\underbrace{\Phi(x^2y^{-2}x)}_{\begin{pmatrix} 3 \\ -2 \end{pmatrix}}, \underbrace{\Phi(y^{-1}x^3yx)}_{=\begin{pmatrix} 4 \\ 0 \end{pmatrix}}) \;=\; \mathbb{Z}^2/\begin{pmatrix} 3 & 4 \\ -2 & 0 \end{pmatrix}\mathbb{Z}^2 \;\underset{\substack{\uparrow \\ \text{can be calculated using the} \\ \text{Smith normal form of the matrix}}}{\cong}\; \mathbb{Z}_8.$$

Proof. It is straightforward to see that there exists a unique homomorphism

$$\Psi\colon \pi/[\pi,\pi] \;\to\; \mathbb{Z}^m/\operatorname{span}(\Phi(r_1), \ldots, \Phi(r_n))$$

with $\Psi([x_i]) = [e_i]$. It remains to show that Ψ is an isomorphism. We write $F = \langle x_1, \ldots, x_m \rangle$ and we write $G = \langle y_1, \ldots, y_n \rangle$. We consider the following diagram

$$\begin{tikzcd}
G_{\mathrm{ab}} = G/[G,G] \arrow[r, "{[y_i] \mapsto e_i}", "\cong"'] \arrow[d, "{[y_i] \mapsto [r_i]}"'] & \mathbb{Z}^n \arrow[d, "e_i \mapsto \Phi(r_i)"] \\
F_{\mathrm{ab}} = F/[F,F] \arrow[r, "{[x_i] \mapsto e_i}", "\cong"'] \arrow[d, two heads] & \mathbb{Z}^m \arrow[d, two heads] \\
\pi_{\mathrm{ab}} = \pi/[\pi,\pi] \arrow[r, "\Psi"] & \mathbb{Z}^m/\operatorname{span}(\Phi(r_1), \ldots, \Phi(r_n)).
\end{tikzcd}$$

Note that the diagram commutes. It follows from the Abelianization Calculation Proposition 19.8 (4) that the top two horizontal maps are isomorphisms. The bottom vertical maps are clearly epimorphisms and it is clear that the images of the top vertical maps lie in the kernel of the bottom vertical maps. It now follows from an elementary "diagram chase" that the bottom horizontal map is also an isomorphism. We leave the details to the reader. ■

21.3. Presentations of groups and constructions of groups.

Lemma 21.6. (Quotient Presentation–Lemma) Let $G = \langle X \mid R\rangle$ be a group and let $\varphi\colon G \to H$ be an epimorphism. If $S \subset \langle X\rangle$ is a subset, such that $\ker(\varphi) = \langle\langle\{[s] \mid s \in S\}\rangle\rangle$, then

$$\begin{array}{rcl} \langle X \mid R \cup S\rangle & \to & H \\ x & \mapsto & \varphi([x]) \end{array}$$

is a presentation for H.

Proof. We consider the following diagram:

$$\begin{tikzcd} \langle X\rangle \arrow[r, "\mathrm{id}"] \arrow[d] & \langle X\rangle \arrow[d] & \\ & \langle X \mid R\rangle \arrow[d] & \\ \langle X \mid R \cup S\rangle \arrow[r] & \langle X \mid R\rangle/\ker(\varphi) \arrow[r, "{[g]\mapsto\varphi(g)}"] & H. \end{tikzcd}$$

Note that the identity homomorphism on the top (in both directions) descends to homomorphisms (in both directions) between the groups on the bottom left. It follows from the Generating Set Properties–Lemma 21.3 (1) that the two homomorphisms are inverses of one another, in particular the homomorphism from the left to the center is an isomorphism. The lemma now follows from the observation that the homomorphism to the bottom right is an isomorphism, since φ is an epimorphism. ■

Notation. Given a set X we sometimes write relator in the form $a = b$ for $a, b \in \langle X\rangle$, this just means that the relator is given by $a \cdot b^{-1} \in \langle X\rangle$.

Example. Let A be a finitely generated abelian group. By the Abelian Group-Classification Theorem 18.11 there exist $a_1, \ldots, a_n \in \mathbb{N}_0$ such that $A \cong \mathbb{Z}_{a_1} \oplus \cdots \oplus \mathbb{Z}_{a_n}$. In other words, we can write A as a quotient of $\mathbb{Z}^n$. Using the Abelianization-of-Presentation Proposition 21.5 and the Quotient Presentation–Lemma 21.6 one can now easily verify that

$$A \cong \bigoplus_{i=1}^{n} \mathbb{Z}_{a_i} \cong \langle x_1, \ldots, x_n \mid [x_i, x_j] = e \text{ for } 1 \le i < j \le n \text{ and } x_i^{a_i} = e \text{ for } i = 1, \ldots, n\rangle.$$

Thus we have shown that every finitely generated abelian group is also finitely presented.

Lemma 21.7. (Amalgamated Product–Presentation Lemma) Let $A = \langle X \mid R\rangle$ and $B = \langle Y \mid S\rangle$ be two groups with $X \cap Y = \varnothing$. Furthermore let G be a group with generating set Z and let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. There is a natural isomorphism

$$\langle X \sqcup Y \mid R \sqcup S \sqcup \{\alpha(z) = \beta(z) \mid z \in Z\}\rangle \xrightarrow{\cong} A *_G B.$$

Example. The case $G = \{e\}$ in the Amalgamated Product–Presentation Lemma 21.7 gives us presentations of free products of two groups.

Proof of the Amalgamated Product–Presentation Lemma 21.7. We first consider the setting of the free product $\langle X \mid R\rangle * \langle Y \mid S\rangle$. We consider the following diagram:

$$\begin{tikzcd}
\langle X \sqcup Y\rangle \arrow[r] \arrow[d] & \langle X\rangle * \langle Y\rangle \arrow[r] \arrow[d] & \langle X \sqcup Y\rangle \arrow[d] \\
\langle X \sqcup Y \mid R \cup S\rangle \arrow[r, "\alpha"] & \langle X \mid R\rangle * \langle Y \mid S\rangle \arrow[r, "\beta"] & \langle X \sqcup Y \mid R \cup S\rangle.
\end{tikzcd}$$

We make the following observations and clarifications:

(1) The top horizontal maps are given by the universal properties of free groups and free products.
(2) The vertical maps are (induced by) the natural projections.
(3) The bottom left horizontal map is induced by the Normal Closure–Factorization Lemma 19.10.
(4) It follows from the Normal Closure–Factorization Lemma 19.10 that the homomorphism $\langle X\rangle \to \langle X \sqcup Y \mid R \cup S\rangle$ descends to a homomorphism starting at $\langle X \mid R\rangle$. The same way we obtain a homomorphism starting at $\langle Y \mid S\rangle$. These two homomorphisms give us the bottom right horizontal homomorphism.
(5) Using the Generating Set–Lemma 21.2 and the Generating Set Properties–Lemma 21.3 (1) one can easily verify that $\beta \circ \alpha = \mathrm{id}$ and $\alpha \circ \beta = \mathrm{id}$.

We have thus proved the desired statement for the case of a free product. The case of a non-trivial amalgamated product follows easily from the above discussion together with the Quotient Presentation–Lemma 21.6. ■

21.4. Tietze transformations.

Definition. Let
$$\phi\colon \langle x_1, \dots, x_k \mid r_1, \dots, r_l\rangle \xrightarrow{\cong} \pi$$
be a presentation of a group π. We obtain new presentations of π as follows:

(T1) For any element s of $\langle\langle r_1, \dots, r_l\rangle\rangle \subset \langle X\rangle$ we obtain the presentation
$$\langle x_1, \dots, x_k \mid r_1, \dots, r_l, s\rangle \underset{\substack{\uparrow \\ \text{since } s \in \langle\langle r_1, \dots, r_l\rangle\rangle \text{ the groups are identically}}}{=} \langle x_1, \dots, x_k \mid r_1, \dots, r_l\rangle \xrightarrow{\phi} \pi.$$

(T2) For any $x \notin \{x_1, \dots, x_k\}$ and any two words s, t in $x_1, \dots, x_k$ we obtain the presentation
$$\begin{aligned} \langle x_1, \dots, x_k, x \mid r_1, \dots, r_l, s \cdot x \cdot t\rangle &\to \langle x_1, \dots, x_k \mid r_1, \dots, r_l\rangle \xrightarrow{\phi} \pi. \\ x_i &\mapsto x_i \\ x &\mapsto s^{-1} \cdot t^{-1} \end{aligned}$$

The above two methods, and their inverses, of obtaining new presentations out of a given one are called Tietze transformations.

Theorem 21.8. (Tietze Theorem) Any two finite presentations for a given group are related by a finite sequence of Tietze transformations.

Proof. We outsource the proof to [Fri23]. ■

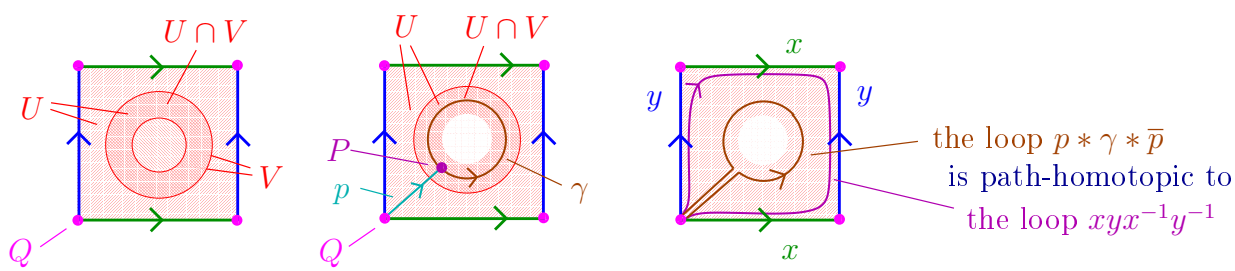
22. The classification of compact 2-dimensional manifolds

22.1. The fundamental groups of the surfaces Σ_g.

(1) We first consider the torus $T = ([0,1] \times [0,1])/\sim$. We will use the following notation:
   (a) We denote by Q the point on T corresponding to one of the four vertices of the square.
   (b) We set

$$U \;=\; T \setminus \overline{B_{\frac{1}{8}}(\tfrac{1}{2},\tfrac{1}{2})} \;\subset\; T \quad \text{and } V \;=\; B_{\frac{1}{4}}(\tfrac{1}{2},\tfrac{1}{2}) \;\subset\; T,$$

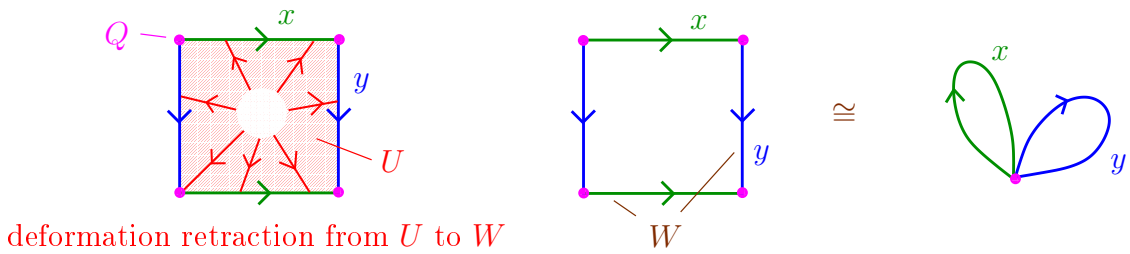
   i.e. U is the complement of a closed disk and V is an open disk. Note that $T = U \cup V$ and note that $U \cap V$ is an open annulus, i.e. homeomorphic to $S^1 \times (0,1)$.
   (c) We denote by $i\colon U \cap V \to U$ the inclusion.
   (d) We let $P = (\frac{1}{3},\frac{1}{3}) \in U \cap V$ and we denote by p the direct path from Q to P.
   (e) We pick a loop γ in $(U \cap V, P)$ that "goes counterclockwise around the annulus". Since $U \cap V$ has an obvious deformation retraction to a circle we obtain from the Circle-π_1-Theorem 11.13 together with the Homotopy-π_1-Proposition 13.4 (2b) that γ represents a generator of the infinite cyclic group $\pi_1(U \cap V, P) \cong \mathbb{Z}$.
   (f) We denote by x and y the two loops in (T, Q) corresponding to the horizontal and the vertical edge of the square. By a slight abuse of notation we denote the corresponding elements in $\pi_1(T, Q)$ by x and y as well.



Now we see that

$$\begin{array}{ccccc}
\pi_1(T,P) & \underset{\uparrow}{\cong} & \pi_1(U,P) *_{\pi_1(U\cap V,P)} \pi_1(V,P) & \underset{\uparrow}{=} & \pi_1(U,P) *_{\pi_1(U\cap V,P)} \{e\} \\
 & \text{Seifert–van Kampen-Theorem 20.1} & & V \text{ is a disk, hence } \pi_1(V,P) = \{e\} & \\
 & \underset{\uparrow}{=} & \pi_1(U,P)/\langle\!\langle i_*(\pi_1(U\cap V,P))\rangle\!\rangle & \underset{\uparrow}{=} & \pi_1(U,P)/\langle\!\langle [\gamma]\rangle\!\rangle \\
 & \text{by the Amalgamated Product–Quotient Lemma 19.13 we have } A *_G \{e\} = A/\langle\!\langle \alpha(G)\rangle\!\rangle & & \text{since } \pi_1(U\cap V,P) = \langle[\gamma]\rangle & \\
 & \underset{\uparrow}{\xrightarrow{\cong}} & \pi_1(U,Q)/\langle\!\langle [p*\gamma*\overline{p}]\rangle\!\rangle & \underset{\uparrow}{=} & \pi_1(U,Q)/\langle\!\langle xyx^{-1}y^{-1}\rangle\!\rangle \;=\; (*). \\
 & \text{by the Change-of-Base Point Proposition 11.6 (1) the path } p \text{ gives rise to an isomorphism } p_* & & \text{since } p*\gamma*\overline{p} \text{ is path-homotopic to } xyx^{-1}y^{-1} &
\end{array}$$

One can easily show that $W := \partial([0,1] \times [0,1])/\sim$ is a deformation retract of U and that W is homeomorphic to the wedge of two circles. By the Wedge-of-Circles–π_1-Lemma 20.7 we can therefore make the identification $\pi_1(W,P) = \langle x, y\rangle$.



Now we continue with the above calculation of $\pi_1(T,P)$:

$$(*) \quad = \quad \pi_1(U,Q)/\langle\langle xyx^{-1}y^{-1}\rangle\rangle \quad \underset{\uparrow}{\overset{\cong}{\longrightarrow}} \quad \pi_1(W,Q)/\langle\langle xyx^{-1}y^{-1}\rangle\rangle$$

by the Homotopy-π_1-Proposition 13.4 (2b), since W is a deformation retract of U

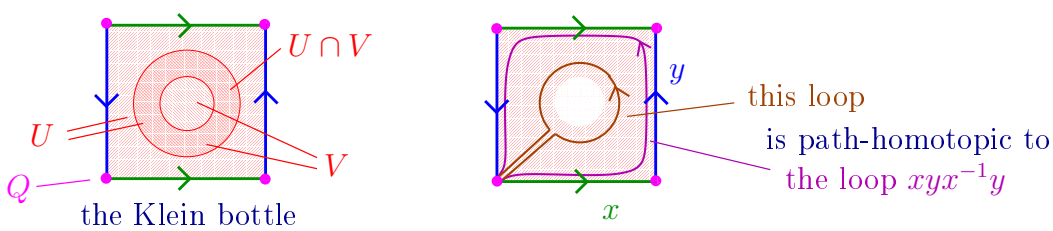
$$\underset{\uparrow}{=} \quad \langle x,y\rangle/\langle\langle xyx^{-1}y^{-1}\rangle\rangle \quad = \quad \langle x,y \mid [x,y]\rangle \quad \underset{\uparrow}{\cong} \quad \mathbb{Z}^2.$$

by the Wedge-of-Circles–π_1-Lemma 20.7 since W is the wedge of the circles x and y

by the Free Abelian Group-Presentation Lemma 21.4

(2) We turn to the Klein bottle $K = ([0,1]\times[0,1])/\sim$ with base point $Q = [(0,0)]$. The approach to calculating the fundamental group of the Klein bottle is almost identical to the above approach to calculating the fundamental group of the torus. We just replace the situation illustrated in the figure above by the situation illustrated in the figure below. The same argument as above shows that

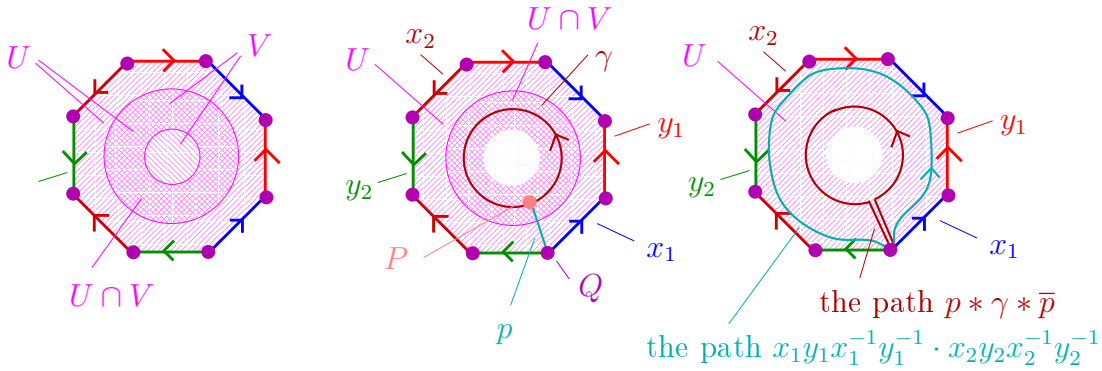
$$\pi_1(K,Q) \;\cong\; \langle x,y \mid xyx^{-1}y\rangle.$$



(3) We consider the surface $F = E_8/\sim$ of genus 2. As in the torus case we write $F = U\cup V$ where V is an open disk and where U is the complement of a closed disk such that $U\cap V$ is an annulus. Similar to the torus case we write $W = \partial E_8/\sim$. As in the torus case we see that W is a deformation retraction of U and that W is homeomorphic to the wedge of four circles corresponding to x_1, y_1, x_2, y_2. Finally almost the same argument as in the torus case shows that

$$\begin{aligned}
\pi_1(E_8/\sim,P) \;&\cong\; \pi_1(U,Q)/\langle\langle \overbrace{\underbrace{x_1y_1x_1^{-1}y_1^{-1}\cdot x_2y_2x_2^{-1}y_2^{-1}}_{=[x_1,y_1]\cdot[x_2,y_2]}}^{\text{path-homotopic to } p*\gamma*\overline{p}} \rangle\rangle \\
&\cong\; \pi_1(W,Q)/\langle\langle [x_1,y_1]\cdot[x_2,y_2]\rangle\rangle \\
&\cong\; \langle x_1,y_1,x_2,y_2\rangle/\langle\langle [x_1,y_1]\cdot[x_2,y_2]\rangle\rangle \;=\; \langle x_1,y_1,x_2,y_2 \mid [x_1,y_1]\cdot[x_2,y_2]\rangle.
\end{aligned}$$

Proposition 22.1. (Surface-π_1-Proposition)



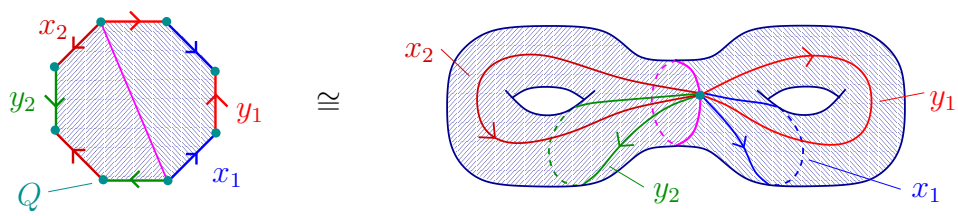
(1) Let $g \in \mathbb{N}$. Let Q be the point in $\Sigma_g = E_{4g}/\sim$ corresponding to the vertex of E_{4g}. We have an isomorphism

$$\pi_1(\text{surface } \Sigma_g \text{ of genus } g, Q) \;\cong\; \langle x_1, y_1, \ldots, x_g, y_g \mid [x_1, y_1] \cdot \ldots \cdot [x_g, y_g] \rangle$$

where the $x_1, y_1, \ldots, x_g, y_g$ correspond to edges of E_{4g}. Furthermore we have

$$\pi_1(\text{surface } \Sigma_g \text{ of genus } g, Q)_{\text{ab}} \;\cong\; \mathbb{Z}^{2g}.$$

(2) For $g \neq h$ the surfaces of genus g and h are not homotopy equivalent.



Proof.

(1) Let $g \in \mathbb{N}_0$. For $g = 0$ we obtain the fundamental group from the Sphere-π_1-Proposition 11.14 and for $g = 1$ we obtain the presentation from the discussion on page 201. For $g \geq 2$ the presentation follows from a straightforward generalization of the calculation preceding the proposition. Finally the stated abelianization of the fundamental group is an immediate consequence of the Abelianization-of-Presentation Proposition 21.5.

(2) For $g \neq h$ we obtain from the Abelian Group-Classification Theorem 18.11 that the abelianizations of the fundamental groups of the surfaces of genus $g \neq h$ are not isomorphic. It follows from the Homotopy-π_1-Proposition 13.4 that the surfaces are not homotopy equivalent. ■

22.2. The fundamental groups of the non-orientable surfaces N_g.

Proposition 22.2. (Surface-π_1-Proposition II)

(1) For any $g \in \mathbb{N}$ we have

$$\pi_1(\text{non-orientable surface } N_g \text{ of genus } g) \;\cong\; \langle x_1, \ldots, x_g \mid x_1^2 \cdot \ldots \cdot x_g^2 \rangle$$

and

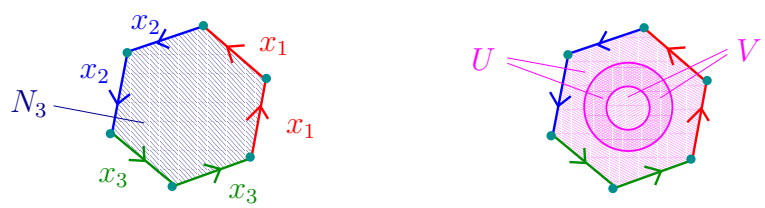
$$\pi_1(\text{non-orientable surface } N_g \text{ of genus } g)_{\text{ab}} \;\cong\; \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2.$$

(2) For $g \neq h$ the non-orientable surfaces of genus g and h are not homeomorphic.

Proof.

(1) Let $g \in \mathbb{N}$. Recall that by the definition on page 89 the non-orientable surface of genus one is the real projective plane. In the Real Projective Space–π_1-Proposition 15.3 we saw that $\pi_1(\mathbb{R}\mathrm{P}^2) \cong \mathbb{Z}_2 \cong \langle x \,|\, x^2 \rangle$. The case $g \geq 2$ follows from a straightforward modification of the calculation of the fundamental group of the surface of genus 2 that we gave on page 202. Using the figure below the reader should have no troubles with assembling a full argument. Finally note that

$$\pi_1(N_g)_{\mathrm{ab}} \underset{\substack{\uparrow \\ \text{by (1)}}}{\cong} \langle x_1, \ldots, x_g \mid x_1^2 \cdot \ldots \cdot x_g^2 \rangle_{\mathrm{ab}} \underset{\substack{\uparrow \\ \text{by the Abelianization-of-Presentation Proposition 21.5}}}{\cong} (\mathbb{Z}^g)/\mathbb{Z} \cdot \begin{pmatrix} 2 \\ \vdots \\ 2 \end{pmatrix} \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2.$$



(2) For $g \neq h$ we obtain from the above calculation and from the Abelian Group-Classification Theorem 18.11 that the abelianizations of the fundamental groups of the non-orientable surfaces of genus g and h are not isomorphic. Hence the non-orientable surfaces of genus $g \neq h$ are not homeomorphic. ■

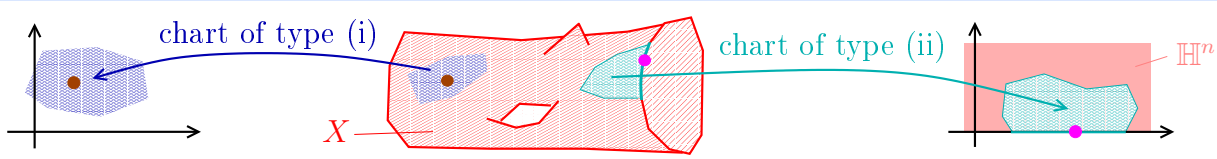
22.3. Interlude: The boundary of 2-dimensional topological manifolds.

Definition. Let X be a topological space.

(a) An n-dimensional chart for X at a point $x \in X$ is a homeomorphism $\Phi\colon U \to V$ where U is an open neighborhood of x and V is one of the following:
  (i) V is an open subset of $\mathbb{R}^n$ or
  (ii) V is an open subset of the upper half-space $\mathbb{H}^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ and $\Phi(x)$ lies on $\partial\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$.
  In the former case we say that Φ is a chart of type (i), in the latter case we say that Φ is a chart of type (ii).

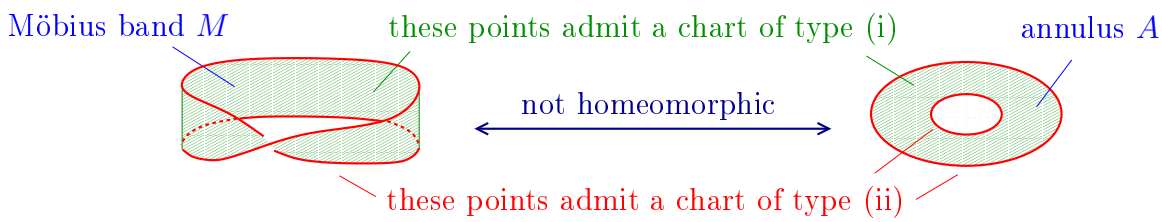
(b) We say X is an n-dimensional topological manifold if every point admits a chart and if X is Hausdorff and second countable.

(c) We say that a point x on a topological manifold X is a boundary point if x does not admit a chart of type (i). We denote by ∂X the set of all boundary points of X.



Proposition 22.4. (Topological Surface–Boundary Proposition) Let F be a 2-dimensional topological manifold. Every point on F admits either a chart of type (i) or a chart of type (ii).

Example. In the figure below we show the Möbius band M and the annulus A. The Topological Surface–Boundary Proposition 22.4 says that the boundary of the Möbius band M and the annulus A, *as a topological manifold*, is exactly what we think it should be, namely precisely the points that "obviously" admit a chart of type (ii). In particular we see that ∂M consists of one component whereas ∂A consists of two components. Since homeomorphisms of topological manifolds induce homeomorphisms of their boundaries and since ∂M is not homeomorphic to ∂A we see that the Möbius band M is not homeomorphic to the annulus A.



Proof. We outsource the proof of the Topological Surface–Boundary Proposition 22.4 (which uses the fundamental group) to [Fri23]. ■

22.4. The fundamental groups of surfaces with boundary.

Definition. Let Σ be a closed connected non-empty 2-dimensional smooth manifold and let $n \in \mathbb{N}_0$. We pick smooth embeddings $\varphi_1, \dots, \varphi_n \colon \overline{B}^2 \to \Sigma$ with disjoint images. We refer to $\Sigma \setminus (\varphi_1(B^2) \cup \dots \cup \varphi_n(B^2))$ as Σ minus n open disks.

Lemma 22.5. (Surface-Minus-Open Disks Lemma) Let Σ be a closed connected non-empty 2-dimensional smooth manifold and let $n \in \mathbb{N}_0$.

(1) The topological space Σ minus n open disks is a compact connected 2-dimensional topological manifold with n boundary components, which are given by the boundaries of the closed disks whose interiors were removed.
(2) Σ minus n open disks is a submanifold of the smooth manifold Σ.
(3) The diffeomorphism type of the smooth manifold Σ minus n open disks does not depend on the choice of the disks.

Proof. The lemma follows from basic facts about smooth manifolds. For example (3) follows from the fact, proved in the Smooth Ball Embedding Theorem of [Fri23], that any two choices of smooth embeddings $\overline{B}^2 \to \Sigma$ are smoothly isotopic (up to a possible reflection of $\overline{B}^2$.) We refer to [Fri23] for details. ■

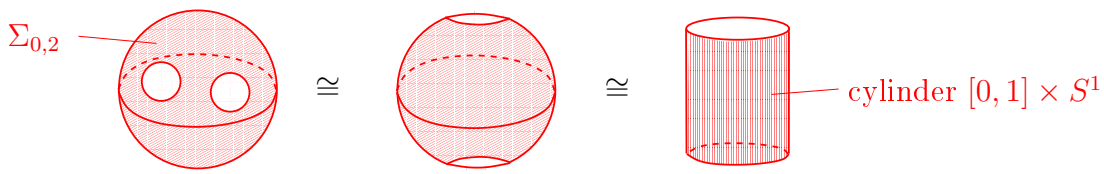
Notation. Given $g, n \in \mathbb{N}_0$ we write

$$\Sigma_{g,n} \quad := \quad \text{the surface of genus } g \text{ minus } n \text{ open disks.}$$

Furthermore for $g \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we write

$$N_{g,n} \quad := \quad \text{the non-orientable surface of genug } g \text{ minus } n \text{ open disks.}$$

Example. The surface $\Sigma_{0,2}$ is the result of removing the interiors of two disjoint closed disks from the 2-sphere. It is pretty elementary to show that the resulting smooth manifold is diffeomorphic to the "cylinder" $[0,1] \times S^1$.



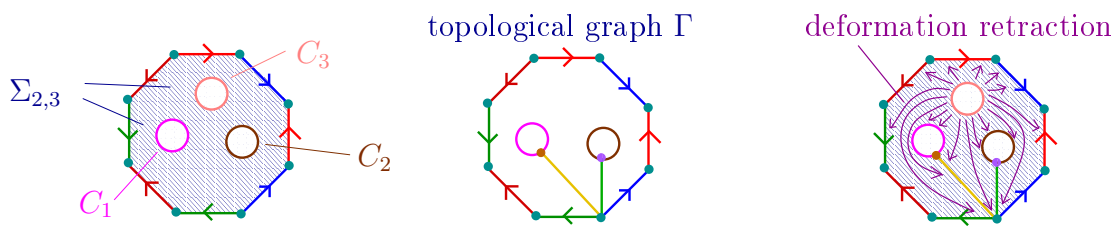
Lemma 22.6. (Surface-with-Boundary–π_1-Lemma)

(1) Given $g \in \mathbb{N}_0$ and $n \in \mathbb{N}$ the surface $\Sigma_{g,n}$ admits a deformation retract to a topological graph of Euler characteristic $2-2g-n$. In particular $\pi_1(\Sigma_{g,n})$ is a free group on $2g+n-1$ generators.

(2) Given $k \in \mathbb{N}$ and $n \in \mathbb{N}$ the surface $N_{k,n}$ admits a deformation retract to a topological graph of Euler characteristic $2-k-n$. In particular $\pi_1(N_{k,n})$ is a free group on $k+n-1$ generators.

Sketch of proof.

(1) We denote by $C_1, \ldots, C_n$ the boundary components of $\Sigma_{g,n}$. Let Γ be the topological space that is given by the wedge of $2g$-circles (which is obtained from identifying as usual the edges of the $4g$-gon pairwise), together with the boundary components $C_1, \ldots, C_{n-1}$ and together with edges $e_1, \ldots, e_{n-1}$ that connect the wedge point to $C_1, \ldots, C_{n-1}$. The deformation retraction from $\Sigma_{g,n}$ to Γ is given by pushing the remaining boundary component C_n "inward". We illustrate this in the figure below.



We equip Γ with the obvious structure of a topological graph which has $n = 1+(n-1)$ vertices and $2g+2(n-1)$ edges. Note that $\chi(\Gamma) = 2-2g-n$. Finally we see that

$$\pi_1(\Sigma_{g,n}) \underset{\substack{\uparrow \\ \text{by the above together with the Homotopy-}\pi_1\text{-Proposition 13.4 (2b)}}}{\cong} \pi_1(\Gamma) \cong \pi_1\left(\begin{array}{c}\text{connected topological graph } \Gamma \\ \text{with } \chi(\Gamma) = 2-2g-n\end{array}\right) \underset{\substack{\uparrow \\ \text{by the Graph-}\pi_1\text{-Proposition 20.9}}}{\cong} \begin{array}{c}\text{free group on} \\ 2g+n-1 \text{ generators}\end{array}$$

(2) The proof of this statement is almost identical to the proof of the first statement. ■

Definition. Let M be a topological manifold. Let "$\sim$" be the equivalence relation on M that is generated by $x \sim y$ whenever x and y lie on the same component of ∂M. We write $\widehat{M} := M/\sim$.

Lemma 22.7. (Surface Recovery Lemma) Let F be a closed connected non-empty 2-dimensional smooth manifold and let $n \in \mathbb{N}_0$. If we set M to be F minus n open disks, then $\widehat{M}$ is homeomorphic to F.

Proof. By the Surface-Minus-Open Disks Lemma 22.5 we can work with any smooth embeddings. Thus let $\varphi_1, \dots, \varphi_n \colon \overline{B}_2^2 \to F$ be smooth embeddings of the closed disks *of radius* 2 with disjoint images. We need to show that for $M := F \setminus (\varphi_1(B^2) \cup \cdots \cup \varphi_n(B^2))$ there exists a homeomorphism $\widehat{M} \to F$. We consider the following map:

$$\begin{aligned} \Theta \colon \widehat{M} &\to F \\ [P] &\mapsto \begin{cases} \varphi_i(Q \cdot 2 \cdot (\|Q\| - 1)), & \text{if } P = \varphi_i(Q) \text{ for some } i \in \{1, \dots, n\} \text{ and } Q \in \overline{B}_2^2 \setminus B^2, \\ [P], & \text{otherwise.} \end{cases} \end{aligned}$$

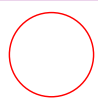
It follows from the Pasting Proposition 2.4 (2) and the Topological-Quotient Proposition 6.1 (1b) that Θ is continuous. Note that by the Compact Image Lemma 2.8 we know that $\widehat{M}$ is compact. Now it follows almost immediately from the Compact-Hausdorff Proposition 2.10 (3) that the given map $\Theta \colon \widehat{M} \to F$ is indeed a homeomorphism. ■

22.5. The Surface Classification Theorem.

Theorem 22.8. (Topological 1-Dimensional Manifold Classification Theorem) Let M be a connected non-empty 1-dimensional topological manifold. The homeomorphism type of M is given by the following table

	without boundary	with non-empty boundary
compact	the circle S^1	the closed interval $[0, 1]$
non-compact	the real line $\mathbb{R}$	the half-open interval $[0, 1)$.

An analogous statement holds for the diffeomorphism type of smooth manifolds.

	without boundary	with boundary
compact	S^1 	$[0, 1]$
non-compact	$\mathbb{R}$	$[0, 1)$

Proof. The proof is relatively elementary, but it is nonetheless not totally obvious. For example it is not entirely clear where in the proof one uses that topological manifolds are Hausdorff and second countable. We provide the proof in [Fri23]. ■

Theorem 22.9. (Surface Classification Theorem) Every compact connected non-empty 2-dimensional topological manifold is homeomorphic to either

(1) the surface $\Sigma_{g,n}$ for *unique* $g, n \in \mathbb{N}_0$ or
(2) to the surface $N_{k,n}$ for *unique* $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$.

Furthermore, if we are given a smooth manifold, then we can upgrade "homeomorphic" to "diffeomorphic".

The Surface Classification Theorem 22.9 is an immediate consequence of the following three theorems.

Theorem 22.10. (Diffeomorphism-to-Standard Surfaces Theorem) Let Σ be a compact connected non-empty 2-dimensional smooth manifold.

(1) If Σ is orientable, then Σ is diffeomorphic to the surface $\Sigma_{g,n}$ for some $g, n \in \mathbb{N}_0$.
(2) If Σ is non-orientable, then Σ is diffeomorphic to the surface $N_{k,n}$ for some $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$.

Theorem 22.11. (Surface-Smooth Structure Existence Theorem) Every compact 2-dimensional topological manifold admits a smooth structure.

Theorem 22.12. (Surfaces-Non Homeomorphic Theorem) The topological spaces $\Sigma_{g,m}$ and $N_{h,n}$ are pairwise non-homeomorphic.

Proof of the Diffeomorphism-to-Standard Surfaces Theorem 22.10. We will prove this theorem later in a few semesters time using "Morse theory". The impatient student can also look up the proof in [Fri23]. ■

Proof of the Surface-Smooth Structure Existence Theorem 22.11. This theorem is technically quite tricky. A proof is for example given in [Hat13, Theorem A]. ■

Proof of the Surfaces-Non Homeomorphic Theorem 22.12. Let M be a topological manifold. As before let "$\sim$" be the equivalence relation on M that is generated by $x \sim y$ whenever x and y lie on the same component of ∂M. As before we write $\widehat{M} := M/\sim$.

Claim. For $g \in \mathbb{N}_0$ and $k \in \mathbb{N}$ we have the following table of invariants:

X	Σ_g	$\Sigma_{g,m}$ with $m \geq 1$	N_k	$N_{k,n}$ with $n \geq 1$
isomorphism type of $\pi_1(X)_{\mathrm{ab}}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{2g+m-1}$	$\mathbb{Z}^{k-1} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{k+n-1}$
number of boundary components	0	m	0	n
isomorphism type of $\pi_1(\widehat{X})_{\mathrm{ab}}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{k-1} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{k-1} \oplus \mathbb{Z}_2$.

Proof. The number of boundary components can be determined using the Topological Surface–Boundary Proposition 22.4. The abelianizations of the fundamental groups of X are given by the Surface-π_1-Propositions 22.1 and 22.2 and the Surface-with-Boundary–π_1-Lemma 22.6. Finally the abelianizations of the fundamental of $\widehat{X}$ are given by the above calculation and the fact, shown in the Surface Recovery Lemma 22.7, that $\widehat{\Sigma_{g,m}} \cong \Sigma_g$ and $\widehat{N_{k,n}} \cong N_k$. ⊞

It follows from the second and third row that the $\Sigma_{g,m}$, $g \in \mathbb{N}_0, m \in \mathbb{N}_0$ and $N_{k,n}$, $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$ are pairwise non-homeomorphic. ■

23. The Seifert–van Kampen Theorem applied to Smooth Manifolds

23.1. The Seifert-van Kampen Theorem for smooth manifolds.

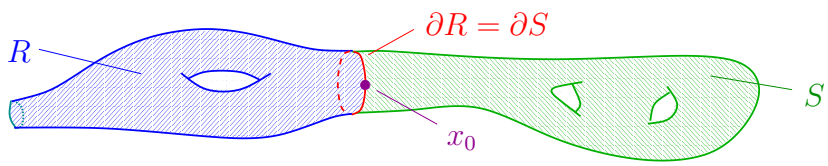
Theorem 23.1. (Manifold-Seifert–van Kampen Theorem) Let M be an m-dimensional smooth or topological manifold and let $R, S \subset M$ be two codimension zero submanifolds such that the following hold:

(1) $M = R \cup S$.
(2) R and S are closed subsets of M. (Note that by the Compact-Closed Lemma 1.9 (2) this condition is satisfied if R and S are compact.)
(3) $R \cap S$ is a single component of ∂R and it is a single component of ∂S.

For every base point $x_0 \in R \cap S$ the inclusion induced map

$$\pi_1(R, x_0) *_{\pi_1(R\cap S, x_0)} \pi_1(S, x_0) \;\to\; \pi_1(M, x_0)$$

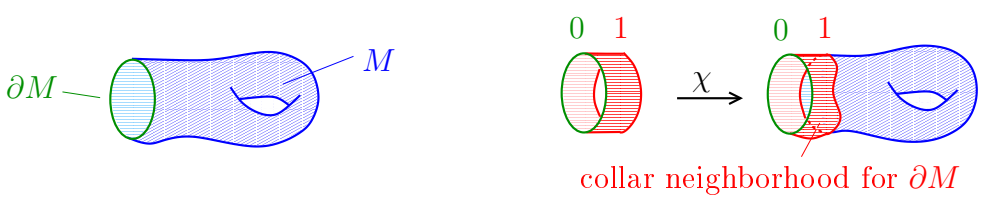
is an isomorphism.



Theorem 23.2. (Collar Neighborhood Theorem) Given any smooth or topological manifold M and given any union R of components of ∂M there exists an embedding $\chi\colon [0,1] \times R \to M$ with the following two properties:

(1) For every $P \in R$ we have $\chi(0, P) = P$.
(2) The image of $[0,1) \times R$ is an open subset of M.

The image of χ is sometimes called a collar neighborhood of R.



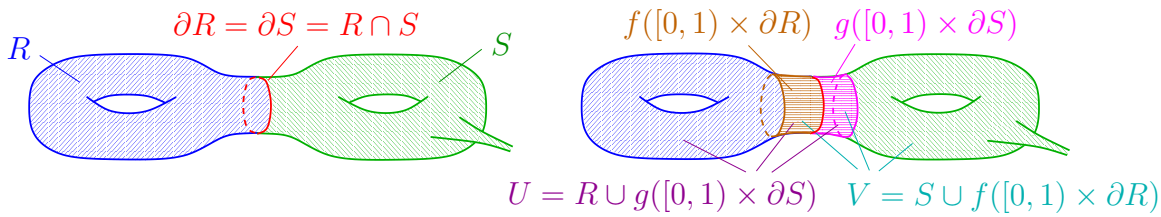
Proof. The proof is provided in [Fri23]. ■

Proof of Theorem 23.1. Let M be an m-dimensional smooth or topological manifold. Furthermore let $R, S \subset M$ be two codimension-zero submanifolds such that the following hold:

(1) $M = R \cup S$.
(2) R and S are closed subsets of M.
(3) $R \cap S$ is a component of ∂R and it is a component of ∂S.

In fact, to simplify the notation a little bit we assume that $R \cap S = \partial R = \partial S$. Let $x_0 \in R \cap S$. We pick embeddings $f\colon [0,1] \times \partial R \to R$ and $g\colon [0,1] \times \partial S \to S$ that are provided by the Collar Neighborhood Theorem 23.2. We set

$$U \;:=\; R \cup \underbrace{g([0,1) \times \partial S)}_{\subset S} \quad \text{and} \quad V \;:=\; S \cup \underbrace{f([0,1) \times \partial R)}_{\subset R}.$$



We consider the following diagram

$$\begin{array}{ccc} \pi_1(R, x_0) *_{\pi_1(R\cap S, x_0)} \pi_1(S, x_0) & \longrightarrow & \pi_1(M, x_0) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) & \longrightarrow & \pi_1(M, x_0). \end{array}$$

The following steps are similar to the ones we performed in the proof of the Wedge-π_1-Proposition 20.6.

(a) Since the vertical maps are induced by inclusions we see that the diagram commutes.

(b) We claim that U is an open subset of $M = R \cup S$. Since R and S are by our hypothesis (2) closed subsets of M we obtain from the Cover-Open Subset Lemma 1.5 (2a) that it suffices to show that $U \cap R$ is open in R and that $U \cap S$ is open in S. Evidently $U \cap R = R$ is open in R and $U \cap S = g([0,1) \times \partial S)$ is open in S by the second property of g, as stated in the Smooth Collar Neighborhood Theorem 23.2.

(c) We want to show that R is a deformation retract of U. To do we consider the following map:

$$\begin{array}{rcl} U \times [0,1] & \to & U \\ (u,t) & \mapsto & \begin{cases} u, & \text{if } u \in R, \\ g(s \cdot t, y), & \text{if } u = g(s,y) \in g([0,1) \times \partial S). \end{cases} \end{array}$$

It remains to show that this map is continuous. By the Homotopy Combination Lemma 8.2 (1) it suffices to show that R and $g([0,1) \times \partial S)$ are closed subsets of U. We see this as follows: By our hypothesis R and S are closed subsets of M. It follows that $R = R \cap U$ and $g([0,1) \times \partial S) = S \cap U$ are closed subsets of U.

(d) The obvious analogues of (b) and (c) hold for V and S.

(e) Similar to (c) one can show that $R \cap S$ is a deformation retract of $U \cap V$.

(f) We consider the following diagram:

$$\begin{array}{ccccc} \pi_1(R, x_0) & \longleftarrow & \pi_1(R \cap S, x_0) & \longrightarrow & \pi_1(S, x_0) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_1(U, x_0) & \longleftarrow & \pi_1(U \cap V, x_0) & \longrightarrow & \pi_1(V, x_0). \end{array}$$

Here all maps are induced by inclusions. It follows that the diagram commutes. Furthermore we obtain from (c), (d), (e) and the Homotopy-π_1-Proposition 13.4 (2) that the vertical maps are all isomorphisms. It follows that the induced map between amalgamated products, shown in the initial diagram to the left, is an isomorphism.

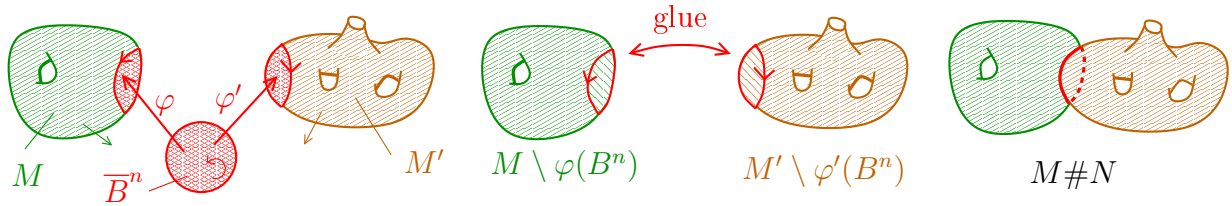
(g) By (b) and (e) we can apply the Seifert–van Kampen-Theorem 20.1 to the decomposition $M = U \cup V$ and we know that the bottom horizontal map of the initial commutative diagram is an isomorphism.

(h) Since the vertical maps are isomorphisms we see that the top horizontal map of the initial diagram is also an isomorphism. ■

23.2. Connected sum of smooth manifolds.

Definition. Let $n \in \mathbb{N}$. Let M and M' be two connected non-empty n-dimensional smooth manifolds. Let $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\varphi'\colon \overline{B}^n \to M' \setminus \partial M'$ be smooth embeddings. We define the connected sum of the smooth manifolds M and M' as

$$M\#M' := \big(M \setminus \varphi(B^n)\big) \sqcup \big(M' \setminus \varphi'(B^n)\big)/\sim \quad \text{where } \varphi(P)\sim\varphi'(P) \text{ for all } P\in S^{n-1}.$$



Example. Note that the connected sum of g tori is diffeomorphic to the surface Σ_g of genus g. Furthermore note that the connected sum of g copies of $\mathbb{R}\mathrm{P}^2$ is diffeomorphic to the non-orientable surface N_g of genus g.

Proposition 23.3. (Connected Sum-π_1-Proposition) Let $n \geq 3$ and let M and M' be two connected non-empty n-dimensional smooth manifolds. Then

$$\pi_1(M\#M') \;\cong\; \pi_1(M) * \pi_1(M').$$

Example. Let $m \geq 2$. We consider the connected sum of k copies of $S^1 \times S^m$. We see that:

$$\pi_1((S^1\times S^m)\#\ldots\#(S^1\times S^m)) \underset{\substack{\uparrow \\ \text{by the Connected Sum-}\pi_1\text{-Proposition 23.3,} \\ \text{since } m\geq 2}}{\cong} \pi_1(S^1\times S^m) *\ldots* \pi_1(S^1\times S^m) \underset{\substack{\uparrow \\ \text{by the Product-}\pi_1\text{-Proposition 13.7} \\ \text{we know that } \pi_1(S^1\times S^m)\cong\langle g_i\rangle}}{\cong} \overbrace{\langle g_1\rangle *\ldots* \langle g_k\rangle}^{=\langle g_1,\ldots,g_k\rangle}.$$

Proof of the Connected Sum-π_1-Proposition 23.3. Let M and M' be two connected non-empty n-dimensional smooth manifolds, and let $\varphi\colon \overline{B}^n \to M$ and $\varphi'\colon \overline{B}^n \to M'$ be two smooth embeddings. Recall that the connected sum of M and M' is defined as

$$M\#M' := \big(\underbrace{(M \setminus \varphi(B^n))}_{=:X} \sqcup \underbrace{(M' \setminus \varphi'(B^n))}_{=:X'}\big)/\sim \quad \text{where } \varphi(P)\sim\varphi'(P) \text{ for } P\in S^{n-1}.$$

We pick a base point x_0 that lies in the intersection of the images of X and X' in $M\#M'$.

Claim. The inclusion induced maps $\pi_1(X,x_0) \to \pi_1(M,x_0)$ and $\pi_1(X',x_0) \to \pi_1(M',x_0)$ are isomorphisms.

Proof. By symmetry it suffices to show that the inclusion induced map $\pi_1(X,x_0) \to \pi_1(M,x_0)$ is an isomorphism. Note that X and $\varphi(\overline{B}^n)$ are n-dimensional submanifolds of M and that they are closed subsets of M. Note that $X \cap \varphi(\overline{B}^n) = \varphi(S^{n-1})$. From the

Manifold-Seifert–van Kampen Theorem 23.1 we obtain the following pushout diagram:

$$\begin{tikzcd} \pi_1(\varphi(S^{n-1}), x_0) \arrow[r] \arrow[d] & \pi_1(X, x_0) \arrow[d] \\ \pi_1(\varphi(\overline{B}^n), x_0) \arrow[r] & \pi_1(M, x_0). \arrow[ul, phantom, "\lrcorner"] \end{tikzcd}$$

Since $n \geq 3$ we obtain from the Sphere-π_1-Proposition 11.14 that $\pi_1(\varphi(S^{n-1}), x_0) = \{e\}$. Thus the vertical map to the left is an isomorphism. It follows from the Amalgamated Product Lemma 19.12 (2a) that the right vertical map is also an isomorphism. ⊞

Now we see that

$$\pi_1(M\#M', x_0) \underset{\substack{\uparrow \\ \text{by the Manifold-Seifert–van Kampen Theorem 23.1,} \\ \text{since } X \cap X' \cong S^{n-1} \text{ is simply connected}}}{\xleftarrow{\cong}} \pi_1(X, x_0) * \pi_1(X', x_0) \underset{\substack{\uparrow \\ \text{by the above claim}}}{\xrightarrow{\cong}} \pi_1(M, x_0) * \pi_1(M', x_0).$$

■

23.3. Fundamental groups of high-dimensional smooth manifolds.

Proposition 23.4. (Manifold-π_1-Realization Proposition) Let π be a finitely presented group. Given any $n \geq 4$ there exists a closed orientable connected non-empty n-dimensional smooth manifold M with $\pi_1(M) \cong \pi$.

Lemma 23.5. (Drilling Out–π_1–Lemma) Let W be an n-dimensional smooth manifold. Let $k \in \mathbb{N}$ and let $\varphi\colon \overline{B}^{n-k} \times S^k \to W \setminus \partial W$ be a smooth embedding.

(1) If $n \geq k+3$, then for every base point $w_0 \in W \setminus \varphi(B^{n-k} \times S^k)$ the inclusion induced map $\pi_1(W \setminus \varphi(B^{n-k} \times S^k), w_0) \to \pi_1(W, w_0)$ is an isomorphism.

(2) If $n = k+2$, then for every base point $w_0 \in W \setminus \varphi(B^{n-k} \times S^k)$ the inclusion induced map $\pi_1(W \setminus \varphi(B^{n-k} \times S^k), w_0) \to \pi_1(W, w_0)$ is an epimorphism.

Proof of the Drilling Out–π_1–Lemma 23.5. First note that it follows from the Change-of-Base Point Proposition 13.2 that it suffices to prove the statement for some base point $w_0 \in \varphi(S^{n-k-1} \times S^k)$. We consider the following pushout diagram:

$$\begin{tikzcd} \pi_1(\varphi(S^{n-k-1} \times S^k), w_0) \arrow[r] \arrow[d] & \pi_1(W \setminus \varphi(B^{n-k} \times S^k), w_0) \arrow[d] \\ \pi_1(\varphi(B^{n-k} \times S^k), w_0) \arrow[r] & \pi_1(W, w_0) \arrow[ul, phantom, "\lrcorner"] \end{tikzcd}$$

It follows from the Manifold-Seifert–van Kampen Theorem 23.1 that this is a pushout diagram.

If $n \geq k+3$, then $n-k-1 \geq 2$. Thus we obtain from the Sphere-π_1-Proposition 11.14 that $\pi_1(S^{n-k-1})$ is trivial. It follows from the Product-π_1-Proposition 13.7 that the left vertical map is an isomorphism, thus it follows from the Amalgamated Product Lemma 19.12 (2a) that the right vertical map is – as promised – an isomorphism.

If $n = k+2$, then the left vertical map is an epimorphism. It follows from the Amalgamated Product Lemma 19.12 (2b) that the right vertical map is also an epimorphism. ■

Lemma 23.6. (Fill-in-Solid Torus-π_1-Lemma) Let W be an n-dimensional smooth manifold. Let $\varphi\colon S^{n-2}\times S^1\to\partial W$ be a diffeomorphism onto a component of ∂W. We set

$$X \;:=\; (W\sqcup(S^{n-2}\times\overline{B}^2))/\sim \qquad \text{where } \varphi(x,y)\sim(x,y) \text{ for } (x,y)\in S^{n-2}\times S^1.$$

We pick $x\in S^{n-2}$ and we set $C:=\varphi(\{x\}\times S^1)\subset\partial W$. If $n\geq 3$, then for any $w_0\in W$ the inclusion induced map $\pi_1(W,w_0)\to\pi_1(X,w_0)$ induces an isomorphism

$$\pi_1(W,w_0)/\langle\langle C\rangle\rangle \;\xrightarrow{\cong}\; \pi_1(X,w_0).$$

Proof of the Fill-in-Solid Torus-π_1-Lemma 23.6. We will only need the case $n\geq 4$. Thus in the following we will provide the (slightly easier) proof of the special case $n\geq 4$. We leave it to the reader to modify the argument below to deal with the case $n=3$. Note that it follows from the Change-of-Base Point Proposition 13.2 that it suffices to prove the statement for some base point $w_0\in\varphi(S^{n-2}\times S^1)$. Next note that we have the following two isomorphisms:

$$\pi_1(W,w_0)/\langle\langle C\rangle\rangle \;\underset{\uparrow}{\xrightarrow{\cong}}\; \pi_1(W,w_0)*_{\pi_1(S^{n-2}\times S^1,w_0)}\pi_1\big(S^{n-2}\times\overline{B}^2,w_0\big) \;\underset{\uparrow}{\xrightarrow{\cong}}\; \pi_1(X,w_0).$$

an isomorphism by the Amalgamated Product–Quotient Lemma 19.13, here we use that $\pi_1(S^{n-2}\times\overline{B}^2,w_0)=0$ and that $\pi_1(S^{n-2}\times S^1,w_0)=\langle C\rangle$, note that for the latter statement we used that $n\geq 4$

an isomorphism by the Manifold-Seifert–van Kampen Theorem 23.1

Since the horizontal maps are induced by the inclusions we see that the inclusion induced map $\pi_1(W,w_0)/\langle\langle C\rangle\rangle\to\pi_1(X,w_0)$ is indeed an isomorphism. ■

Proof of the Manifold-π_1-Realization Proposition 23.4. Throughout the proof we will mostly ignore base points. But at the end of the proof we will argue, why being a little sloppy with base points in between does not affect the outcome.

Let $\pi=\langle g_1,\dots,g_k\,|\,r_1,\dots,r_l\rangle$ be a finitely presented group. Furthermore let $n\geq 4$. We consider

$$X \;:=\; \underbrace{(S^1\times S^{n-1})\#\dots\#(S^1\times S^{n-1})}_{k \text{ copies of } S^1\times S^{n-1}}.$$

Note that X is a closed orientable connected n-dimensional smooth manifold. As on page 211 we see that

$$\pi_1(\overbrace{(S^1\times S^{n-1})\#\dots\#(S^1\times S^{n-1})}^{=X}) \;\underset{\uparrow}{\cong}\; \pi_1(S^1\times S^{n-1})*\dots*\pi_1(S^1\times S^{n-1}) \;\underset{\uparrow}{\cong}\; \overbrace{\langle g_1\rangle*\dots*\langle g_k\rangle}^{=\langle g_1,\dots,g_k\rangle}.$$

by the Connected Sum-π_1-Proposition 23.3, since $n\geq 3$

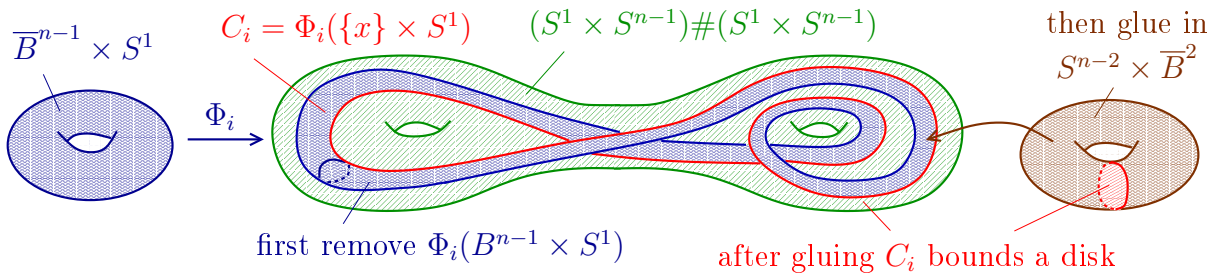
by the Product-π_1-Proposition 13.7 and since $n\geq 3$ we know that $\pi_1(S^1\times S^{n-1})\cong\mathbb{Z}$

We use this isomorphism to make the identification $\pi_1(X,x_0)=\langle g_1,\dots,g_k\rangle$. Next we pick loops $b_1,\dots,b_l\colon(S^1,*)\to(X,x_0)$ that represent $r_1,\dots,r_l\in\pi_1(X,x_0)=\langle g_1,\dots,g_k\rangle$.

Since $\dim(X)=n\geq 3$ we can apply the Circle–Smooth Embedding Theorem from [Fri23] and we obtain that the map $b_1\sqcup\dots\sqcup b_l\colon S^1\sqcup\dots\sqcup S^1\to X$ is in fact homotopic to a smooth embedding $c_1\sqcup\dots\sqcup c_l\colon S^1\sqcup\dots\sqcup S^1\to X$. It follows from the Smooth Embedding Theorem and the Tubular Neighborhood Theorem from [Fri23] that there exist smooth embeddings $\Phi_i\colon\overline{B}^{n-1}\times S^1\to X$, $i=1,\dots,l$ with disjoint images such that for

each $i \in \{1, \dots, l\}$ and each $y \in S^1$ we have $\Phi_i(0, y) = c_i(y)$. Next we consider

$$Y \;:=\; X \setminus \bigcup_{i=1}^{l} \Phi_i(B^{n-1} \times S^1) \qquad \text{and} \qquad Z \;:=\; \Big(Y \sqcup \bigsqcup_{i=1}^{l} (S^{n-2} \times \overline{B}^2)\Big) \Big/ \underset{\substack{\uparrow \\ \text{where } \Phi_i(x,y) \sim (x,y) \text{ for} \\ x \in S^{n-2} \text{ and } y \in S^1}}{\sim}$$



Finally we see that

$$\begin{aligned} \pi_1(Z) \;&\overset{\substack{\downarrow \\ \cong}}{\longleftarrow}\; \pi_1(Y)/\langle\!\langle C_1, \dots, C_l \rangle\!\rangle && \overset{\substack{\downarrow \\ \cong}}{\longleftarrow}\; \pi_1(X)/\langle\!\langle C_1, \dots, C_l \rangle\!\rangle \\ &\underset{\substack{\uparrow \\ \text{by the above, the } x_i \text{ are necessary since} \\ \text{we lost track of the base points}}}{\cong}\; \langle g_1, \dots, g_k \rangle / \langle\!\langle x_1 r_1 x_1^{-1}, \dots, x_l r_l x_l^{-1} \rangle\!\rangle && =\; \underbrace{\langle g_1, \dots, g_k \rangle / \langle\!\langle r_1, \dots, r_l \rangle\!\rangle}_{=\langle g_1, \dots, g_k \,|\, r_1, \dots, r_l \rangle}. \end{aligned}$$

the Fill-in-Solid Torus-π_1-Lemma 23.6 applied l times, hereby we set $C_i := \Phi_i(\{x\} \times S^1)$ for some $x \in S^{n-2}$

the Drilling Out–π_1–Lemma 23.5 applied k times

We have thus found the promised smooth manifold. ∎

24. Decision problems

Question 24.1.

(1) Triviality Problem: Does there exist an algorithm that can decide whether or not a given finite presentation $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ represents the trivial group?
(2) Isomorphism Problem: Does there exist an algorithm that can decide whether or not two given finite presentations $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ and $\Gamma = \langle h_1, \dots, h_m \mid s_1, \dots, s_n \rangle$ represent isomorphic groups?
(3) Word Problem: Does there exist an algorithm that can determine, given
  (a) a finite presentation $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ and
  (b) a word w in $g_1, \dots, g_k$, i.e. an element $w \in \langle g_1, \dots, g_k \rangle$,
  whether or not w represents the trivial element in the group π?
(4) Conjugacy Problem: Does there exist an algorithm that can determine, given
  (a) a finite presentation $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ and
  (b) and two words v and w in $g_1, \dots, g_k$,
  whether or not v and w represent conjugate elements in the group π?

Theorem 24.2. (Non-Solvability Theorem) The answer to all four questions raised in Question 24.1 is no.

Proof.

(1) Sergei Adyan [Ady55] and Michael Rabin [**Rabi1985**] showed in 1955 that the answer to the first question is no. Alternatively see [**MillC1992**] for a proof.
(2) An algorithm that performs (2) would also give an algorithm for determining whether or not a given presentation is isomorphic to the trivial group with the empty presentation $\langle | \rangle$. But since the answer to (1) is no, it follows that the answer to (2) is also no.
(3) If there did exist an algorithm that dealt with the third problem, then we could apply it in particular to the generators. Since a presentation presents the trivial group if and only if all generators are trivial, it follows that an algorithm for (3) would also give an algorithm for (1). Since the latter cannot exist, the former cannot exist either.
(4) An element in a group is trivial if and only if it is conjugate to the trivial element. Hence the fact that (3) has a negative answer leads to a negative answer to (4). ■

Theorem 24.3. (Manifold–Non-Classification Theorem) Let $n \geq 4$.

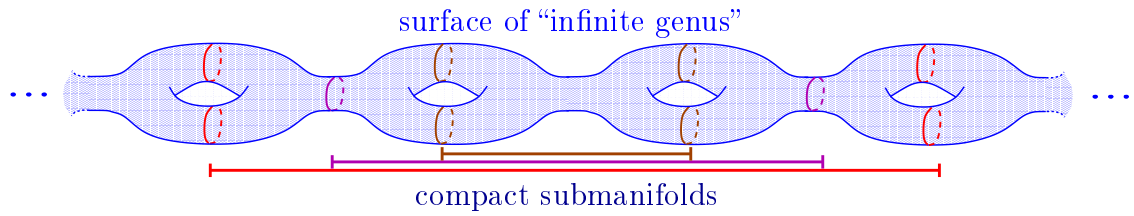
(1) There is no algorithm that can decide whether or not two closed orientable n-dimensional smooth manifolds are homeomorphic.
(2) There is no algorithm that can decide whether or not two closed orientable n-dimensional smooth manifolds are diffeomorphic.

Proof. The first statement was first proved by Aleksandr Markov [Mar58]. We refer to [BHP68, Theorem 1] and [Hak73] for a detailed discussion of the proofs of both statements. In particular these references also explain in more detail how one can describe a closed smooth manifold with finite data. We also refer to [Sti93, Chapter 9.4.2], [CZ93, Chapter 7.2] and [McA21] for more information and variations on the above results. ■

Theorem 24.4. (Smooth 3-Manifold Recognition Theorem) There exists an algorithm that can decide whether or not two closed orientable connected 3-manifolds are diffeomorphic.

Proof. The proof of this theorem is considerably more complicated than in the 1-dimensional and the 2-dimensional case. The proof that such an algorithm exists is due to the work of many mathematicians, first and foremost William Thurston [Thu82] and Grigori Perelman. The proof of the theorem was completed about 2003. It is impossible to mention here all the work that goes into proving Theorem 24.4. We refer to [AFW, Theorem 4.27] for details and precise references. ■

25. Direct limits

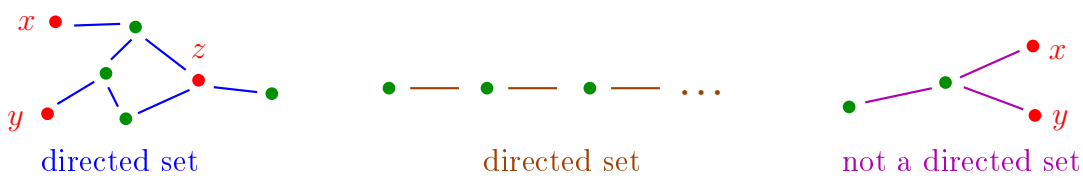


25.1. Directed sets.

Definition. A directed set is a set X together with a partial order $\leq$ that has the extra property, that for any $x, y \in X$ there exists a $z \in X$ with $x \leq z$ and $y \leq z$.

Examples.

(1) We consider the three partial orders that are illustrated in the figure below. Here it is understood that $P \leq Q$ if P is connected to Q via a path that goes everywhere from left to right. We see that the two partial orders to the left form directed sets whereas the one on the right does not form a directed set.



(2) Let X be a subset of $\mathbb{R}$. If we denote by $\leq$ the usual "less or equal" relation on the real numbers, then $(X, \leq)$ is a directed set. In particular $(\mathbb{N}, \leq)$ is a directed set.

25.2. Direct systems and direct limits.

Definition. Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be a category. A direct system in the category $\mathcal{C}$ over $(I \leq)$ is a diagram (in the sense of the definition on page 65) in $\mathcal{C}$ over the directed set $(I, \leq)$. In other words, a direct system in $\mathcal{C}$ over $(I, \leq)$ is a family of objects $\{X_i\}_{i \in I}$ in $\mathcal{C}$, together with a family of morphisms $\{f_{ji} \colon X_i \to X_j\}_{i \leq j}$ such that the following two conditions are satisfied:

$$\begin{array}{lrcll} (1) & f_{ii} & = & \mathrm{id}_{X_i} & \text{for all } i \in I, \\ (2) & f_{ki} & = & f_{kj} \circ f_{ji} & \text{for all } i, j, k \in I \text{ with } i \leq j \leq k. \end{array}$$

Definition. Let $\mathcal{C}$ be a category and let $w \in \mathbb{Z}$. A sequence in the category $\mathcal{C}$ consists of a family $\{X_i\}_{i \in \mathbb{Z}_{\geq w}}$ of objects and a family of morphisms $\{\varphi_i \colon X_i \to X_{i+1}\}_{i \in \mathbb{Z}_{\geq w}}$. A sequence of morphisms as above gives rise to a direct system over the directed set $(\mathbb{Z}_{\geq w}, \leq)$ by considering for $i \leq j$ the morphism $f_{ji} := \varphi_{j-1} \circ \cdots \circ \varphi_{i+1} \circ \varphi_i \colon X_i \to X_j$.

Examples.

(A) In the category of abelian groups we consider the sequence

$$\mathbb{Z} \xrightarrow{v \mapsto (v,0)} \mathbb{Z}^2 \xrightarrow{v \mapsto (v,0)} \mathbb{Z}^3 \xrightarrow{v \mapsto (v,0)} \dots$$

(B) In the category of groups we consider the sequence

$$\langle x_1 \rangle \hookrightarrow \langle x_1, x_2 \rangle \hookrightarrow \langle x_1, x_2, x_3 \rangle \hookrightarrow$$

of free groups where the morphisms are the obvious inclusion homomorphisms.

Definition. Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be a category. Suppose we are given a direct system $(\{X_i\}_{i\in I}, \{f_{ji\colon X_i \to X_j}\}_{i\leq j})$ in the category $\mathcal{C}$. A direct limit of the direct system is a colimit , in the sense of the definition on page 67, of the direct system viewed as a diagram. In other words, a direct limit is an object Y in $\mathcal{C}$ together with a family of morphisms $\{g_i\colon X_i \to Y\}_{i\in I}$ in the category $\mathcal{C}$ such that the following two conditions are satisfied:

(1) For all $i \leq j$ we have $g_j \circ f_{ji} = g_i\colon X_i \to Y$.
(2) If we are given another object Y' and a family of morphisms $\{g'_i\colon X_i \to Y'\}_{i\in I}$ that satisfy (1), then there exists a unique morphism $F\colon Y \to Y'$ such that for all $i \in I$ we have $g'_i = F \circ g_i\colon X_i \to Y'$.

By the Colimit Uniqueness Lemma 4.7 we know that if a direct limit exists, then it is unique in the usual sense. We denote the direct limit, if it exists, by $\varinjlim X_i$ or sometimes by $\operatorname{colim}_{\substack{\longrightarrow \\ I}} X_i$ or $\operatorname{colim}_{\substack{\longrightarrow \\ i}} X_i$.

Examples. We discuss the direct limits of the direct systems we had discussed above.

(A) As on page 217 we consider in the category of abelian groups the sequence of monomorphisms $\mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z}^3 \to \dots$ that is given by the inclusion maps $\varphi_i(v) := (v, 0)$. We claim that

$$\operatorname{colim}_{\longrightarrow} \mathbb{Z}^i \;=\; \mathbb{Z}^{(\mathbb{N})} \;=\; \{(x_1, x_2, \dots) \mid x_i \in \mathbb{Z} \text{ but only finitely many } x_i\text{'s are non-zero}\},$$

where the maps $g_j\colon \mathbb{Z}^j \to \operatorname{colim}_{\longrightarrow} \mathbb{Z}^i = \mathbb{Z}^{(\mathbb{N})}$, $j \in \mathbb{N}$ are the obvious monomorphisms. Indeed, if we are given an abelian group Y' and homomorphisms $g'_j\colon \mathbb{Z}^j \to Y'$ that satisfy $g'_{j+1} \circ \varphi_j = g'_j$ for all $j \in \mathbb{N}_0$, then we set

$$\begin{aligned} F\colon \operatorname{colim}_{\longrightarrow} \mathbb{Z}^i \;=\; \mathbb{Z}^{(\mathbb{N})} &\;\to\; Y' \\ (x_1, \dots, x_j, 0, \dots) &\;\mapsto\; g'_j(x_1, \dots, x_j). \end{aligned}$$

It is straightforward to see that this definition is well-defined, i.e. independent of the choice of x_j. It is clear that for all $i \in \mathbb{N}_0$ we have $g'_i = F \circ g_i$. Furthermore, since any $x \in \mathbb{Z}^{(\mathbb{N})}$ is contained in the image of some g_j we see that F is the unique homomorphism that makes the diagram (2) commute.

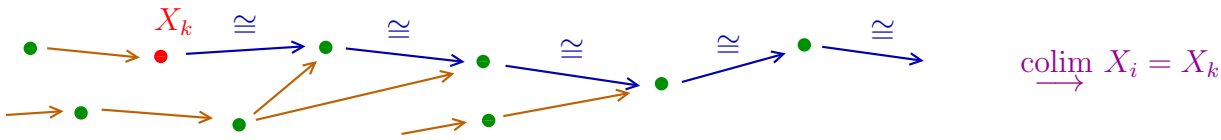
(B) As on page 217 we consider in the category of groups the sequence of monomorphisms $\langle x_1 \rangle \to \langle x_1, x_2 \rangle \to \dots$. Similar to the argument in (A) one can show that

$$\operatorname{colim}_{\longrightarrow} \langle x_1, x_2, \dots, x_i \rangle \;=\; \langle x_1, x_2, \dots \rangle \;=\; \text{free group on the generators } x_1, x_2, x_3, \dots$$

where the maps $F_j \to \operatorname{colim}_{\longrightarrow} F_i = \langle x_1, x_2, \dots \rangle$, $j \in \mathbb{N}$ are the obvious homomorphisms.

Lemma 25.1. (Direct Limit–Stabilization Lemma) Let $(I, \leq)$ be a directed set, let $\mathcal{C}$ be a category and let $(\{X_i\}_{i\in I}, \{f_{ji}\}_{i\leq j})$ be a direct system in the category $\mathcal{C}$. Suppose that there exists a $k \in I$ such that for all $l \geq k$ the morphism $f_{lk}\colon X_k \to X_l$ is in fact an isomorphism. Then $\operatorname{colim}_{\longrightarrow} X_i$ exists and it is naturally isomorphic to X_k. More precisely,

a direct limit is given by $\varinjlim X_i := X_k$ and for each $l \in I$ we consider the morphism $f_{mk}^{-1} \circ f_{ml} \colon X_l \to X_k$ where $m \in I$ is chosen such that $m \geq k$ and $m \geq l$.



Example.

(1) If $(I, \leq)$ is a finite directed set, then there exists a maximal element $i_{\max}$. It follows from the Direct Limit–Stabilization Lemma 25.1 that the direct limit is given by $X_{i_{\max}}$.

(2) We have

$$\varinjlim \left(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot(-1)} \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{\cdot(-1)} \mathbb{Z} \xrightarrow{\mathrm{id}} \dots\right) \underset{\substack{\uparrow \\ \text{by the Direct Limit–Stabilization Lemma 25.1}}}{\cong} \mathbb{Z}.$$

Proof. We leave the proof as an exercise to the reader. ■

Proposition 25.2. (Sets–Direct Limit Existence Proposition) Let $(I, \leq)$ be a directed set and let $(\{X_i\}_{i \in I}, \{f_{ji} \colon X_i \to X_j\}_{i \leq j})$ be a direct system in the category $\mathcal{Set}$ of sets.

(1) The direct limit is given by the set

$$\varinjlim X_i \;:=\; \Big(\bigsqcup_{i \in I} X_i\Big)/\sim \quad \text{where} \quad x \sim f_{ji}(x) \text{ for all } i \leq j,\, x \in X_i$$

together with the natural maps from each X_i into $\varinjlim X_i$.

(2) If all of the maps $f_{ji} \colon X_i \to X_j$ are inclusion maps, then we can also take

$$\varinjlim X_i \;:=\; \bigcup_{i \in I} X_i$$

together with the natural maps from each X_i into $\bigcup_{i \in I} X_i$.

Proof. Both statements follows easily from the definitions and elementary arguments. ■

Proposition 25.3. (Algebraic–Direct Limit Existence Proposition)

(1) Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be one of the following two categories:
   (a) the category $\mathcal{Group}$ of groups,
   (b) the category $\mathcal{AbGroup}$ of abelian groups.

   Let $(\{X_i\}_{i \in I}, \{f_{ji} \colon X_i \to X_j\}_{i \leq j})$ be a direct system in $\mathcal{C}$. A direct limit is given by

$$\text{(a)} \qquad \varinjlim X_i \;=\; \Big(\mathop{\ast}_{i \in I} X_i\Big)/\,\langle\langle f_{ji}(x) \cdot x^{-1} \mid \; i \leq j \text{ and } x \in X_i \rangle\rangle,$$

$$\text{(b)} \qquad \varinjlim X_i \;=\; \Big(\bigoplus_{i \in I} X_i\Big)/\,\langle f_{ji}(x) - x \mid \; i \leq j \text{ and } x \in X_i \rangle,$$

   with the obvious algebraic structure. Alternatively in each of the two categories a direct limit can be chosen whose underlying set is given by

$$\varinjlim X_i \;=\; \Big(\bigsqcup_{i \in I} X_i\Big)/\sim \quad \text{where } x \sim f_{ji}(x) \text{ for all } i \leq j,\, x \in X_i.$$

In particular in either of the two categories any direct system admits a direct limit.

(2) Suppose we are in the setting of (1). If all of the homomorphisms $f_{ji}\colon X_i \to X_j$ are inclusion maps, then as underlying set we can always take

$$\varinjlim X_i \;=\; \bigcup_{i\in I} X_i.$$

Proof. Let $(I,\leq)$ be a directed set and let $\mathcal{C}$ be one of the two given categories. Furthermore let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_i \to X_j\}_{i\leq j})$ be a direct system in $\mathcal{C}$.

(1) The first statement for the category of groups follows easily from the Coproduct-of-Groups Lemma 19.2. Similarly we see that the first statement for the category of abelian groups follows from the Coproduct-of-Abelian Groups Lemma 18.1.

Now we turn to the second statement. We set

$$\varinjlim X_i \;:=\; \Big(\bigsqcup_{i\in I} X_i\Big)/\sim \quad \text{where } x\sim f_{ji}(x) \text{ for all } i\leq j,\ x\in X_i$$

and we denote by $g_j\colon X_j \to \varinjlim X_i$, $j\in I$, the obvious maps. If we work in the category of (abelian) groups, then $\varinjlim X_i = \big(\bigsqcup_{i\in I} X_i\big)/\sim$ is naturally an (abelian) group as follows: Let $a,b \in \varinjlim X_i = \big(\bigsqcup_{i\in I} X_i\big)/\sim$. This means that there exist $i,j\in I$ and $\widetilde{a}\in X_i$ and $\widetilde{b}\in X_j$ with $a=[\widetilde{a}]$ and $b=[\widetilde{b}]$. By definition of a directed set there exists a $k\in I$ with $i\leq k$ and $j\leq k$. We set

$$a\cdot b \;:=\; \big[\underbrace{f_{ki}(\widetilde{a})}_{\in X_k}\cdot\underbrace{f_{kj}(\widetilde{b})}_{\in X_k}\big] \;\in\; \varinjlim X_i.$$

It follows easily from the definition of a direct system that this definition of the product structure is independent of the choice of k and that this defines indeed an (abelian) group structure on $\varinjlim X_i$.

(2) This statement follows easily from the definitions. ■

Example. We consider the two direct systems that are given by $\{\mathbb{Z}^i\}_{i\in\mathbb{N}}$ and $\{\mathbb{Z}^i\oplus 0\}_{i\in\mathbb{N}}$ where $\mathbb{Z}^i\oplus 0\subset\mathbb{Z}^\infty$. We have the following isomorphisms:

$$\varinjlim \mathbb{Z}^i \;\underset{\substack{\uparrow\\ \text{since the two direct systems}\\ \text{are isomorphic}}}{\cong}\; \varinjlim (\mathbb{Z}^i\oplus 0) \;\underset{\substack{\uparrow\\ \text{Algebraic–Direct Limit Existence}\\ \text{Proposition 25.3}}}{\cong}\; \mathbb{Z}^\infty.$$

We have thus obtained the result from page 218 in a more systematic fashion.

25.3. The direct limit in the topological category.

Proposition 25.4. (Topological-Direct Limit Proposition) Let

$$X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$$

be a sequence of topological spaces such that each $\varphi_i\colon X_i\to X_{i+1}$ is an embedding.

(0) We consider the set

$$\varinjlim X_i \;:=\; \Big(\bigsqcup_{i\in\mathbb{N}_0} X_i\Big)/\sim \quad \text{where } x\sim\varphi_i(x) \text{ for all } i\in\mathbb{N}_0 \text{ and } x\in X_i.$$

Given $i \in \mathbb{N}_0$ we denote by $g_i \colon X_i \to \varinjlim X_i$ the natural map.

(a) We say $U \subset \varinjlim X_i$ is open if and only if for each $i \in \mathbb{N}_0$ the preimage $g_i^{-1}(U)$ is an open subset of X_i. This defines a topology on $\varinjlim X_i$. In the remainder of this proposition we equip $\varinjlim X_i$ with this topology.

(b) A subset $U \subset \varinjlim X_i$ is closed with respect to the above topology if and only if for each $i \in \mathbb{N}_0$ the preimage $g_i^{-1}(U)$ is a closed subset of X_i.

(1) (a) For each $i \in \mathbb{N}_0$ the natural map $g_i \colon X_i \to \varinjlim X_i$ is an embedding.

(b) Let Y' be a topological space. Given a family of continuous maps map $g_i' \colon X_i \to Y'$ with the property that for any $i \in \mathbb{N}_0$ we have $g_{i+1}' \circ \varphi_i = g_i'$ the map

$$\begin{array}{rcll} G' \colon \varinjlim X_i & \to & Y' & \\ x & \mapsto & g_i'(x_i) & \quad \text{if } x = g_i(x_i) \text{ for some } x_i \in X_i \end{array}$$

is well-defined and it is continuous.

($*$) A map $h \colon \varinjlim X_i \to Y'$ to some topological space is continuous if and only if each map $h \circ g_i \colon X_i \to Y'$ is continuous.

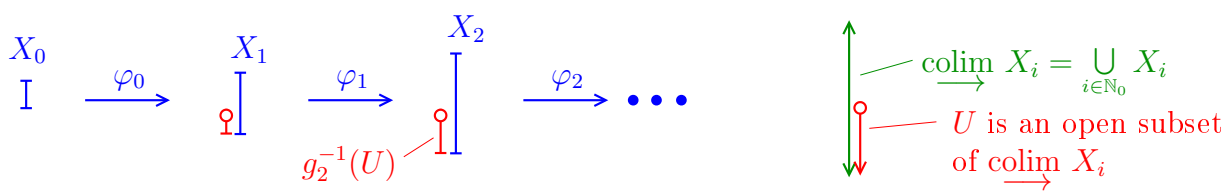
(2) The topological space $\varinjlim X_i$, together with the natural inclusions $g_i \colon X_i \to \varinjlim X_i$, is the direct limit of the above sequence in the category of topological spaces.

If all the embeddings $\varphi_i \colon X_i \to X_{i+1}$ are in fact inclusion maps, then we can replace (0) by

(0') We consider the set

$$\varinjlim X_i \ := \ \bigcup_{i \in \mathbb{N}_0} X_i.$$

We say $U \subset X$ is open if and only if for each $i \in \mathbb{N}_0$ the intersection $U \cap X_i$ is an open subset of X_i. This defines a topology on X which again satisfies (1a), (1b) and (2).



Proof. Let

$$X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$$

be a sequence of topological spaces such that each $\varphi_i \colon X_i \to X_{i+1}$ is an embedding. We set

$$\varinjlim X_i \ := \ \Big(\bigsqcup_{i \in \mathbb{N}_0} X_i \Big) / \sim \quad \text{where } x \sim \varphi_i(x) \text{ for all } i \in \mathbb{N}_0 \text{ and } x \in X_i.$$

For each $i \in \mathbb{N}_0$ we consider the natural inclusion $g_i \colon X_i \to \varinjlim X_i$.

(0) (a) We set

$$\mathcal{T} \ := \ \big\{ U \subset \varinjlim X_i \,|\, \text{for each } i \in \mathbb{N} \text{ the preimage } g_i^{-1}(U) \text{ is open in } X_i \big\}.$$

One can easily verify, say using the Image-Preimage Lemma 0.3, that $\mathcal{T}$ defines a topology on $\varinjlim X_i$.

(b) This statement follows immediately from the definition of being closed and the observation that by the Image-Preimage Lemma 0.3 (8) we always have the equality $g_i^{-1}(\varinjlim X_i \setminus U) = g_i^{-1}(\varinjlim X_i) \setminus g_i^{-1}(U) = X_i \setminus g_i^{-1}(U)$.

(1) (a) Let $j \in \mathbb{N}_0$. We need to show that the natural inclusion $g_j \colon X_j \to \varinjlim X_i$ is an embedding. The map is evidently injective. It follows immediately from the definition of $\mathcal{T}$ that $g_j \colon X_j \to \varinjlim X_i$ is continuous. To show that $g_j \colon X_j \to X$ is actually an embedding it remains to prove the following claim.

Claim. Given any open subset $U \subset X_j$ there exists an open subset $V \subset \varinjlim X_i$ with $V \cap g_j(X_j) = g_j(U)$.

Proof. Let $U_j \subset X_j$ be an open subset. Since $\varphi_j \colon X_j \to X_{j+1}$ is an embedding there exists an open subset $U_{j+1} \subset X_{j+1}$ such that $\varphi_j(X_j) \cap U_{j+1} = \varphi_j(U_j)$. We iterate this procedure to obtain subsets $U_j, U_{j+1}, U_{j+2}, \ldots$. We set $V := \bigcup_{k \geq j} g_k(U_k)$. Evidently we have $V \cap g_j(X_j) = g_j(U_j)$. It remains to show that V is open in $\varinjlim X_i$. This means that we need to show that for every $i \in \mathbb{N}_0$ the preimage $g_i^{-1}(V)$ is an open subset of X_i. For $i \geq j$ this follows immediately from the construction of V. It remains to consider the case $i < j$. First note that we have $g_{j-1}^{-1}(V) = (g_j \circ \varphi_{j-1})^{-1}(V) = \varphi_{j-1}^{-1}(g_j^{-1}(V)) = \varphi_{j-1}^{-1}(U_j)$. Since $U_j \subset X_j$ is open and since $\varphi_{j-1} \colon X_{j-1} \to X_j$ is continuous we see that $g_{j-1}^{-1}(U_j)$ is indeed an open subset of X_{j-1}. Iterating this argument we see that for all $i \in \{0, \ldots, j-1\}$ the preimage $g_i^{-1}(V)$ is an open subset of X_i. ⊞

(b) As in the proof of the Sets–Direct Limit Existence Proposition 25.2 we see that the map $G' \colon \varinjlim X_i \to Y'$ is well-defined. By construction we have $G' \circ g_i = g_i'$ for each $i \in \mathbb{N}_0$. It remains to show that the map G' is continuous. Thus let V be an open subset of Y'. We need to show that $(G')^{-1}(V)$ is an open subset of $\varinjlim X_i$. By definition of the topology on $\varinjlim X_i$ this means that we have to show that for each $i \in \mathbb{N}_0$ the preimage $g_i^{-1}((G')^{-1}(V))$ is an open subset of X_i. But by construction of G' we have $G' \circ g_i = g_i'$. Thus we see that $g_i^{-1}((G')^{-1}(V)) = (g_i')^{-1}(V)$, which is of course an open subset of X_i since g_i' is continuous.

(∗) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) This statement follows from (1a) and (1b) together with the observation, which we proved in the Sets–Direct Limit Existence Proposition 25.2, that the map G in (1b) is the unique map which satisfies $G \circ g_i = g_i'$ for all $i \in \mathbb{N}_0$.

Finally note that if all the maps $f_{ji} \colon X_i \to X_j$ are in fact inclusion maps, then it follows by the same argument as in the Sets–Direct Limit Existence Proposition 25.2 (2) that we can work with $\varinjlim X_i := \bigcup_{i \in \mathbb{N}_0} X_i$. We leave it to the reader to make the few adjustments to finish off this proof. ■

Lemma 25.5. (Direct Limit–Open Subset Lemma) Let X be a topological space and let $X_0 \subset X_1 \subset X_2 \subset \ldots$ be subspaces of X with $\varinjlim X_i = X$. Suppose that we are given $w \in \mathbb{N}_0$ and suppose that for each $i \geq w$ we are given an open subset $U_i \subset X_i$. If for each $i \geq w$ we have $U_i \subset U_{i+1}$, then $\bigcup_{i \geq w} U_i$ is an open subset of X.

Proof. We set $U := \bigcup_{i \geq w} U_i$. Since $X = \varinjlim X_i$ it suffices to show that for each $j \in \mathbb{N}_0$ the intersection $X_j \cap U$ is open in X_j. Thus let $j \in \mathbb{N}_0$. We set $m := \max\{j, w\}$. Note that

$$X_j \cap U \;=\; X_j \cap \bigcup_{i \geq w} U_i \;\underset{\substack{\uparrow \\ \text{since for each } i \geq w \text{ we have } U_i \subset U_{i+1}}}{=}\; X_j \cap \bigcup_{i \geq m} U_i \;=\; \bigcup_{i \geq m} (X_j \cap U_i).$$

Next note that for $i \geq m$ it follows from the fact that U_i is open in X_i that $X_j \cap U$ is open in X_j. In summary we have shown that $X_j \cap U$ is the union of open subsets of X_j, thus $X_j \cap U$ is an open subset of X_j. ■

Criterion 25.6. (Union-equals-Direct Limit Criterion) Let X be a topological space and let $X_0 \subset X_1 \subset X_2 \subset \ldots$ be subspaces of X. We assume that the following two conditions are satisfied:

(1) We have $\bigcup_{i \in \mathbb{N}_0} X_i = X$.

(2) For each $i \in \mathbb{N}_0$ we have $X_i \subset \mathring{X}_{i+1}$.

Then $X = \varinjlim X_i$ as topological spaces.

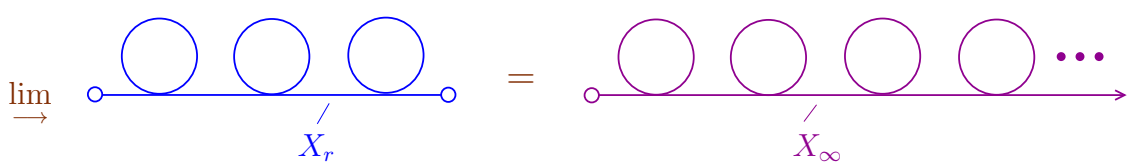
Example. Given $r \in \mathbb{N}_0$ we set

$$X_r \;:=\; \big(0, r + \tfrac{1}{2}\big) \;\cup\; \bigcup_{j=1}^{r} \underbrace{\big\{z \in \mathbb{C} \,\big|\, \big|z - (j + \tfrac{1}{4}\mathrm{i})\big| = \tfrac{1}{4}\big\}}_{\text{circle of radius } \frac{1}{4} \text{ around } j + \frac{1}{4}\mathrm{i} \in \mathbb{C}} \;\subset\; \mathbb{C}.$$

In other words, for $r \in \mathbb{N}_0$ the set X_r is the subspace of $\mathbb{C} = \mathbb{R}^2$ that is given by the interval $(0, r + \frac{1}{2})$ with r circles attached. We obtain a sequence of continuous maps by considering the inclusions $X_r \to X_{r+1}$. It follows fairly easily from the Union-equals-Direct Limit Criterion 25.6 that the direct limit is the interval $(0, \infty)$ with infinitely many circles attached, i.e.

$$\varinjlim X_r \;=\; X_\infty \;:=\; \big(0, \infty\big) \;\cup\; \bigcup_{j=1}^{\infty} \big\{z \in \mathbb{C} \,\big|\, \big|z - (j + \tfrac{1}{4}\mathrm{i})\big| = \tfrac{1}{4}\big\} \;\subset\; \mathbb{C},$$

where the maps $X_s \to \varinjlim X_r = X_\infty$, $s \in \mathbb{N}_0$ are the obvious inclusion maps.



Proof. If $U \subset X$ is open, then, by definition of the subspace topology, each $U \cap X_i$ is open in X_i. This implies that $U \subset \varinjlim X_i$ is open.

Now assume that $U \subset \varinjlim X_i$ is open. We need to show that $U \subset X$ is open. Since U is open in $\varinjlim X_i$ we know that each $U \cap X_i$ is open in X_i. By definition of the subspace topology this means that for each $i \in \mathbb{N}_0$ there exists an open subset $V_i \subset X$ such that $V_i \cap X_i = U \cap X_i$. Now we see that

$$\begin{aligned}
\bigcup_{i\in\mathbb{N}_0} (V_i \cap \mathring{X}_i) &\subset \bigcup_{i\in\mathbb{N}_0} (V_i \cap X_i) \overset{\substack{\text{since } V_i \cap X_i = U \cap X_i \\ \downarrow}}{=} \bigcup_{i\in\mathbb{N}_0} (U \cap X_i) = U \cap \bigcup_{i\in\mathbb{N}_0} X_i \overset{\substack{\text{follows from (1)} \\ \downarrow}}{=} U \\
&\underset{\substack{\uparrow \\ \text{since it follows from (1) and (2)} \\ \text{that } X = \bigcup_{i\in\mathbb{N}_0} \mathring{X}_i}}{=} U \cap \bigcup_{i\in\mathbb{N}_0} \mathring{X}_i = \bigcup_{i\in\mathbb{N}_0} (U \cap \mathring{X}_i) \underset{\substack{\uparrow \\ \text{since } V_i \cap X_i = U \cap X_i \text{ and since } \mathring{X}_i \subset X_i}}{=} \bigcup_{i\in\mathbb{N}_0} (V_i \cap \mathring{X}_i).
\end{aligned}$$

We see that all inclusions are equalities. This implies that $U = \bigcup_{i\in I} (V_i \cap \mathring{X}_i)$. Thus we have shown that U is the union of open subsets, which implies that U itself is open. ■

25.4. Examples and properties of the direct limit of topological spaces.

Definition.

(1) We consider the *set*

$$\mathbb{R}^\infty := \mathbb{R}^{(\mathbb{N})} = \{(x_1, x_2, \dots) \mid x_i \in \mathbb{R} \text{ but only finitely many } x_i\text{'s are non-zero}\}$$

(2) Given a subset $X \subset \mathbb{R}^k$ we write

$$X \times \{0\} := \{(x, 0, 0, \dots) \mid x \in X\} \subset \mathbb{R}^\infty.$$

We have an obvious bijection of $X \times \{0\}$ to X and we use it to equip $X \times \{0\}$ with a topology.

(3) Unless we explicitly say something else we equip the set $\mathbb{R}^\infty$ with the topology given by $\mathbb{R}^\infty = \varinjlim (\mathbb{R}^i \times \{0\})$. In other words, $U \subset \mathbb{R}^\infty$ is open if and only if for each $i \in \mathbb{N}_0$ the intersection $U \cap (\mathbb{R}^i \times \{0\})$ is open in $\mathbb{R}^i \times \{0\}$.

(4) Similarly we define $\mathbb{C}^\infty := \varinjlim (\mathbb{C}^i \times \{0\})$ as a topological space.

Example.

(1) To practice the above definition we will show that any linear map $\mathbb{R}^\infty \to \mathbb{R}^\infty$ is continuous with respect to the above topology. The proof proceeds in two steps:

(a) By the Topological-Direct Limit Proposition 25.4 (1b) it suffices to that show for each $n \in \mathbb{N}_0$ the map

$$\begin{aligned} f_n \colon \mathbb{R}^n \times \{0\} &\to \mathbb{R}^\infty \\ x &\mapsto f(x, 0) \end{aligned}$$

is continuous.

(b) Let $n \in \mathbb{N}_0$. Note that there exists a $k \in \mathbb{N}$ such that $f(e_1), \dots, f(e_n) \in \mathbb{R}^k \times \{0\}$. Thus we obtain the following commutative diagram

$$\begin{tikzcd}
\mathbb{R}^n \times \{0\} \arrow[r, "x \mapsto f(x,0)"] \arrow[d, "(x,0) \mapsto x"] & \mathbb{R}^\infty \\
\mathbb{R}^n \arrow[r, "\widetilde{f}"] & \mathbb{R}^k \arrow[u, "y \mapsto (y,0)"']
\end{tikzcd}$$

for a suitable linear map $\widetilde{f}\colon \mathbb{R}^n \to \mathbb{R}^k$. We make three observations:

- The left vertical map is a homeomorphism.
- Each linear map $\mathbb{R}^n \to \mathbb{R}^k$ is continuous.
- By the Topological-Direct Limit Proposition 25.4 (1a) we know that the right vertical map is continuous.

It follows that the top horizontal map is continuous.

(2) We consider the map

$$\begin{array}{rcl} \mathbb{C}^\infty & \to & \mathbb{R}^\infty \\ (x_1 + \mathrm{i}y_1, x_2 + \mathrm{i}y_2, \dots) & \mapsto & (x_1, y_1, x_2, y_2, \dots). \end{array}$$

This map has an obvious inverse. Using the approach from (1) one can easily show that both maps are continuous, thus they are both homeomorphisms. At times we will use these homeomorphisms to make the identification $\mathbb{C}^\infty =_i \mathbb{R}^\infty$

Definition. We consider the topological space

$$S^\infty \;:=\; \Big\{(x_1, x_2, \dots) \in \mathbb{R}^\infty \,\Big|\, \sum_{j=1}^{\infty} x_j^2 = 1\Big\} \;=\; \varinjlim\, (S^{i-1} \times \{0\}).$$

Definition. Let G be a group that acts on two sets X and Y. A map $f\colon X \to Y$ is called G-equivariant if given any $x \in X$ and $g \in G$ we have $f(g \cdot x) = g \cdot f(x)$.

Lemma 25.7. (Direct Limit–Action Lemma) Let

$$X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$$

be a sequence of topological spaces such that each $\varphi_i\colon X_i \to X_{i+1}$ is an embedding. Let G be a group. Suppose that for each $i \in \mathbb{N}_0$ we are given a continuous action of G on X_i such that each φ_i is G-equivariant. It follows almost immediately from this data that G also acts continuously on $\varinjlim X_i$.

(1) The natural $g_i\colon X_i \to \varinjlim X_i$ maps are G-equivariant.

(2) The resulting natural map $\varinjlim\, (X_i/G) \to \big(\varinjlim X_i\big)/G$ is a homeomorphism.

Sketch of proof. For $g \in G$ let $\theta_i^g\colon X_i \to X_i$ be the continuous map that is given by multiplication by g. We consider the continuous maps $X_i \xrightarrow{\theta_i^g} X_i \to \varinjlim X_i$. By our hypothesis on the actions and the universal property of direct limits we get a unique induced continuous map

$$\theta^g\colon \varinjlim X_i \;\to\; \varinjlim X_i.$$

Using the uniqueness of the universal property one can easily verify that these maps define a continuous G-action on $\varinjlim X_i$ and that the natural inclusions $X_i \to \varinjlim X_i$ are G-equivariant. Thus we get induced maps

$$X_i/G \;\to\; (\varinjlim X_i)/G.$$

It follows from the Twice Quotient Lemma 6.3 that these maps are continuous. By the universal property of direct limits we obtain a continuous map

$$\Xi\colon \varinjlim\, (X_i/G) \;\to\; \big(\varinjlim X_i\big)/G.$$

It follows from the explicit description of the direct limit in the category of topological spaces that we gave in the Topological-Direct Limit Proposition 25.4 that this map Ξ is a bijection. We want to show that Ξ is a homeomorphism. It remains to prove the following claim.

Claim. The map $\Xi\colon \varinjlim (X_i/G) \to \big(\varinjlim X_i\big)/G$ is an open map.

Proof. For $i \in \mathbb{N}_0$ we denote by $p_i\colon X_i \to X_i/G$ the natural projection. Furthermore we denote by $q\colon \varinjlim X_i \to \big(\varinjlim X_i\big)/G$ the natural projection. Let $U \subset \varinjlim (X_i/G)$ be an open subset. By definition of the direct limit topology this means that for each $i \in \mathbb{N}_0$ the intersection $U_i := (X_i/G) \cap U$ is an open subset of X_i/G. Note that by definition of the quotient topology this means that $\widetilde{U}_i := p_i^{-1}(U_i)$ is an open subset of X_i. We set $\widetilde{U} := \bigcup_{i\in\mathbb{N}_0} \widetilde{U}_i \subset \varinjlim X_i$. It follows easily from the definition of the direct limit topology that $\widetilde{U}$ is an open subset of $\varinjlim X_i$. Next note that it follows from an elementary argument that $q(\widetilde{U})$ is an open subset of $\big(\varinjlim X_i\big)/G$. Finally note that it follows easily from the various definitions that $\Xi(U) = q(\widetilde{U})$. We have thus shown that Ξ is indeed an open map. ■

Definition.

(1) We consider the sequence

$$\mathbb{R}\mathrm{P}^0 \xrightarrow{[x]\mapsto[x:0]} \mathbb{R}\mathrm{P}^1 \xrightarrow{[x]\mapsto[x:0]} \mathbb{R}\mathrm{P}^2 \xrightarrow{[x]\mapsto[x:0]} \dots$$

of embeddings. We refer to

$$\mathbb{R}\mathrm{P}^\infty := \varinjlim \mathbb{R}\mathrm{P}^i$$

as the infinite-dimensional real projective space.

(2) The same way we also define the infinite-dimensional complex projective space $\mathbb{C}\mathrm{P}^\infty$.

Lemma 25.8. (Infinite Projective Space-Sphere Quotient Lemma)

(1) The obvious action of the group $\{\pm 1\}$ on S^∞ is continuous and discrete.

(2) There exists a homeomorphism $S^\infty/\{\pm 1\} \to \mathbb{R}\mathrm{P}^\infty$.

Proof.

(1) We leave the verification of this statement to the reader.

(2) We consider the following commutative diagram:

$$\begin{tikzcd}
\varinjlim (S^n/\{\pm 1\}) \arrow[r] \arrow[d] & \varinjlim \mathbb{R}\mathrm{P}^n \arrow[d] \\
S^\infty/\{\pm 1\} \arrow[r] & \mathbb{R}\mathrm{P}^\infty.
\end{tikzcd}$$

It follows from the Projective Space-Sphere Quotient Lemma 6.9 that the top horizontal map is a homeomorphism. The right vertical map is a homeomorphism by definition. Finally the left vertical map is a homeomorphism by the Direct Limit–Action Lemma 25.7. It follows that the bottom map is a homeomorphism. ■

26. Fundamental groups and direct limits

Proposition 26.1. (Direct Limit-π_1-Proposition) Let

$$(X_0, w_0) \xrightarrow{\varphi_0} (X_1, w_1) \xrightarrow{\varphi_1} (X_2, w_2) \xrightarrow{\varphi_2} \dots$$

be a sequence of pointed topological spaces such that each $\varphi_i \colon X_i \to X_{i+1}$ is an embedding with $\varphi_i(w_i) = w_{i+1}$. For each $j \in \mathbb{N}_0$ we denote by $g_j \colon X_j \to \varinjlim X_i$ the natural map. If for each $i \in \mathbb{N}_0$ we have $g_i(X_i) \subset \operatorname{int}(g_{i+1}(X_{i+1}))$, then we have a natural isomorphism

$$\varinjlim \pi_1(X_i, w_i) \xrightarrow{\cong} \pi_1\big(\varinjlim X_i, g_0(w_0)\big).$$

Proposition 26.2. (Exhaustion-π_1-Proposition) Let X be a topological space and let

$$X_0 \subset X_1 \subset X_2 \subset X_3 \subset \dots$$

be a sequence of subspaces of X. We assume that the following two conditions are satisfied:

(1) We have $\bigcup_{i \in I} X_i = X$.

(2) For each $i \in \mathbb{N}_0$ we have $X_i \subset \mathring{X}_{i+1}$.

Then for any $w \in X_0$ we have a natural isomorphism

$$\varinjlim \pi_1(X_i, w) \xrightarrow{\cong} \pi_1(X, w).$$

Proof of Proposition 26.2 assuming Proposition 26.1. We consider the following natural commutative diagram:

$$\begin{tikzcd}
\varinjlim \pi_1(X_i, w) \arrow[rr] \arrow[dr] & & \pi_1\big(\varinjlim X_i, w\big) \arrow[dl] \\
& \pi_1(X, w). &
\end{tikzcd}$$

It follows from our hypothesis that we can apply the Direct Limit-π_1-Proposition 26.1, which then tells us, that the horizontal map is an isomorphism. Since we know that for each $i \in \mathbb{N}_0$ we have $X_i \subset \mathring{X}_{i+1}$ we obtain from the Union-equals-Direct Limit Criterion 25.6 that we have a natural homeomorphism $\varinjlim X_i \to X$. Thus the right diagonal map is an isomorphism. It follows from this discussion that the left diagonal map is an isomorphism. ■

Lemma 20.11. (Compact Subspace–Exhaustion Lemma) Let X be a topological space and let

$$X_0 \subset X_1 \subset X_2 \subset X_3 \subset \dots$$

be a sequence of subspaces of X such the following two conditions are satisfied:

(1) We have $\bigcup_{i \in \mathbb{N}_0} X_i = X$.

(2) For each $i \in \mathbb{N}_0$ we have $X_i \subset \mathring{X}_{i+1}$.

Given any compact subset K of X there exists an $i \in \mathbb{N}_0$ with $K \subset X_i$.

Proof of the Direct Limit-π_1-Proposition 26.1. Let

$$(X_0, w_0) \xrightarrow{\varphi_0} (X_1, w_1) \xrightarrow{\varphi_1} (X_2, w_2) \xrightarrow{\varphi_2} \dots$$

be a sequence of topological spaces such that each $\varphi_i\colon X_i \to X_{i+1}$ is an embedding. For each $i \in \mathbb{N}_0$ we denote by $g_i\colon X_i \to \varinjlim X_i$ the natural map. We assume that for each $i \in \mathbb{N}_0$ we have $g_i(X_i) \subset \operatorname{int}(g_{i+1}(X_{i+1}))$. We set $X := \varinjlim X_i$. It follows from the Topological-Direct Limit Proposition 25.4 that the natural maps $g_i\colon X_i \to X$ are embeddings and that the union of the images of the X_i is all of X. Thus we can and will view each X_i as a subspace of X. In summary we can now assume that we have a sequence

$$X_0 \;\subset\; X_1 \;\subset\; X_2 \;\subset\; X_3 \;\subset\; \dots$$

of subspaces of X such the following two conditions are satisfied:

(a) We have $\bigcup_{i\in\mathbb{N}_0} X_i = X$.

(b) For each $i \in \mathbb{N}_0$ we have $X_i \subset \mathring{X}_{i+1}$.

Let $w \in X_0$. Given $k < l$ we denote by $\theta_{lk}\colon \pi_1(X_k, w) \to \pi_1(X_l, w)$ the inclusion induced homomorphism. Given $k \in \mathbb{N}_0$ we denote by $\psi_k\colon \pi_1(X_k, w) \to \varinjlim \pi_1(X_i, w)$ the natural homomorphism.

Note that by the universal property of a direct limit there exists a unique homomorphism $\Phi\colon \varinjlim \pi_1(X_i, w) \to \pi_1(X, w)$ that makes the following diagram commute for any $k < l$:

$$\begin{tikzcd}
\pi_1(X_l, w) \arrow[dr, "\psi_l"] \arrow[drr, bend left] & & \\
& \varinjlim \pi_1(X_i, w) \arrow[r, dashed, "\Phi", "\exists!"'] & \pi_1(X, w). \\
\pi_1(X_k, w) \arrow[uu, "\theta_{lk}"] \arrow[ur, "\psi_k"] \arrow[urr, bend right] & &
\end{tikzcd}$$

We need to show that Φ is an isomorphism.

(1) First we show that the map Φ is surjective. Let $g \in \pi_1(X, w)$.

Claim. There exists a compact subset K of X with $w \in K$ such that g lies in the image of the inclusion induced map $\pi_1(K, w) \to \pi_1(X, w)$.

Proof. We pick a loop $\gamma\colon [0,1] \to X$ that represents $g \in \pi_1(X, w)$. By the Compact Image Lemma 2.8 the image $\gamma([0,1])$ is compact. Thus $K := \gamma([0,1])$ has the desired property. ⊞

Note that it follows from the claim together with the Compact Subspace–Exhaustion Lemma 20.11 that there exists a $k \in \mathbb{N}_0$ such that g lies in the image of the inclusion induced map $\pi_1(X_k, w) \to \pi_1(X, w)$. We consider the following commutative diagram

$$\begin{tikzcd}
& \varinjlim \pi_1(X_i, w) \arrow[r, "\Phi"] & \pi_1(X, w). \\
\pi_1(X_k, w) \arrow[ur, "\psi_k"] \arrow[urr, bend right, "{[\gamma]\mapsto g}"'] & &
\end{tikzcd}$$

By our choice of k we see that g lies in the image of the curved map, hence it also lies in the image of Φ. This concludes the proof that Φ is surjective.

(2) Now we show that the map Φ is injective. So let $g \in \ker\big(\Phi\colon \varinjlim \pi_1(X_i, w) \to \pi_1(X, w)\big)$. By the Algebraic–Direct Limit Existence Proposition 25.3 we can write

$$\varinjlim \pi_1(X_i, w) \;=\; \Big(\bigsqcup_{i\in\mathbb{N}_0} \pi_1(X_i, w)\Big)/\sim \quad \text{where } g \sim \theta_{lk}(g) \text{ for all } k \le l,\ g \in \pi_1(X_k, w).$$

It follows in particular from this explicit description that there exists a $k \in \mathbb{N}_0$ and a $g_k \in \pi_1(X_k, w)$ with $g = \psi_k(g_k)$.

Claim. There exists a compact subset K of X such that g_k lies in the kernel of the inclusion induced map $\pi_1(X_k, w) \to \pi_1(X_k \cup K, w)$.

Proof. We pick a loop $\gamma\colon [0,1] \to X_k$ that represents g_k. By the above we know that $[\gamma] \in \ker(\pi_1(X_k, w) \to \pi_1(X, w))$. By definition this means that there exists a path-homotopy $H\colon [0,1]\times[0,1] \to X$ from the path γ to the constant path. We appeal again to the Compact Image Lemma 2.8 to deduce that $K := H([0,1]\times[0,1])$ is a compact subset of X. Evidently K has the desired property. ⊞

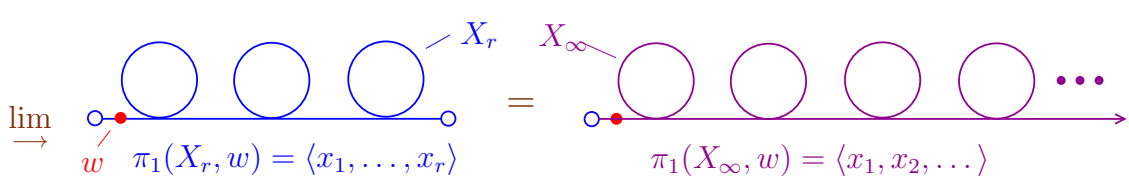
It follows from the claim, the Compact Subspace–Exhaustion Lemma 20.11 and the fact that we always have $X_i \subset X_{i+1}$ that there exists an $l \ge k$ such that $\theta_{lk}([\gamma])$ is trivial in $\pi_1(X_l, w)$.
We obtain the following diagram

$$\begin{tikzcd}
\pi_1(X_l, w) \arrow[dr, "\psi_l"] & & \\
& \varinjlim \pi_1(X_i, w) \arrow[r, "\Phi", "g \mapsto e"'] & \pi_1(X, w). \\
\pi_1(X_k, w) \arrow[uu, "\theta_{lk}", "g \mapsto e"'] \arrow[ur, "\psi_k", "g_k \mapsto g"'] & &
\end{tikzcd}$$

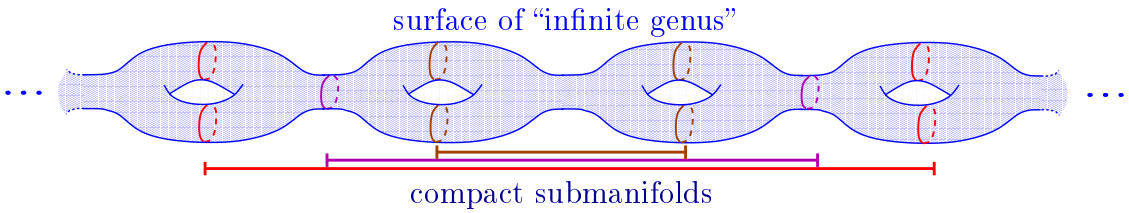
It follows immediately that g itself is trivial. This concludes the proof that Φ is injective. ■

Example. As on page 223 we denote by X_∞ the interval $(0, \infty)$ with infinitely many circles attached. We pick the base point $w = \frac{1}{4}$. We obtain that

$$\pi_1\big(X_\infty, w\big) \underset{\substack{\uparrow \\ \text{see page 218}}}{=} \pi_1\big(\varinjlim X_r, w\big) \underset{\substack{\uparrow \\ \text{Direct Limit-}\pi_1\text{-Proposition 26.1}}}{=} \varinjlim \pi_1\big(X_r, w\big) \overset{\substack{\text{Graph-}\pi_1\text{-Proposition 20.9} \\ \downarrow}}{=} \varinjlim \langle x_1, \dots, x_r\rangle \underset{\substack{\uparrow \\ \text{see page 218}}}{=} \langle x_1, x_2, \dots\rangle.$$



Example. In [Fri23] we will use the Direct Limit-π_1-Proposition 26.1 to show that the fundamental group of the surface of infinite genus is the free group on infinitely many generators.


surface of “infinite genus”
compact submanifolds

27. Universal coverings

27.1. Lifting maps to coverings.

Definition. Let $p\colon (X,x_0) \to (B,b_0)$ be a covering of pointed topological spaces. Furthermore let $f\colon (Z,z_0) \to (B,b_0)$ be a continuous map. A lift of f is a continuous map $\widetilde{f}\colon (Z,z_0) \to (X,x_0)$ that makes the following diagram commute

$$\begin{tikzcd} & (X,x_0) \arrow[d, "p"] \\ (Z,z_0) \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & (B,b_0). \end{tikzcd}$$

Example.

(1) We consider the maps:

$$\begin{tikzcd} & (S^1,1) \arrow[d, "p"', "z\mapsto z^3"] \\ (S^1,1) \arrow[ur, dashed, "???\ \text{lift}\ \widetilde{f}???"] \arrow[r, "z\mapsto z^2", "f"'] & (S^1,1). \end{tikzcd}$$

Here the map p is a 3-fold covering. Does the map f lift to a map $\widetilde{f}$?

(2) To answer the question we just posed in (1) it is helpful to consider a more general setup. Thus let $p\colon (X,x_0) \to (B,b_0)$ be a covering of pointed topological spaces and let $f\colon (Z,z_0) \to (B,b_0)$ be a continuous map. Let us suppose that there exists a lift $\widetilde{f}\colon (Z,z_0) \to (X,x_0)$. We obtain the commutative diagram

$$\begin{tikzcd} & (X,x_0) \arrow[d, "p"] \\ (Z,z_0) \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & (B,b_0) \end{tikzcd}$$

from the functoriality of the fundamental group it follows that $f_* = \big(p \circ \widetilde{f}\big)_* = p_* \circ \widetilde{f}_*$, i.e. we get the commutative diagram

$$\begin{tikzcd} & \pi_1(X,x_0) \arrow[d, "p_*"] \\ \pi_1(Z,z_0) \arrow[ur, "\widetilde{f}_*"] \arrow[r, "f_*"'] & \pi_1(B,b_0). \end{tikzcd}$$

Thus it follows that if f lifts, then

$$\mathrm{im}(f_*) \;=\; \mathrm{im}((p \circ \widetilde{f})_*) \;=\; \mathrm{im}(p_* \circ \widetilde{f}_*) \subset \mathrm{im}(p_*).$$

(3) In the above example (1) we have $\pi_1(S^1,1) = \mathbb{Z}$. Note that $\mathrm{im}(f_*) = 2\mathbb{Z}$ and we have $\mathrm{im}(p_*) = 3\mathbb{Z}$. Since $2\mathbb{Z} \not\subset 3\mathbb{Z}$ we see that f does *not* lift.

Proposition 27.1. (Map Lifting Criterion) Let $p\colon (X,x_0) \to (B,b_0)$ be a covering of pointed topological spaces. Furthermore let Z be a path-connected topological space and let $f\colon (Z,z_0) \to (B,b_0)$ be a continuous map. If Z is locally path-connected, then we have the following equivalence of statements:

$$\begin{tikzcd} & (X,x_0) \arrow[d, "p"] \\ (Z,z_0) \arrow[ur, dashed, "\text{there exists a lift}\ \widetilde{f}"] \arrow[r, "f"'] & (B,b_0) \end{tikzcd} \qquad \Longleftrightarrow \qquad \mathrm{im}(f_*) \subset \mathrm{im}(p_*).$$

Example. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. The condition on the fundamental groups in the Map Lifting Criterion 27.1 is automatically satisfied if Z is simply connectede.g. if $Z = S^n$ with $n \geq 2$.

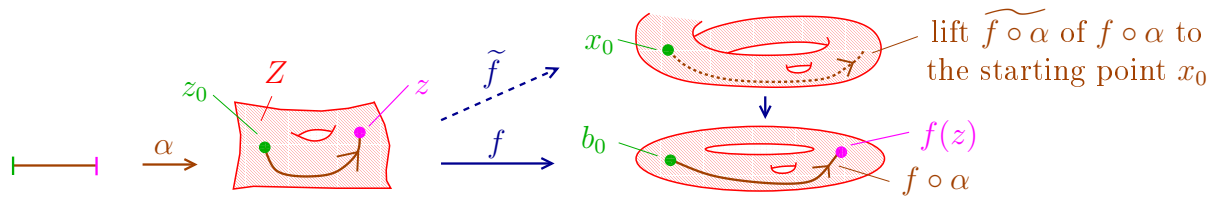
Proof of the Map Lifting Criterion 27.1. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. Furthermore let Z be a path-connected and locally path-connected topological space and let $f\colon (Z, z_0) \to (B, b_0)$ be a continuous map.

By the discussion on page 231 we only have to prove the "$\Longleftarrow$"-direction of the proposition. So suppose that we have

$$\operatorname{im}(f_*\colon \pi_1(Z, z_0) \to \pi_1(B, b_0)) \ \subset\ \operatorname{im}(p_*\colon \pi_1(X, x_0) \to \pi_1(B, b_0)).$$

We have to construct a lift $\widetilde{f}\colon (Z, z_0) \to (X, x_0)$. We will do so as follows. Let $z \in Z$ be a point. Since Z is path-connected we can find a path $\alpha\colon [0,1] \to Z$ with $\alpha(0) = z_0$ and $\alpha(1) = z$. According to the Path-Lifting Proposition 14.10 (1) we can lift the path $f \circ \alpha\colon [0,1] \to B$ to a path $\widetilde{(f \circ \alpha)}\colon [0,1] \to X$ with starting point x_0. We set

$$\widetilde{f}(z) \ := \ \widetilde{(f \circ \alpha)}(1).$$



It remains to prove the following claim.

Claim.
(1) The map $\widetilde{f}\colon Z \to X$ is well-defined.
(2) The map $\widetilde{f}$ is continuous.
(3) The map $\widetilde{f}$ is a lift.

Proof.
(1) This statement follows from $\operatorname{im}(f_*) \subset p_*(\pi_1(X, x_0))$.
(2) For this statement we need that B is locally path-connected.
(3) This statement follows immediately from the definition.

We carry out the details in [Fri23].

■

27.2. Existence of covering spaces.

Theorem 27.2. (Covering Existence Theorem) Let Y be a topological space that is path-connected and locally simply connected, let $y_0 \in Y$ and let $\Gamma \subset \pi = \pi_1(Y, y_0)$ be a subgroup.

(1) There exists a path-connected covering $p\colon (X, x_0) \to (Y, y_0)$ of pointed topological spaces such that $p_*(\pi_1(X, x_0)) = \Gamma$.
(2) For the covering p from (1) we have

$$[X : Y] \ = [\pi_1(Y, y_0) : \Gamma].$$

Example. We consider $(Y, y_0) = (S^1, 1)$. We pick an identification $\pi_1(S^1, 1) = \mathbb{Z}$. Given $n \in \mathbb{N}$ we consider $\Gamma = n\mathbb{Z} \subset \mathbb{Z} = \pi_1(S^1, 1)$. It follows from the discussion on page **??** that the covering

$$\begin{array}{rcl} p\colon S^1 & \to & S^1 \\ z & \mapsto & z^n \end{array}$$

has the property that $p_*(\pi_1(S^1, 1)) = n\mathbb{Z}$.

27.3. **Proof of the Covering Existence Theorem 27.2 I.** Note that we only need to show the existence of such a covering, the statement about the degree of the covering is an immediate consequence of the Covering-Degree Proposition 14.13.

Definition. Let Y be a topological space, let $y_0 \in Y$ and let $\Gamma \subset \pi = \pi_1(Y, y_0)$ be a subgroup.

(1) We denote by $Y^{[0,1]}$ the set of all continuous maps $[0, 1] \to Y$.

(2) We consider the path space

$$P_{y_0} \;:=\; \{\gamma \in Y^{[0,1]} \,|\, \gamma(0) = y_0\} \;=\; \{\text{all paths in } Y \text{ with starting point } y_0\}.$$

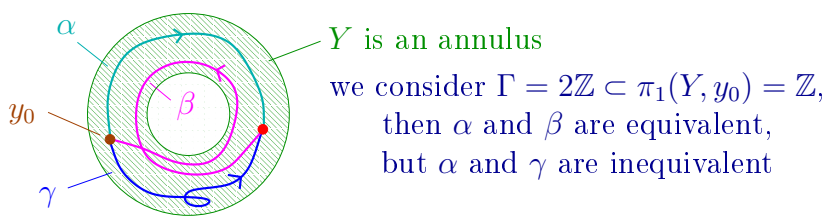
(3) For $u, v \in P_{y_0}$ we write

$$u \sim v \quad :\Longleftrightarrow \quad u \text{ and } v \text{ have the same endpoint and } [\ \underbrace{u * \overline{v}}_{\text{loop in } (Y, y_0)}\ ] \in \Gamma.$$

(4) We write $X_{y_0,\Gamma} := P_{y_0}/\sim$ and $x_0 := [e_{y_0}]$.

(5) We consider the evaluation map

$$\begin{array}{rcl} p\colon X_{y_0,\Gamma} = P_{y_0}/\sim & \to & Y \\ [v] & \mapsto & v(1). \end{array}$$



The following lemma defines such a topology and thus completes the proof of the Covering Existence Theorem 27.2.

Lemma 27.3. (Whisker Topology Lemma) Let Y be a path-connected topological space, let $y_0 \in Y$ and let $\Gamma \subset \pi_1(Y, y_0)$ be a subgroup. We write $P := P_{y_0}$ and we write $X := X_{y_0,\Gamma}$. Given $x = [f] \in X = P/\sim$ and given an open neighborhood V of $p(x) = f(1) \in Y$ we consider

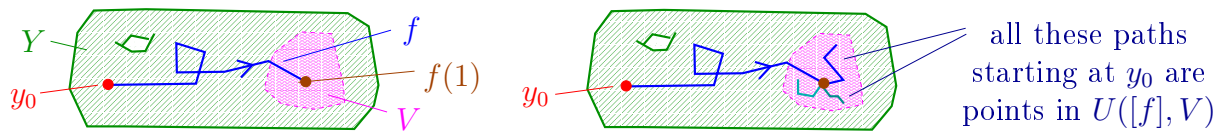
$$U(x, V) \;:=\; \{[f * u] \,|\, u \text{ is a path in } V \text{ with starting point } f(1)\} \subset X = P/\sim.$$

With this notation the following statement holds:

(0) The family of all such subsets $U(x, V)$ of X satisfy the "basis property" that we introduced on page 36.

The topology that is generated on $X = X_{Y_0,\Gamma}$ by the sets $U(x, V)$ is called the whisker topology. If Y is locally simply connected, and if we equip $X = X_{Y_0,\Gamma}$ with the whisker topology, then the following statements hold:

(1) The above evaluation map $p\colon X \to Y$ is continuous and open.
(2) The map $p\colon X \to Y$ is a covering.
(3) The topological space X is path-connected.
(4) We have $p_*(\pi_1(X, x_0)) = \Gamma$.



Proof. We work out the details in [Fri23]. ■

27.4. Uniqueness of coverings.

Theorem 27.4. (Covering–Covering Theorem) Let B be a locally path-connected topological space, let $b_0 \in B$ and let $p\colon (X, x_0) \to (B, b_0)$ be a path-connected covering. For every other path-connected covering $q\colon (Y, y_0) \to (B, b_0)$ with $p_*(\pi_1(X, x_0)) \subset q_*(\pi_1(Y, y_0))$ there exists a *unique* covering $r\colon (X, x_0) \to (Y, y_0)$ such that the following diagram commutes:

$$\begin{tikzcd} (X, x_0) \arrow[d, "p"'] \arrow[dr, dashed, "\exists!\, r"] & \\ (B, b_0). & (Y, y_0) \arrow[l, "q"] \end{tikzcd}$$

Proof. Let B be a locally path-connected topological space and let $b_0 \in B$ and let $p\colon (X, x_0) \to (B, b_0)$ and $q\colon (Y, y_0) \to (B, b_0)$ be two path-connected coverings such that $p_*(\pi_1(X, x_0)) \subset q_*(\pi_1(Y, y_0))$. Note that X is also locally path-connected. Thus it follows from the Map Lifting Criterion 27.1 together with the Lift Uniqueness Lemma 11.10 that there exists a unique map $r\colon (X, x_0) \to (Y, y_0)$ that makes the following diagram commute

$$\begin{tikzcd} & (Y, y_0) \arrow[d, "q"] \\ (X, x_0) \arrow[ur, "r"] \arrow[r, "p"'] & (B, b_0). \end{tikzcd}$$

It remains to prove that $r\colon X \to Y$ is a covering. We work out the details in [Fri23]. ■

Definition. Let $p\colon X \to B$ and $q\colon Y \to B$ be two coverings of a topological space B. We say p and q are equivalent, if there exists a homeomorphism $\Phi\colon X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "\Phi", "\cong"'] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B. & \end{tikzcd}$$

Similarly we define the equivalence of coverings of pointed topological spaces.

Example. Given $n \in \mathbb{N}$ the two coverings

$$\begin{array}{rcl} q\colon \mathbb{R}/n\mathbb{Z} & \to & S^1 \\ t + n\mathbb{Z} & \mapsto & \exp(2\pi \mathrm{i} t) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} p\colon S^1 & \to & S^1 \\ z & \mapsto & z^n \end{array}$$

are equivalent. In fact the desired map $\Phi\colon \mathbb{R}/n\mathbb{Z} \to S^1$ is given by $[t] \mapsto \exp(2\pi \mathrm{i} \frac{t}{n})$.

Theorem 27.5. (Covering Unique Existence Theorem) Let B be a path-connected topological space that is locally simply connected. Furthermore let $b_0 \in B$ and let $\Gamma \subset \pi_1(B, b_0)$ be a subgroup. There exists, up to equivalence, a *unique* path-connected covering $p\colon (X, x_0) \to (B, b_0)$ of pointed topological spaces such that $p_*(\pi_1(X, x_0)) = \Gamma$.

Proof. Let $b_0 \in B$ and let $\Gamma \subset \pi_1(B, b_0)$ be a subgroup. The existence of such a covering is just the statement of the Covering Existence Theorem 27.2. The uniqueness statement is an immediate consequence of the Covering–Covering Theorem 27.4 and the usual proof of uniqueness for an object satisfying a universal property. ■

27.5. The universal covering.

Proposition 27.6. (Universal Covering Proposition) Let X be a topological space that is path-connected and locally simply connected. Furthermore let $x_0 \in X$. The following statements hold:

(1) There exists, up to equivalence, a unique path-connected covering

$$p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$$

such that $\widetilde{X}$ is simply connected.

(2) Let $q\colon (Y, y_0) \to (X, x_0)$ be another path-connected covering. There exists a unique covering $r\colon (\widetilde{X}, \widetilde{x}_0) \to (Y, y_0)$ such that the following diagram commutes:

$$\begin{tikzcd} (\widetilde{X}, \widetilde{x}_0) \arrow[d, "p"'] \arrow[dr, dashed, "\exists! r"] & \\ (X, x_0). & (Y, y_0) \arrow[l, "q"] \end{tikzcd}$$

Proof.

(1) Note that if we apply the Covering Unique Existence Theorem 27.5 to the trivial subgroup of $\pi_1(X, x_0)$ and we obtain, up to equivalence, a unique path-connected covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ is the trivial group. It follows from the Covering-π_1-Monomorphism Proposition 14.11 that $\widetilde{X}$ is simply connected.

(2) The second statement is an immediate consequence of the Covering–Covering Theorem 27.4. ■

Definition. Let X be a topological space that is path-connected, locally simply connected and let $x_0 \in X$. We refer to any covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that $\widetilde{X}$ is simply connected as a universal covering of (X, x_0).

Example. For any $n \in \mathbb{N}$ the map

$$\begin{array}{rcl} p\colon \mathbb{R}^n & \to & (S^1)^n \\ (t_1, \dots, t_n) & \mapsto & \big(\exp(2\pi\,\mathrm{i}\,t_1), \dots, \exp(2\pi\,\mathrm{i}\,t_n)\big) \end{array}$$

is a universal covering of the n-dimensional torus $(S^1)^n$.

Convention. In practice one often suppresses the actual map p from the language. For example, often one says that "$\mathbb{R}^n$ is a universal covering of $(S^1)^n$".

Example. Let $n \geq 2$. It follows from the Projective Space-Sphere Quotient Lemma 6.9 together with the Sphere-π_1-Proposition 11.14 that the universal covering of $\mathbb{R}\mathrm{P}^n$ is given by S^n.

27.6. Explicit descriptions of the universal covering.

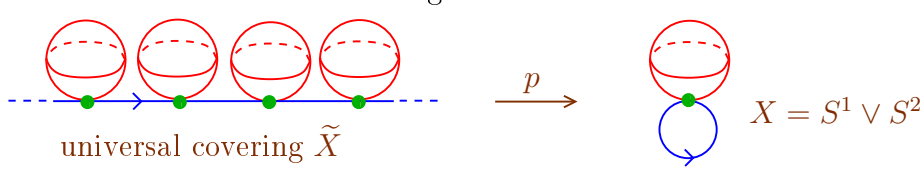
Example. We consider $X = S^1 \vee S^2$ where we identify $\mathrm{i} \in S^1$ with $(0,0,-1) \in S^2$. We also consider the topological space

$$\widetilde{X} \;=\; \big(\mathbb{R} \cup (\mathbb{Z} \times S^2)\big) \,/\, n \sim (n, (0,0,-1)), \quad n \in \mathbb{Z}.$$

It is pretty straightforward to see that the map

$$\begin{array}{rcl} p\colon \big(\mathbb{R} \cup (\mathbb{Z} \times S^2)\big) \,/\, n \sim (n, (0,0,-1)) & \to & S^1 \vee S^2 \\ [P] & \mapsto & \begin{cases} \exp(\mathrm{i}(2\pi P + \frac{\pi}{2})), & \text{if } P \in \mathbb{R} \\ Q, & \text{if } P = (n, Q) \in \mathbb{Z} \times S^2 \end{cases} \end{array}$$

is well-defined and in fact a covering. On page 193 we showed that $\widetilde{X}$ is simply connected. Therefore $p\colon \widetilde{X} \to X$ is a universal covering of X.



Now we turn to the universal covering of the wedge $X = S^1 \vee S^1$. By the discussion on page 156 we know that X is homeomorphic to the topological realization of the unique abstract graph G that is given by the single vertex $*$ and the edge set $\{x, y\}$.

Convention. Let V be a set and let $E \subset V \times V$ be a subset. The pair (V, E) defines an abstract graph (V, E, i, t) by setting $i(g,h) = g$ and $t(g,h) = h$. We use this observation to view such (V, E) as an abstract graph.

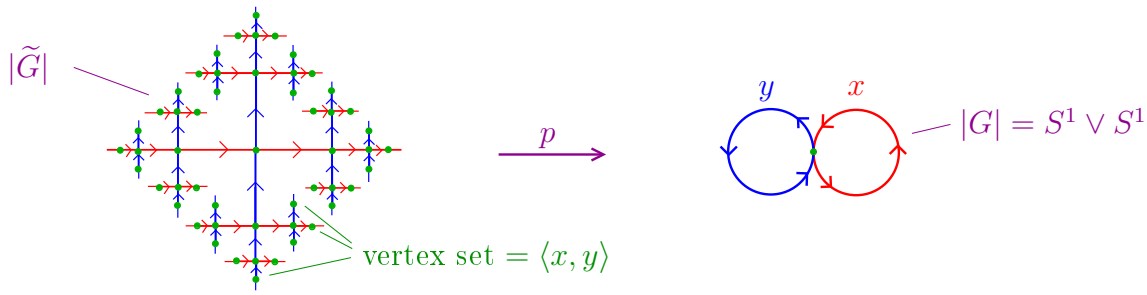
Proposition 27.7. (Wedge-of-Circles–Universal Covering Proposition) We consider the abstract graph $G = (\{*\}, \{x, y\})$. We also consider the abstract graph

$$\widetilde{G} \;=\; (\underbrace{\widetilde{V} = \text{ free group } \langle x, y\rangle}_{\text{vertex set}}, \underbrace{\widetilde{E} = \{(g,h) \in \widetilde{V} \times \widetilde{V} \,|\, h = g \cdot x \text{ or } h = g \cdot y\}}_{\text{edge set}}).$$

The maps

$$\begin{array}{rcl} \text{vertex set of } \widetilde{G} & \to & \text{vertex set of } G \\ g & \mapsto & * \end{array} \quad \text{and} \quad \begin{array}{rcl} \text{edge set of } \widetilde{G} & \to & \text{edge set of } G \\ (g,h) & \mapsto & g^{-1} \cdot h \end{array}$$

define a morphism of graphs in the sense of the definition on page 95. The corresponding map $p\colon |\widetilde{G}| \to |G|$ is a universal covering of $|G| \cong S^1 \vee S^1$.



Proof. We leave it to the reader to verify that the map $p\colon |\widetilde{G}| \to |G|$ is a covering. It remains to show that $|\widetilde{G}|$ is simply connected. This requires some preparations.

Given $g \in \widetilde{V} = \langle x, y\rangle$ we define its length to be

$$l(g) \;:=\; \min\{k \in \mathbb{N}_0 \,|\, g = a_1 \cdot \ldots \cdot a_k \text{ for some } a_1, \ldots, a_k \in \{x^{\pm 1}, y^{\pm 1}\}\}.$$

Given $k \in \mathbb{N}_0$ we consider the subgraph $\widetilde{G}_k = (\widetilde{V}_k, \widetilde{E}_k)$ of $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ that is given by $\widetilde{V}_k := \{g \in \widetilde{V} = \langle x, y\rangle \,|\, l(g) \le k\}$ and $\widetilde{E}_k := \widetilde{E} \cap \mathcal{P}(\widetilde{V}_k)$.

Claim. For each $k \in \mathbb{N}_0$ we have $\chi(\widetilde{G}_k) = 1$.

Proof. We prove the claim by induction on $k \in \mathbb{N}_0$. First note that $\widetilde{G}_0$ has a single vertex and no edge. Thus we see that $\chi(\widetilde{G}_0) = 1$. Now suppose that we know that $\chi(\widetilde{G}_k) = 1$. We make the following two observations:

(1) It follows easily from the definition of the free product $\langle x, y\rangle = \langle x\rangle * \langle y\rangle$ as a set of reduced words, see page 166, that each $g \in \widetilde{V}_{k+1} \setminus \widetilde{V}_k$ is uniquely of the form $a_1 \cdot \ldots \cdot a_{k+1}$ with $a_1, \ldots, a_{k+1} \in \{x^{\pm 1}, y^{\pm 1}\}$ and with $g := a_1 \cdot \ldots \cdot a_k \in \widetilde{V}_k$. Note that *precisely one* of $(g, g \cdot a_{k+1})$ or $(g \cdot a_{k+1}, g)$ lies in $\widetilde{E}_{k+1}$.

(2) Let $(g, h) \in \widetilde{E}_{k+1}$. It follows again easily from the definition of $\langle x, y\rangle = \langle x\rangle * \langle y\rangle$ as a set of reduced words that precisely one of g and h lies in $\widetilde{V}_k$ and $\widetilde{V}_{k+1}$.

Thus we see that

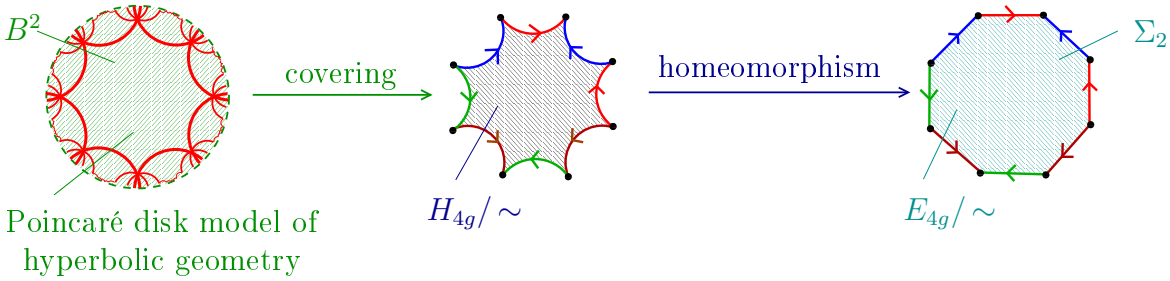
$$\chi(\widetilde{G}_{k+1}) \;=\; \chi(\widetilde{V}_k) + \underbrace{\#(\widetilde{V}_{k+1} \setminus \widetilde{V}_k) - \#(\widetilde{E}_{k+1} \setminus \widetilde{E}_k)}_{\substack{= 0\text{, since by the above two observations} \\ \text{we have a bijection between the} \\ \text{sets } \widetilde{V}_{k+1} \setminus \widetilde{V}_k \text{ and } \widetilde{E}_{k+1} \setminus \widetilde{E}_k}} \;=\; \chi(\widetilde{G}_k) \;=\; 1.$$

⊞

It follows from the claim together with the Tree Characterization Proposition 7.7 that each $\widetilde{G}_k$ is a tree. The Finite Tree–Contractible Proposition 17.4 together with the Homotopy-π_1-Proposition 13.4 now implies that each $|\widetilde{G}_k|$ is simply connected. It follows easily from the Exhaustion–Simply Connected Proposition 20.10 (or alternatively the Exhaustion-π_1-Proposition 26.2) that $|\widetilde{G}| = \bigcup_{k \in \mathbb{N}} |\widetilde{G}_k|$ itself is simply connected. ■

Theorem 27.8. (Hadamard's Theorem) Given any $g \ge 2$ there exists a covering map $B^2 \to \Sigma_g = E_{4g}/\sim$.

Proof. The key idea of the proof is to replace the "Euclidean" octagon E_{4g} by a "hyperbolic" octagon H_{4g} and to equip the open disk B^2 with a "hyperbolic structure" via the Poincaré disk model of hyperbolic geometry. The long and difficult details are worked out in [Fri23]. ■



28. Higher homotopy groups: Definition and basics

28.1. Definition of the higher homotopy groups.

Definition. Let W and X be topological spaces and let $A \subset W$ and $B \subset X$ be subsets.

(1) A continuous map $(W, A) \to (X, B)$ of pairs of topological spaces is a continuous map $f\colon W \to X$ with $f(A) \subset B$.

(2) Let $f, g\colon (W, A) \to (X, B)$ be two continuous maps of pairs of topological spaces.

(a) A homotopy between f and g is a continuous map

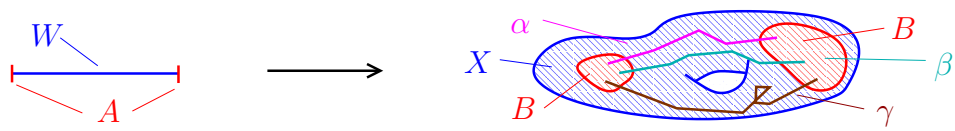
$$H\colon W \times [0,1] \to X,$$

such that $H_0 = f$ and $H_1 = g$ and such that each H_t is a map $(W, A) \to (X, B)$.

(b) If there exists a homotopy, then we say f and g are homotopic and we write $f \simeq g$.

(3) We set

$$[(W, A), (X, B)] := \{\text{continuous maps } (W, A) \to (X, B)\}/\simeq .$$



$\alpha, \beta\colon (W, A) \to (X, B)$ are homotopic but $\alpha, \gamma\colon (W, A) \to (X, B)$ are not homotopic

Notation. Throughout this chapter we write $I = [0, 1]$ for the unit interval. Given $n \in \mathbb{N}$ we denote by $I^n \subset \mathbb{R}^n$ the cube in $\mathbb{R}^n$ and we denote by ∂I^n its boundary as a subspace of $\mathbb{R}^n$.

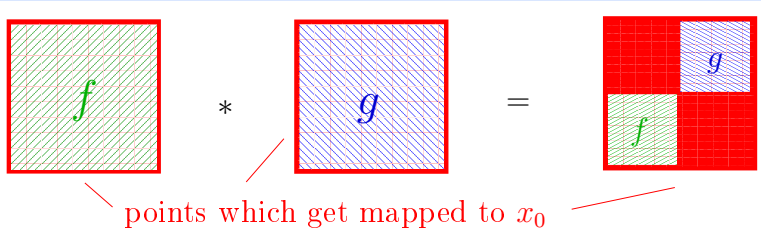
Definition. Given a pointed topological space (X, x_0) and given $n \in \mathbb{N}_0$ we define

$$\pi_n(X, x_0) \;:=\; [(I^n, \partial I^n), (X, x_0)].$$

For $n \geq 1$ we refer to $\pi_n(X, x_0)$ as the n-th homotopy group. We refer to any $\pi_n(X, x_0)$ with $n \geq 2$ as a higher homotopy group.

Definition. Let (X, x_0) be a pointed topological space and let $n \geq 1$. Furthermore let $f, g\colon (I^n, \partial I^n) \to (X, x_0)$ be two continuous maps. We define

$$\begin{array}{rcl} f * g\colon (I^n, \partial I^n) & \to & (X, x_0) \\ (x_1, \ldots, x_n) & \mapsto & \begin{cases} f(2x_1, \ldots, 2x_n), & \text{if } (x_1, \ldots, x_n) \in [0, \frac{1}{2}]^n, \\ g(2x_1 - 1, \ldots, 2x_n - 1), & \text{if } (x_1, \ldots, x_n) \in [\frac{1}{2}, 1]^n, \\ x_0, & \text{otherwise.} \end{cases} \end{array}$$



Definition. Let (X, x_0) be a pointed topological space and let $n \geq 1$. We define

$$\begin{array}{rcl} \pi_n(X, x_0) \times \pi_n(X, x_0) & \to & \pi_n(X, x_0) \\ ([f], [g]) & \mapsto & [f] \cdot [g] := [f * g]. \end{array}$$

Proposition 28.1. (π_n-Group Proposition) Let (X, x_0) be a pointed topological space and let $n \geq 1$. The map

$$\begin{array}{rcl} \pi_n(X, x_0) \times \pi_n(X, x_0) & \to & \pi_n(X, x_0) \\ ([f], [g]) & \mapsto & [f] \cdot [g] := [f * g] \end{array}$$

defines a group structure on $\pi_n(X, x_0)$.

Definition.

(1) A hyperrectangle is any subset of $\mathbb{R}^n$ of the form $[c_1, d_1] \times \cdots \times [c_n, d_n]$.

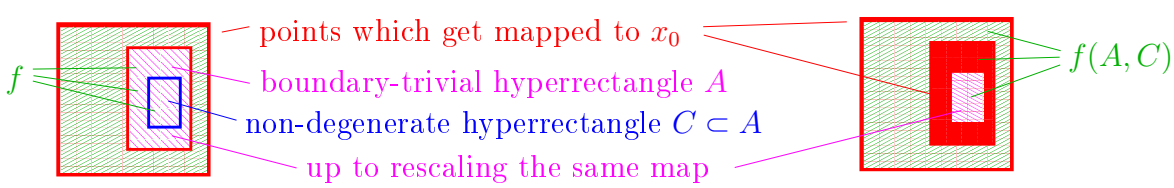
(2) We say a hyperrectangle $[c_1, d_1] \times \cdots \times [c_n, d_n]$ is non-degenerate if for all $i = 1, \ldots, n$ we have $d_i > c_i$.

Let X be a topological space, let $x_0 \in X$ and let $f\colon I^n \to X$ be a continuous map.

(3) We say a hyperrectangle $A \subset I^n$ is boundary-trivial if f sends all points on the boundary ∂A of A to x_0.

(4) Let $A = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset I^n$ be a boundary-trivial hyperrectangle. Furthermore let $C = [c_1, d_1] \times \cdots \times [c_n, d_n]$ be a non-degenerate hyperrectangle that is contained in A. We define

$$\begin{array}{rcl} f(A, C)\colon I^n & \to & X \\ x = (x_1, \ldots, x_n) & \mapsto & \begin{cases} f(x), & \text{if } x \notin A, \\ x_0, & \text{if } x \in A \setminus C, \\ f\left(a_1 + (x_1 - c_1) \cdot \frac{b_1 - a_1}{d_1 - c_1}, \cdots, a_n + (x_n - c_n) \cdot \frac{b_n - a_n}{d_n - c_n}\right) & \text{if } x \in C. \end{cases} \end{array}$$



Lemma 28.2. (Shrinking Lemma) Let (X, x_0) be a pointed topological space. Furthermore let $f\colon (I^n, \partial I^n) \to (X, x_0)$ be a continuous map. Furthermore let $A \subset I^n$ be a boundary-trivial hyperrectangle and let C be a non-degenerate hyperrectangle that is contained in A. Then the maps f and $f(A, C)$ from $(I^n, \partial I^n)$ to (X, x_0) are homotopic.

Proof. Let $f\colon (I^n, \partial I^n) \to (X, x_0)$ be a continuous map, let $A = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset I^n$ be a boundary-trivial hyperrectangle and let $C = [c_1, d_1] \times \cdots \times [c_n, d_n]$ be a non-degenerate hyperrectangle that is contained in A. For $t \in [0, 1]$ we write

$$C_t = [a_1 \cdot (1 - t) + c_1 \cdot t, b_1(1 - t) + d_1 \cdot t] \times \cdots \times [a_n \cdot (1 - t) + c_n \cdot t, b_n \cdot (1 - t) + d_n \cdot t].$$

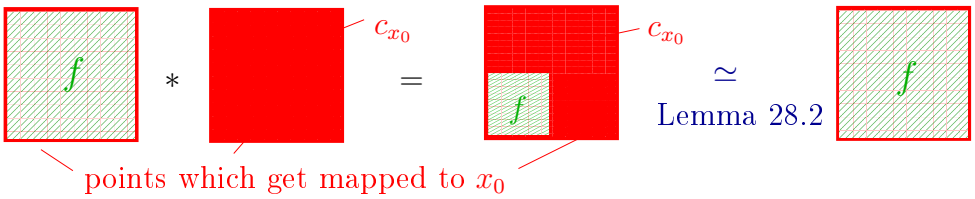
Note that $C_0 = A$ and $C_1 = C$. Using the Pasting Proposition 2.4 (2) one can show fairly easily that the map

$$\begin{array}{rcl} F\colon I^n \times [0, 1] & \to & X \\ (x, t) & \mapsto & f(A, C_t)(x) \end{array}$$

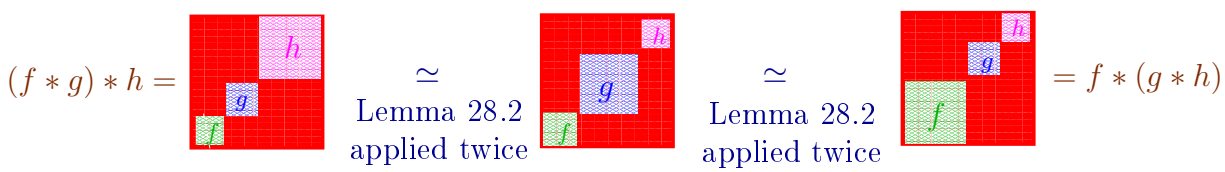
is continuous. By construction we have $F_0 = F$ and $F_1 = f(A, C)$, thus F is a homotopy between f and $f(A, C)$. ■

Proof of the π_n-Group Proposition 28.1.

- The neutral element of the product structure is given by the equivalence class of the constant map
$$\begin{aligned} c_{x_0}\colon I^n &\to X \\ x &\mapsto x_0. \end{aligned}$$
Indeed, it follows immediately from the Shrinking Lemma 28.2, applied to $A = [0,1]^n$ and $C = [0, \frac{1}{2}]^n$, that given any continuous map $f\colon (I^n, \partial I^n) \to (X, x_0)$ we have $f * c_{x_0} \simeq f$. Basically the same argument shows also that $c_{x_0} * f \simeq f$.

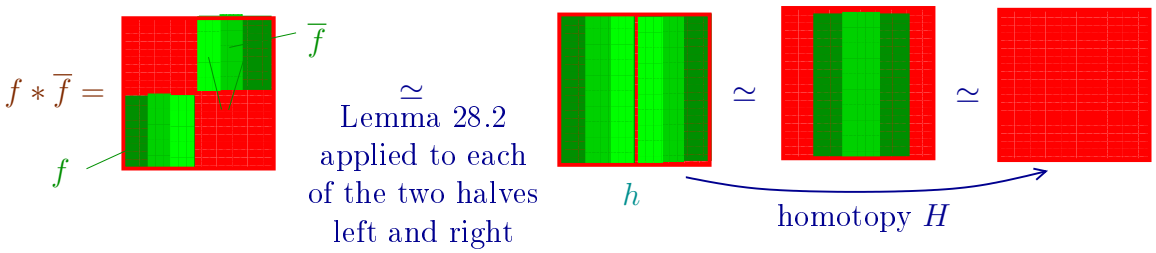


- We consider three maps $f, g, h\colon (I^n, \partial I^n) \to (X, x_0)$. By applying the Shrinking Lemma 28.2 several times one can easily show that $(f * g) * h \simeq f * (g * h)$, see e.g. the figure below. This proves that the multiplication on $\pi_n(X, x_0)$ is associative.



- Now we turn to the most interesting question, namely finding the inverse of a given element in $\pi_n(X, x_0)$. Thus suppose we are given a continuous map $f\colon (I^n, \partial I^n) \to (X, x_0)$. We consider the map
$$\begin{aligned} \overline{f}\colon I^n &\to X \\ (x_1, \dots, x_n) &\mapsto f(1 - x_1, x_2, \dots, x_n). \end{aligned}$$
We claim that $[f] \cdot [\overline{f}] = [f * \overline{f}] = [c_{x_0}]$. Thus we need to show that $f * \overline{f}$ is homotopic to c_{x_0}. It follows from the Shrinking Lemma 28.2 that $f * \overline{f}$ is homotopic to the map
$$\begin{aligned} h\colon (I^n, \partial I^n) &\to (X, x_0) \\ (x_1, \dots, x_n) &\mapsto \begin{cases} f(2x_1, x_2, \dots, x_n), & \text{if } x_1 \in [0, \frac{1}{2}], \\ \underbrace{\overline{f}(2x_1 - 1, x_2, \dots, x_n)}_{= f(2-2x_1, x_2, \dots, x_n)}, & \text{if } x_1 \in [\frac{1}{2}, 1]. \end{cases} \end{aligned}$$
Thus it suffices to show that h is homotopic to c_{x_0}. It is in fact straightforward to see that the map
$$\begin{aligned} H\colon I^n \times [0,1] &\to X \\ (x_1, \dots, x_n, t) &\mapsto \begin{cases} f(2(x_1 - \frac{t}{2}), x_2, \dots, x_n), & \text{if } x_1 \in [\frac{t}{2}, \frac{1}{2}], \\ f(2 - 2(x_1 + \frac{t}{2}), x_2, \dots, x_n), & \text{if } x_1 \in [\frac{1}{2}, 1 - \frac{t}{2}], \\ x_0, & \text{otherwise} \end{cases} \end{aligned}$$
is a homotopy from h to c_{x_0}. We refer to the figure below for an illustration of the argument. Basically the same argument also shows that $[\overline{f}] \cdot [f] = [\overline{f} * f] = [c_{x_0}]$. ■

28.2. Alternative descriptions of homotopy groups.



Identification. Let $n \in \mathbb{N}$.

(1) We pick and fix a homeomorphism $\Phi\colon (I^n, \partial I^n) \to (\overline{B}^n, S^{n-1})$.

(2) We equip S^n throughout with the base point $* = (0, \ldots, 0, 1) \in S^n$.

(3) We denote by $\Psi\colon (\overline{B}^n/S^{n-1}, [S^{n-1}]) \to (S^n, *)$ the explicit homeomorphism from the Ball Quotient-Sphere Lemma 6.5.

(4) Let X be a topological space and let $x_0 \in X$. It follows easily from the Topological-Quotient Proposition 6.1 (1) together with the Homotopy Quotient Theorem 8.5 that the above homeomorphisms Φ and Ψ define bijections

$$\begin{array}{ll} & \text{(homotopy classes) of continuous maps } (I^n, \partial I^n) \to (X, x_0) \\ \Longleftrightarrow & \text{(homotopy classes) of continuous maps } (\overline{B}^n, S^{n-1}) \to (X, x_0) \\ \Longleftrightarrow & \text{(homotopy classes) of continuous maps } (\overline{B}^n/S^{n-1}, [S^{n-1}]) \to (X, x_0) \\ \Longleftrightarrow & \text{(homotopy classes) of continuous maps } (S^n, *) \to (X, x_0). \end{array}$$

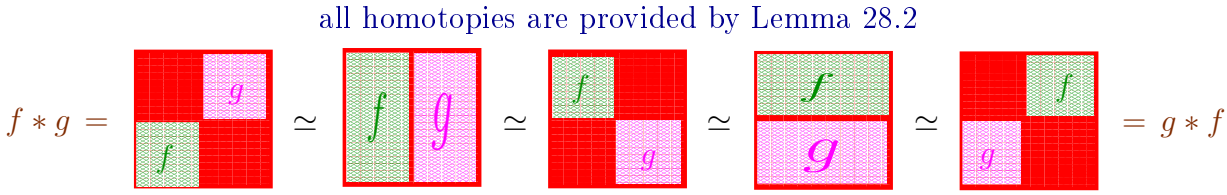
The four equivalences combined say that we can view elements in $\pi_n(X, x_0)$ as homotopy classes of continuous maps $(S^n, *) \to (X, x_0)$.

28.3. Properties and basic calculations of higher homotopy groups.

Proposition 28.3. (π_n-Abelian Proposition) For any pointed topological space (X, x_0) and any $n \geq 2$ the homotopy group $\pi_n(X, x_0)$ is abelian.

Proof. Suppose we are given two continuous maps $f, g\colon (I^n, \partial I^n) \to (X, x_0)$. We have to show that $f * g$ is homotopic to $g * f$. The homotopy is described in the figure below for $n = 2$. It is straightforward to write down an explicit homotopy between $f * g$ and $g * f$ by composing four instances of the homotopy provided by the Shrinking Lemma 28.2. ■

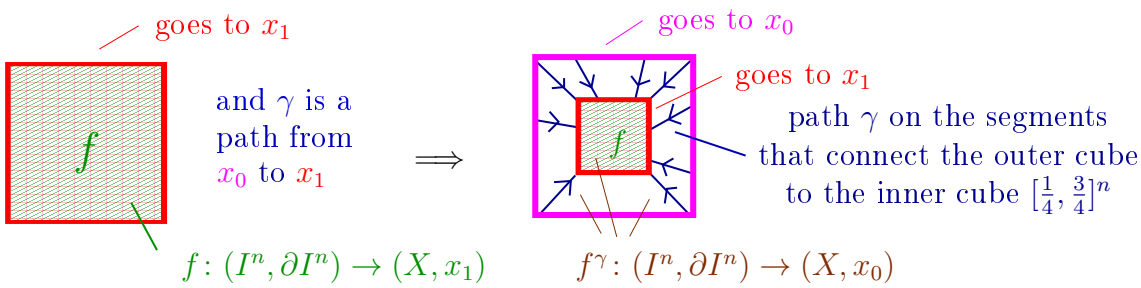
all homotopies are provided by Lemma 28.2



Proposition 28.4. (Change-of-Base Point Proposition) Let X be a topological space, let x_0 and x_1 be two points in X and let $\gamma\colon [0,1] \to X$ be a path from x_0 to x_1. Given a continuous map $f\colon (I^n, \partial I^n) \to (X, x_1)$ we consider the map

$$\begin{array}{rcl} f^\gamma\colon (I^n, \partial I^n) & \to & (X, x_0) \\ (t_1, \ldots, t_n) & \mapsto & \begin{cases} f\big(2t_1 - \frac{1}{2}, \ldots, 2t_n - \frac{1}{2}\big), & \text{if } (t_1, \ldots, t_n) \in \big[\frac{1}{4}, \frac{3}{4}\big]^n, \\ \gamma(2 - 4t), & \text{if } t := \max\{|t_1 - \frac{1}{2}|, \ldots, |t_n - \frac{1}{2}|\} \in [\frac{1}{4}, \frac{1}{2}]. \end{cases} \end{array}$$

The map
$$\begin{array}{rcl} \gamma_*\colon \pi_n(X,x_1) & \to & \pi_n(X,x_0) \\ [f] & \mapsto & [f^\gamma] \end{array}$$
is well-defined and it is a group isomorphism.



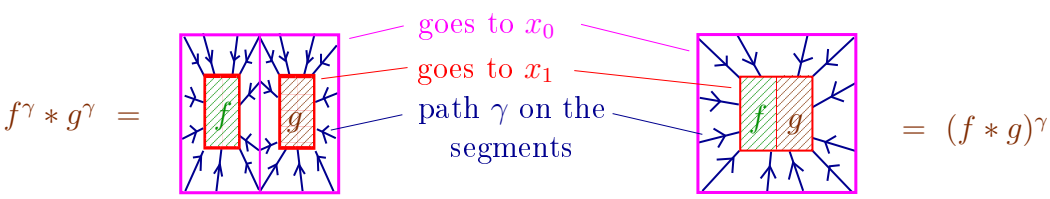
Proof of the Change-of-Base Point Proposition 28.4. We need to prove the following claim.

Claim.

(1) The map $\Phi\colon \pi_n(X,x_1)\to\pi_n(X,x_0)$ given by $[f]\mapsto[f^\gamma]$ is well-defined.
(2) The map Φ is a bijection.
(3) The map Φ is a group homomorphism.

Proof.

(1) This statement can be verified easily by hand.
(2) As usual we denote by $\overline{\gamma}$ the path given by $\overline{\gamma}(t)=\gamma(1-t)$. It is straightforward to verify that the map $\Psi\colon \pi_n(X,x_0)\to\pi_n(X,x_1)$ given by $[g]\mapsto[f^{\overline{\gamma}}]$ is an inverse to Φ.
(3) Let $f,g\colon (I^n,\partial I^n)\to(X,x_0)$ be continuous maps. We need to show that $f^\gamma * g^\gamma$ is homotopic to $(f*g)^\gamma$.



We leave it to the reader to write down an explicit homotopy. ■

28.4. Functoriality of homotopy groups.

Definition. Let $f\colon X\to Y$ be a continuous map between two topological spaces, let $n\in\mathbb{N}$ and let $x_0\in X$. We write $y_0=f(x_0)$. One can easily verify that the map
$$\begin{array}{rcl} f_*\colon \pi_n(X,x_0) & \to & \pi_n(Y,y_0) \\ [\sigma\colon (I^n,\partial I^n)\to(X,x_0)] & \mapsto & [f\circ\sigma\colon (I^n,\partial I^n)\xrightarrow{\sigma}(X,x_0)\xrightarrow{f}(Y,y_0)] \end{array}$$
is well-defined.

Proposition 28.5. (π_n-Functor Proposition) Let $n\geq 1$. Recall that we denote by $\mathcal{PTop}$ the category of pointed topological spaces and that we denote by $\mathcal{Group}$ be the category of groups. The map

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{PTop}) & \to & \mathrm{Ob}(\mathcal{Group}) \\ (X, x_0) & \mapsto & \pi_n(X, x_0) \end{array}$$

and the maps

$$\begin{array}{rcl} \mathrm{Mor}(\mathcal{PTop}) & \to & \mathrm{Mor}(\mathcal{Group}) \\ (f\colon (X, x_0) \to (Y, y_0)) & \mapsto & (f_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)) \end{array}$$

form a covariant functor.

Proof. First note that it follows immediately from the definitions that given a continuous map $f\colon (X, x_0) \to (Y, y_0)$ the induced map $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is actually a group homomorphism. Furthermore note that it follows immediately from the definitions that

$$\begin{array}{rcll} \mathrm{id}_* & = & \mathrm{id} & \text{for all pointed pairs } (X, x_0) \\ (g \circ f)_* & = & g_* \circ f_* & \text{for all continuous maps } f\colon (X, x_0) \to (Y, y_0) \text{ and } g\colon (Y, y_0) \to (Z, z_0). \end{array}$$

But this is exactly what we needed to show. ■

Proposition 28.6. (Homotopy-π_n-Proposition) Let $n \geq 1$.

(1) Let $f, g\colon X \to Y$ be two continuous maps between topological spaces and let $x_0 \in X$. If f and g are homotopic, then

$$\begin{array}{c} f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0)) \\ \text{is an isomorphism} \end{array} \iff \begin{array}{c} g_*\colon \pi_n(X, x_0) \to \pi_n(Y, g(x_0)) \\ \text{is an isomorphism.} \end{array}$$

The analogous statement holds if we replace "isomorphism" by "epimorphism" or "monomorphism".

(2) (a) Let $f\colon X \to Y$ be a continuous map between topological spaces and let $x_0 \in X$ be a point. If f is a homotopy equivalence, then

$$f_*\colon \pi_n(X, x_0) \ \to \ \pi_n(Y, f(x_0))$$

is an isomorphism.

(b) Let X be a topological space and let $A \subset X$ be a subspace. We denote by $i\colon A \to X$ the inclusion. If A is a deformation retract of X, then for any $a_0 \in A$ the induced map

$$i_*\colon \pi_n(A, a_0) \ \to \ \pi_n(X, a_0)$$

is an isomorphism.

(3) If X is a contractible topological space, then for any $x_0 \in X$ we have $\pi_n(X, x_0) = 0$.

Proof. The proof of Statement (1) is quite similar to the proof of Statement (1) of the Homotopy-π_1-Proposition 13.4. The only problem is that the notation gets a little more messy. As in the Homotopy-π_1-Proposition 13.4 one can easily deduce Statements (2) and (3) from Statement (1). ■

Example. The topological space $\mathbb{R}^k$ is homotopy equivalent to a point $\{P\}$. Thus it follows from the Homotopy-π_n-Proposition 28.6 (2) that $\pi_n(\mathbb{R}^k) \cong \pi_n(\{P\}) \cong 0$. Exactly the same way, using the discussion on page 111, one can show that all the higher homotopy groups of convex subsets of $\mathbb{R}^k$ vanish, e.g. we have $\pi_n\big(\overline{B}^k\big) = \pi_n(B^k) = 0$.

Proposition 28.7. (Product-π_n-Proposition) Let $n \in \mathbb{N}$. Let $(X_i, x_i)_{i \in I}$ be a family of pointed topological spaces. For each $j \in I$ we denote by $p_j\colon \prod_{i \in I} X_i \to X_j$ the natural

projection. The induced homomorphism

$$\begin{array}{rcl} \Phi\colon \pi_n\Big(\prod\limits_{i\in I} X_i, (x_i)_{i\in I}\Big) & \to & \prod\limits_{i\in I} \pi_n(X_i, x_i) \\ [f] & \mapsto & (p_{i*}([f]))_{i\in I} \end{array}$$

is an isomorphism.

Proof. The proof is basically identical to the proof of the Product-π_1-Proposition 13.7. ■

Proposition 28.8. (Sphere–Low-π_n-Proposition) Let $k \in \mathbb{N}$. For any $n < k$ we have

$$\pi_n(S^k) \;=\; 0.$$

Proof. Let $k \geq 2$, let $x_0 \in S^k$ and let $\gamma\colon (S^n, *) \to (S^k, x_0)$ be a continuous map. It follows immediately from the Whitney Approximation Theorem [Fri23], applied to the closed subset $\{*\} \subset S^n$, that there exists a homotopy rel $\{*\}$ from $\gamma\colon S^n \to S^k$ to a smooth map $\delta\colon S^n \to S^k$ with $\delta(*) = x_0$. Since δ is smooth and since $k > n$ we see that the image of δ has measure zero, which implies that the smooth map $\delta\colon S^n \to S^k$ is not surjective. We pick $P \in S^n \setminus \delta(S^n)$.

Note that $S^k \setminus \{P\}$ is homeomorphic to $\mathbb{R}^k$, in particular contractible, which implies that $[\delta] = 0 \in \pi_n(S^k \setminus \{P\}, *)$, which implies that $[\delta] = 0 \in \pi_n(S^k, *)$. ■

Proposition 28.9. (Exhaustion-π_n-Zero-Proposition) Let $n \in \mathbb{N}$, let X be a path-connected topological space and let

$$X_0 \;\subset\; X_1 \;\subset\; X_2 \;\subset\; X_3 \;\subset\; \ldots$$

be a sequence of subspaces of X such the following two conditions are satisfied:

(1) We have $\bigcup\limits_{i\in\mathbb{N}_0} X_i = X$.

(2) For each $i \in \mathbb{N}_0$ we have $X_i \subset \mathring{X}_{i+1}$.

(3) For each $i \in \mathbb{N}_0$ the following holds: X_i is path-connected and we have $\pi_n(X_i) = 0$.

Then for each $x_0 \in X$ we have $\pi_n(X, x_0) = 0$.

Proof. The proof is basically the same as the proof of the Exhaustion–Simply Connected Proposition 20.10. ■

28.5. Covering spaces and higher homotopy groups.

Proposition 28.10. (Covering-π_n-Isomorphism Proposition) Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. Given any $n \geq 2$ the map $p_*\colon \pi_n(X, x_0) \to \pi_n(B, b_0)$ is an isomorphism.

Examples.

(1) For $n \geq 2$ we have

$$\pi_n((S^1)^k) \;\cong\; \pi_n(\mathbb{R}^k/\mathbb{Z}^k) \;\underset{\substack{\uparrow \\ \text{Covering-}\pi_n\text{-Isomorphism Proposition 28.10}}}{\xleftarrow{\cong}}\; \pi_n(\mathbb{R}^k) \;\underset{\substack{\uparrow \\ \text{page 244}}}{=}\; 0.$$

(2) Let Σ be a surface of genus ≥ 2. For any $n \geq 2$ we see that

$$\pi_n(\Sigma) \;\underset{\substack{\uparrow \\ \text{Covering-}\pi_n\text{-Isomorphism Proposition 28.10}}}{\cong}\; \pi_n(\text{universal covering of } \Sigma) \;\underset{\substack{\uparrow \\ \text{Hadamard's Theorem 27.8}}}{\cong}\; \pi_n(B^2) \;\underset{\substack{\uparrow \\ \text{see page 244}}}{=}\; 0.$$

Proof of the Covering-π_n-Isomorphism Proposition 28.10. Let $p\colon (X,x_0) \to (B,b_0)$ be a covering of pointed topological spaces and let $n \geq 2$. Our goal is to construct an inverse to the map $p_*\colon \pi_n(X,x_0) \to \pi_n(B,b_0)$.

We saw on page 242 that we can think of elements in $\pi_n(B,b_0)$ as homotopy classes of continuous maps $(S^n,*) \to (B,b_0)$. Thus let $f\colon (S^n,*) \to (B,b_0)$ be a continuous map. We are in the following setting

$$\begin{tikzcd} & (X,x_0) \arrow[d, "p"] \\ (S^n,*) \arrow[r, "f"'] & (B,b_0). \end{tikzcd} \quad \xrightarrow[\text{since by } n\geq 2 \text{ we have } \pi_1(S^n)=0]{\text{by the Map Lifting Criterion 27.1}} \quad \begin{tikzcd} & (X,x_0) \arrow[d, "p"] \\ (S^n,*) \arrow[r, "f"'] \arrow[ur, dashed, "\exists!\,\widetilde{f}"] & (B,b_0). \end{tikzcd}$$

Furthermore it follows again from the Map Lifting Criterion 27.1, applied to the pointed topological space $(Z,z_0) = (S^n \times [0,1],(*,0))$, that the lifts of two homotopic maps $(S^n,*) \to (B,b_0)$ are two homotopic maps $(S^n,*) \to (X,x_0)$. Summarizing we just constructed a map

$$\begin{aligned} \lambda\colon \pi_n(B,b_0) &\to \pi_n(X,x_0) \\ [f] &\mapsto [\widetilde{f}]. \end{aligned}$$

We have to show that λ is an inverse to p_*. First of all, given any $[f] \in \pi_n(B,b_0)$ we have

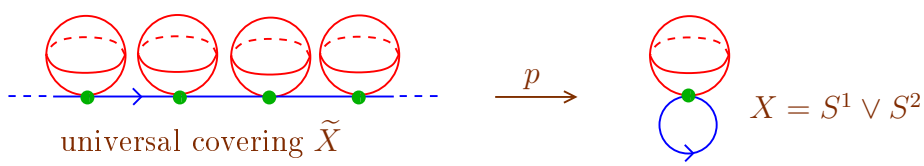
$$(p_* \circ \lambda)([f]) \;=\; p_*\big([\widetilde{f}]\big) \;=\; \big[p \circ \widetilde{f}\big] \;=\; [f].$$

Now let $[g] \in \pi_n(X,x_0)$. We then have

$$(\lambda \circ p_*)([g]) \;=\; \lambda([p \circ g]) \;\underset{\substack{\uparrow \\ \text{since } g \text{ is the unique lift of } p\circ g \text{ to } x_0.}}{=}\; [g].$$

■

Example. We consider $X = S^1 \vee S^2$. On page 236 we determined its universal covering $\widetilde{X}$, which is given by the wedge of $\mathbb{R}$ with infinitely many spheres. It follows from the Covering-π_n-Isomorphism Proposition 28.10 that $\pi_2(S^1 \vee S^2)$ is isomorphic to $\pi_2(\widetilde{X})$. At the moment we cannot determine $\pi_2(\widetilde{X})$, but since $\widetilde{X}$ has "infinitely many spheres" one might expect $\pi_2(\widetilde{X})$ to be pretty big.



28.6. Are there any higher homotopy groups that are non-trivial?

Question 28.11. Let $n \geq 2$.

(1) Does the identity map $\mathrm{id}\colon S^n \to S^n$ define a non-trivial element in the homotopy group $\pi_n(S^n,*)$?

(2) Do we have an isomorphism $\pi_n(S^n,*) \cong \mathbb{Z}$?

We will answer this question in Algebraic Topology II using singular homology.

Definition. We refer to

$$\begin{array}{rcl} H\colon S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^2 =_i \mathbb{C}\mathrm{P}^1 \\ (z_1, z_2) & \mapsto & [z_1 : z_2] \end{array}$$

as the Hopf map . The homotopy class of H defines an element in $\pi_3(S^2, N)$.

Question 28.12. Does the Hopf map $H\colon S^3 \to S^2$ represent a non-trivial element in the homotopy group $\pi_3(S^2, N)$?

We will answer this question in Algebraic topology III

Definition.

(1) Let X be a non-empty topological space. We define the suspension of X to be the topological space

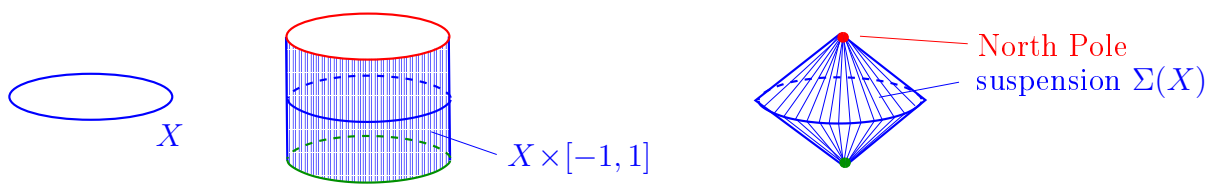
$$\Sigma(X) \;:=\; (X \times [-1, 1])/(X \times \{-1\})/\sim$$

where we identify all points in $X \times \{-1\}$ to a single point and we identify all points in $X \times \{1\}$ to a single point. We refer to the point $[X \times \{1\}]$ as the North Pole of $\Sigma(X)$.

(2) Given a continuous map $f\colon X \to Y$ between topological spaces we denote by

$$\begin{array}{rcl} \Sigma(f)\colon \Sigma(X) & \to & \Sigma(Y) \\ [(x, t)] & \mapsto & [(f(x), t)] \end{array}$$

the suspension of f.



Lemma 28.13. (Suspension Homomorphism Lemma) Let $k \in \mathbb{N}_0$. Let $n \in \mathbb{N}_0$. We denote by $*$ the North Pole of S^n and S^{n+1}. The map

$$\begin{array}{rcl} \pi_k(S^n, *) & \to & \pi_{k+1}(S^{n+1}, *) \\ [f\colon S^k \to S^n] & \mapsto & \left[\Sigma(f)\colon \Sigma(S^k) =_i S^{k+1} \to \Sigma(S^n) =_i S^{n+1}\right] \end{array}$$

is a well-defined group homomorphism. We refer to it as the suspension homomorphism.

Proof. We leave the proof of the lemma to the reader. ■

Question 28.14. Let $H\colon S^3 \to S^2$ be the Hopf map. Is the image of $[H] \in \pi_3(S^2)$ under the iterated suspension homomorphisms $\pi_3(S^2, x_0) \to \pi_{3+k}(S^{2+k}, x_1)$ non-trivial?

Part III

CW-complexes

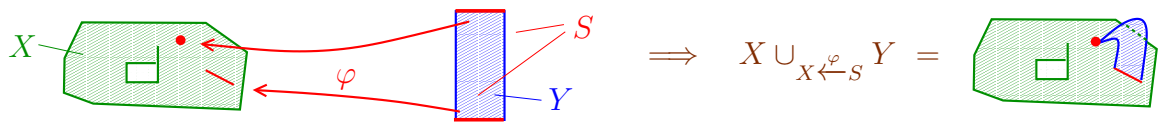
29. Cell attachments

29.1. Attaching a cell a to topological space.

Definition. Let X and Y be topological spaces, let $S \subset Y$ be a subspace and let $\varphi\colon S \to X$ be a continuous map. We write

$$X \cup_\varphi Y \;:=\; X \cup_{X \xleftarrow{\varphi} S} Y \;:=\; (X \sqcup Y)/\sim$$

where $\sim$ is the equivalence relation generated by $\varphi(s) \sim s$ for $s \in S$. We call $X \cup_\varphi Y$ the topological space obtained by gluing Y to X via the map attaching φ.



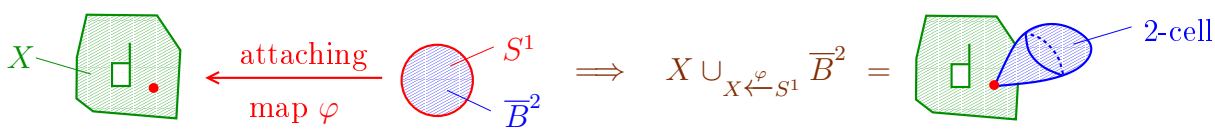
Definition. Let X be a topological space and let $\varphi\colon S^{n-1} \to X$ be a continuous map. We consider the topological space

$$X \cup_{X \xleftarrow{\varphi} S^{n-1}} \overline{B}^n.$$

We introduce the following language:

(1) We refer to the image of B^n as an open n-cell.
(2) We refer to the image of $\overline{B}^n$ as an n-cell.
(3) We refer to φ as the attaching map of the n-cell.

We say that $X \cup_\varphi \overline{B}^n$ is obtained from X by attaching an n-cell along the attaching map φ. It follows from the fact that the inclusion $S^{n-1} \to \overline{B}^n$ is a closed embedding together with the Topological Pushout-Maps Lemma 6.14 (4) that the natural map $X \to X \cup_\varphi \overline{B}^n$ is a closed embedding. Often we will use this fact as an excuse to view X as a closed subset of $X \cup_\varphi \overline{B}^n$.

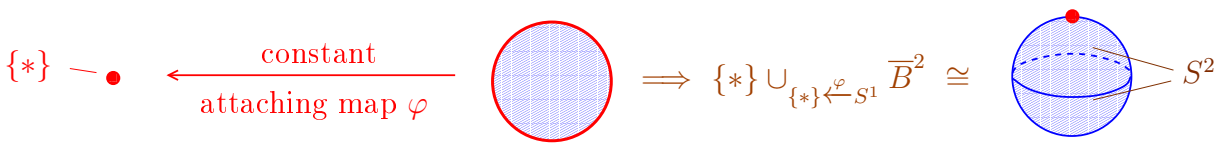


Examples.

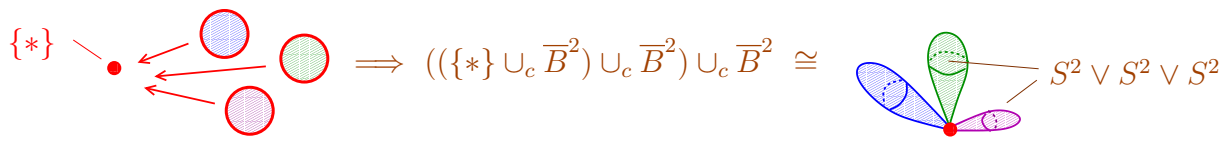
(1) We consider the effect of adding a 0-cell to a topological space X. Since $S^{-1} = \varnothing$ and since $\overline{B}^0 = \{0\}$ we see that attaching a 0-cell is the same as taking the disjoint union of X with a point.
(2) Let $X = \{*\}$ and let $\varphi\colon S^{n-1} \to \{*\}$ be the constant map. We have homeomorphisms

$$\{*\} \cup_{\{*\} \xleftarrow{\varphi} S^{n-1}} \overline{B}^n \quad \underset{\substack{\uparrow \\ \text{see the homeomorphism} \\ \text{from page ??}}}{\xrightarrow{\cong}} \quad \overline{B}^n/S^{n-1} \quad \underset{\substack{\uparrow \\ \text{identification from the} \\ \text{Ball Quotient-Sphere Lemma 6.5}}}{\xrightarrow{\cong}} \quad S^n.$$

In fact, looking at these explicit homeomorphisms we see that these homeomorphisms send $\{*\}$ to the North Pole $(0, \ldots, 0, 1)$.
(3) Again let $X = \{*\}$. Iteratively we attach k n-cells according to the constant attaching map $c\colon S^{n-1} \to \{*\}$. It follows easily from the Topological Pushout Proposition 6.12 (1$*$), the Compact-Hausdorff Proposition 2.10 (3), and the Wedge-Properties

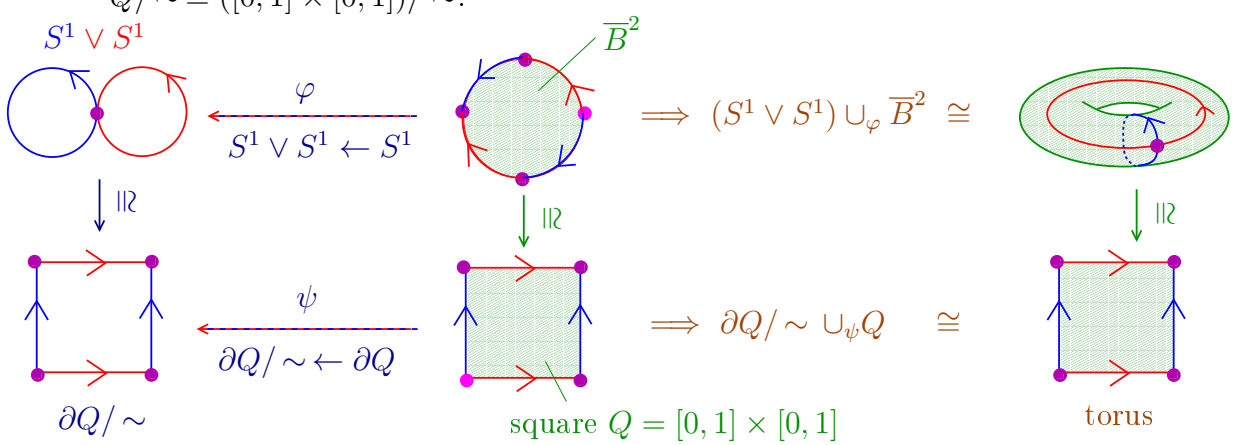


Lemma 17.2 (3) that the resulting topological space is homeomorphic to the wedge of k copies of S^n.

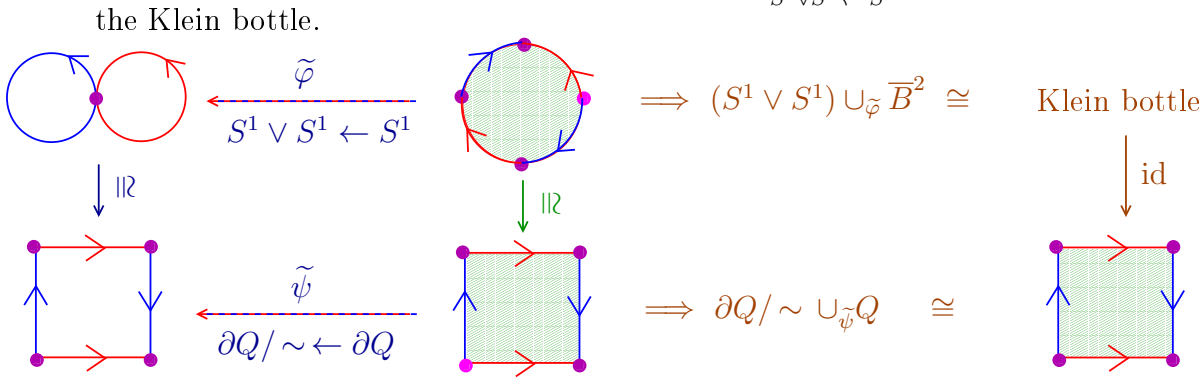


(4) We consider $X = S^1 \vee S^1$.

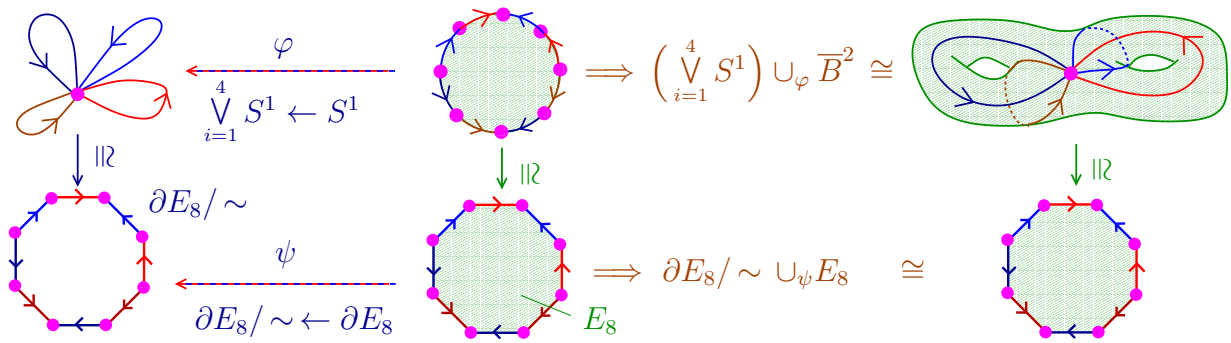
(a) We attach a 2-cell along the attaching map $\varphi\colon S^1 \to S^1 \vee S^1$ that is illustrated below. The Convex-to-Ball Proposition 2.12 gives us an explicit homeomorphism $\overline{B}^2 \to Q := [0,1] \times [0,1]$. This homeomorphism, together with the Topological Pushout Proposition 6.12 (1∗) and the Compact-Hausdorff Proposition 2.10 (3) can be used to show that $(S^1 \vee S^1) \cup_{S^1 \vee S^1 \xleftarrow{\varphi} S^1} \overline{B}^2$ is homeomorphic to the 2-torus $Q/\sim \cong ([0,1] \times [0,1])/\sim$.



(b) If instead we consider the attaching map $\widetilde{\varphi}\colon S^1 \to S^1 \vee S^1$ that is shown in the figure below, then one easily sees that $(S^1 \vee S^1) \cup_{S^1 \vee S^1 \xleftarrow{\widetilde{\varphi}} S^1} \overline{B}^2$ is homeomorphic to the Klein bottle.



(c) In the figure below we show that if we attach a 2-cell in a suitable way to the wedge of four circles, then we obtain a topological space that is homeomorphic to the surface of genus 2.



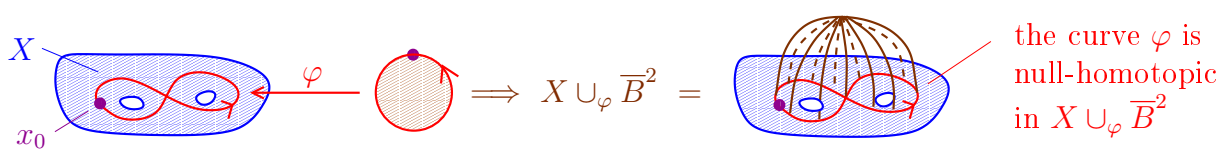
29.2. Attaching cells and fundamental groups.

Proposition 29.1. (Cell Attachment-π_1-Proposition) Let X be a topological space, let $n \geq 1$ and let $\varphi\colon S^{n-1} \to X$ be a continuous map. We set $x_0 := \varphi(\underbrace{(0,\ldots,0,1)}_{\in S^{n-1}})$. The following statements hold:

($n > 2$) The inclusion induced map $\pi_1(X, x_0) \to \pi_1(X \cup_\varphi \overline{B}^n, x_0)$ is an isomorphism.

($n = 2$) The map $[0,1] \to X$ given by $t \mapsto \varphi(\exp(2\pi i t))$ defines an element in $\pi_1(X, x_0)$ that we denote by $[\varphi]$. The inclusion induced map $\pi_1(X, x_0) \to \pi_1(X \cup_\varphi \overline{B}^n, x_0)$ descends to an isomorphism

$$\pi_1(X, x_0)/\langle\langle[\varphi]\rangle\rangle \xrightarrow{\cong} \pi_1\big(X \cup_\varphi \overline{B}^2, x_0\big).$$



Proposition 29.1. (Cell Attachment-π_1-Proposition) We continue with the above notation.

($n = 1$) We set $x_0 := \varphi(1)$ and $y_0 := \varphi(-1)$.

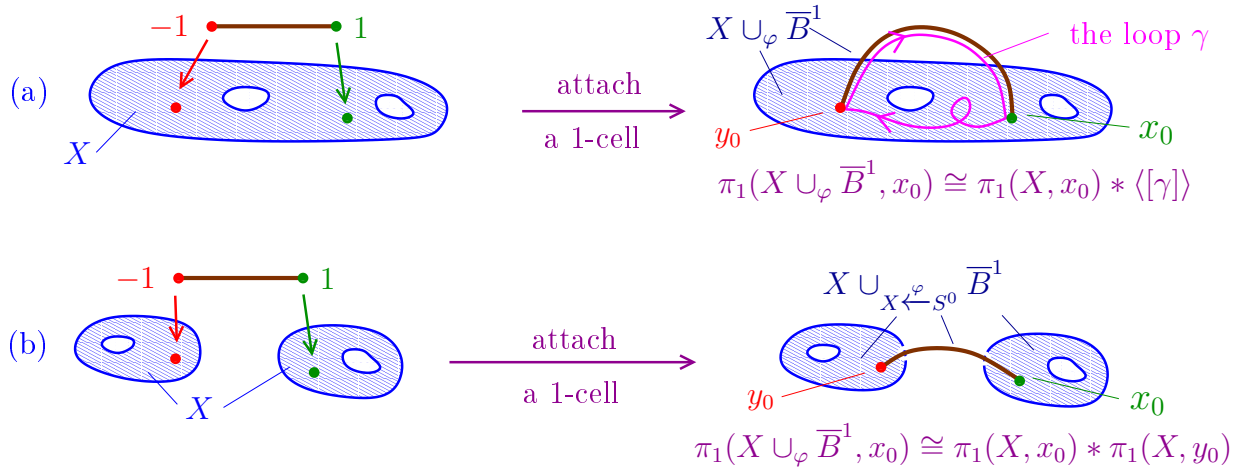
(a) Suppose that x_0 and y_0 lie in the same path-component of X. Let γ be a loop in $X \cup_\varphi \overline{B}^1$ that is given by the concatenation of the obvious path in the 1-cell from $x_0 = \varphi(1)$ to $y_0 = \varphi(-1)$ and some path in X from $y_0 = \varphi(-1)$ to $x_0 = \varphi(1)$. With this notation the map

$$\pi_1(X, x_0) * \langle t\rangle \;\to\; \pi_1(X \cup_\varphi \overline{B}^1, x_0)$$

that is given by the inclusion induced map $\pi_1(X, x_0) \to \pi_1(X \cup_\varphi \overline{B}^1, x_0)$ and given by $t \mapsto [\gamma]$ is an isomorphism.

(b) Suppose that x_0 and y_0 lie in two different path components of X. Then we get an isomorphism

$$\pi_1(X, x_0) * \pi_1(X, y_0) \;\to\; \pi_1(X \cup_\varphi \overline{B}^1, x_0).$$

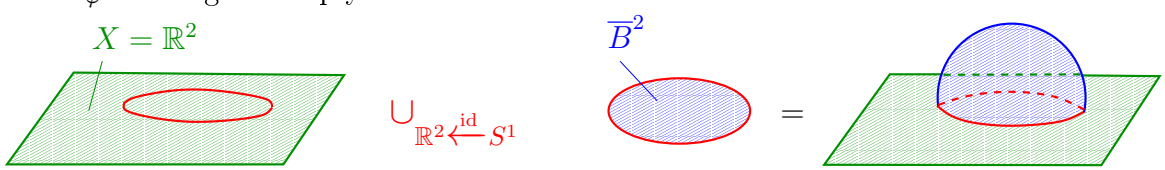


Examples.

(1) Let $X = S^1$ and let $n \in \mathbb{N}$. We attach to S^1 a 2-cell via the attaching map $\varphi\colon S^1 \to S^1$ that is given by $\varphi(z) = z^n$. Then

$$\pi_1\big(S^1 \cup_\varphi \overline{B}^2, 1\big) \underset{\substack{\uparrow \\ \text{Cell Attachment-}\pi_1\text{-Proposition 29.1}}}{=} \pi_1(S^1, 1)/\langle\!\langle [\varphi] \rangle\!\rangle \underset{\substack{\uparrow \\ \text{see the Circle-}\pi_1\text{-Theorem 11.13}}}{=} \langle x \rangle / \langle\!\langle x^n \rangle\!\rangle \;\cong\; \mathbb{Z}_n.$$

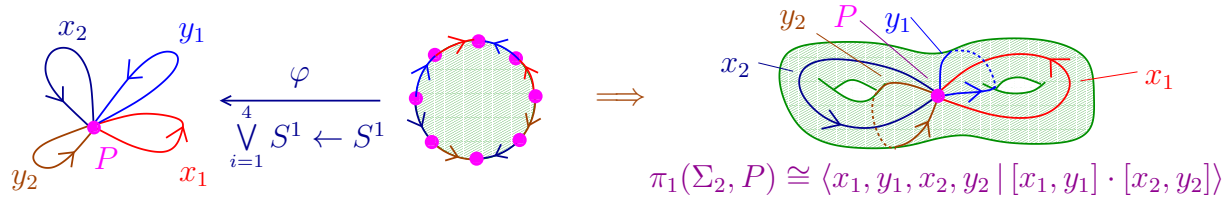
(2) We attach a 2-cell to $X = \mathbb{R}^2$ via the attaching map $\varphi := \mathrm{id}_{S^1}\colon S^1 \to S^1$ that is given by the identity. By the Cell Attachment-π_1-Proposition 29.1 the new topological space $\mathbb{R}^2 \cup_\varphi \overline{B}^2$ is again simply connected



(3) In the previous section we already saw that many interesting topological spaces can be built out of easier topological spaces by attaching cells. In many cases we can use the Cell Attachment-π_1-Proposition 29.1 to calculate the fundamental group. For example, on page 252 we showed that we can obtain the surface of genus 2 from the wedge of four circles by attaching a 2-cell. With the notation from the figure below we now see that

$$\begin{aligned} \pi_1(\Sigma_2, P) \;\cong\; \pi_1\Big(\Big(\bigvee_{i=1}^{4} S^1\Big) \cup_\varphi \overline{B}^2, P\Big) &\underset{\substack{\downarrow \\ \text{by the Cell Attachment-}\pi_1\text{-Proposition 29.1}}}{\cong} \pi_1\Big(\bigvee_{i=1}^{4} S^1, P\Big)/\langle\!\langle [\varphi] \rangle\!\rangle \\ &\underset{\substack{\uparrow \\ \text{by the Wedge-}\pi_1\text{-Proposition 20.6 and since } [\varphi] = [x_1, y_1] \cdot [x_2, y_2]}}{\cong} \langle x_1, y_1, x_2, y_2 \rangle / \langle\!\langle [x_1, y_1] \cdot [x_2, y_2] \rangle\!\rangle. \end{aligned}$$

Of course we obtained the same result much earlier. The advantage of the current approach is that the Cell Attachment-π_1-Proposition 29.1 absorbs many of the technical arguments we had to deal with in the proof of the Surface-π_1-Proposition 22.1, i.e. in our initial calculation of $\pi_1(\Sigma_2, P)$.

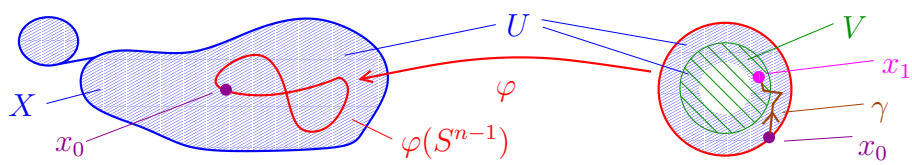


Proof of the Cell Attachment-π_1-Proposition 29.1 for $n \geq 2$. Let X be a topological space, let $n \geq 2$ and let $\varphi\colon S^{n-1} \to X$ be a continuous map. We set $x_0 := \varphi((0, \ldots, 0, 1))$. The key idea of the proof is to apply the Seifert–van Kampen-Theorem 20.1 to the following two subsets of $X \cup_{X \xleftarrow{\varphi} S^{n-1}} \overline{B}^n$

$$U \;=\; X \cup_{X \xleftarrow{\varphi} S^{n-1}} \left\{ z \in \overline{B}^n \,\middle|\, \|z\| > \tfrac{1}{4} \right\} \quad \text{and} \quad V \;=\; \left\{ z \in \overline{B}^n \,\middle|\, \|z\| < \tfrac{3}{4} \right\}.$$

The remainder of the proof consists in showing that the resulting isomorphism implies the desired, a priori slightly different, isomorphism.

We pick a point $x_1 \in U \cap V$ and we pick a path γ in U from x_0 to x_1.



Claim. The following diagram of maps

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[rr] \arrow[d, "\cong"'] & & \pi_1(X \cup_\varphi \overline{B}^n, x_0) \arrow[d, "\mathrm{id}"] \\
\pi_1(U, x_0) \arrow[rr] & & \pi_1(X \cup_\varphi \overline{B}^n, x_0) \\
\pi_1(U, x_1) \arrow[rr] \arrow[u, "\gamma_*", "\cong"'] \arrow[d] & & \pi_1(X \cup_\varphi \overline{B}^n, x_1) \arrow[u, "\gamma_*", "\cong"'] \arrow[d, "\mathrm{id}"] \\
\pi_1(U, x_1)/\langle\langle \pi_1(U \cap V, x_1) \rangle\rangle \arrow[rr] & & \pi_1(X \cup_\varphi \overline{B}^n, x_1) \\
\pi_1(U, x_1) *_{\pi_1(U \cap V, x_1)} \underbrace{\pi_1(V, x_1)}_{=\{e\}} \arrow[rr, "\cong", "\text{Seifert–van Kampen Theorem 20.1}"'] \arrow[u, "\cong", "\text{Amalgamated Product–Quotient Lemma 19.13}"'] & & \pi_1(X \cup_\varphi \overline{B}^n, x_1) \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

has the following three properties:

(i) The diagram commutes.
(ii) All vertical maps except possibly the third vertical map on the left are isomorphisms.
(iii) The bottom horizontal map is an isomorphism.

Proof.

(i) The two vertical maps from the third to the second level are the homomorphisms induced by the choice of base point coming from the path γ from x_0 to x_1. It follows from the Change-of-Base Point Proposition 13.2 that the second square commutes. All

the other vertical and horizontal maps are induced by inclusions or they are the obvious algebraic map. It follows easily from functoriality that the diagram commutes.

(ii) First note that it follows from the Change-of-Base Point Proposition 11.6 (1) that the vertical maps from the third to the second level are isomorphisms. Using the Homotopy Pushout Lemma from [Fri23] one can easily show that X is a deformation retract of U. Thus it follows from the Homotopy-π_1-Proposition 13.4 (2b) that the top left vertical map is an isomorphism.

The natural map $\{z \in B^n \mid \|z\| < \frac{3}{4}\} \to X \cup_{X \xleftarrow{\varphi} S^{n-1}} \overline{B}^n$ is easily seen to be an embedding. It follows that V is homeomorphic to an open ball. Hence $\pi_1(V, x_1)$ is the trivial group. It follows from this observation and the Amalgamated Product–Quotient Lemma 19.13 that the bottom vertical map on the left is an isomorphism.

(iii) It follows easily from the Topological-Pushout Proposition 6.12 (0) and the Topological-Coproduct Proposition 5.8 (0) that U and V are indeed open subsets of $X \cup_{X \xleftarrow{\varphi} S^{n-1}} \overline{B}^n$. We can therefore apply the Seifert–van Kampen-Theorem 20.1 and we obtain that the bottom horizontal map is an isomorphism. ⊞

Now we consider the case $n \geq 3$. In this case $\pi_1(U \cap V, x_1)$ is trivial since $U \cap V$ is homotopy equivalent to S^{n-1}. It follows from the Homotopy-π_1-Proposition 13.4 and the Sphere-π_1-Proposition 11.14 that $\pi_1(U \cap V, x_1) \cong \pi_1(S^{n-1}) = 0$. It is now clear that the one missing vertical map on the left is also an isomorphism. Since all vertical maps are isomorphisms, since the diagram commutes and since the bottom horizontal map is an isomorphism it follows that the top horizontal map is an isomorphism.

Finally we consider the case $n = 2$. In this case $U \cap V$ is homotopy equivalent to S^1 which implies, by the explicit isomorphism from the Circle-π_1-Theorem 11.13 and the Homotopy-π_1-Proposition 13.4, that $\pi_1(U \cap V, x_1) \cong \mathbb{Z}$. Note that under the explicit isomorphism $\Phi\colon \pi_1(X, x_0) \to \pi_1(U, x_1)$ this infinite cyclic subgroup corresponds to $\langle[\varphi]\rangle$. It is straightforward to see that we get a commutative diagram

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[rr] \arrow[dr] \arrow[d, "\gamma_*"', "\cong"] & & \pi_1\big(X \cup_\varphi \overline{B}^n, x_0\big) \arrow[d] \\
& \pi_1(X, x_0)/\langle\langle[\varphi]\rangle\rangle \arrow[ur] \arrow[dl, "\cong"] & \\
\pi_1(U, x_1)/\langle\langle\pi_1(U \cap V, x_1)\rangle\rangle \arrow[rr, "\cong"'] & & \pi_1\big(X \cup_\varphi \overline{B}^n, x_1\big).
\end{tikzcd}$$

This concludes the proof of the proposition. ■

Proof of the Cell Attachment-π_1-Proposition 29.1 for $n = 1$.

(a) We consider the case $\varphi(-1)$ and $\varphi(1)$ lie in the same path-component of X. This case is treated in [Fri23].

(b) We consider the case $\varphi(-1)$ and $\varphi(1)$ lie in the same path-component of X. This case follows once again fairly easily from the Seifert–van Kampen-Theorem 20.1. ■

29.3. **Projective spaces.** We just saw that the Cell Attachment-π_1-Proposition 29.1 can be very useful for calculating fundamental groups. Before we had already been quite successful with calculating fundamental groups, but it turns out that there is one interesting example of topological spaces that we could not handle yet, namely the complex projective spaces $\mathbb{C}\mathrm{P}^n$.

For the reader's convenience we recall the definition of real and complex projective spaces which we defined on page 86.

Definition. Let $\mathbb{A} = \mathbb{R}$ or $\mathbb{A} = \mathbb{C}$.

(1) For $n \in \mathbb{N}_0$ we set
$$\mathbb{A}\mathrm{P}^n \;:=\; (\mathbb{A}^{n+1} \setminus \{0\})/\sim \qquad \text{where } v \sim \lambda \cdot w \text{ for } \lambda \in \mathbb{A} \setminus \{0\}.$$
Given a point $(v_0, \ldots, v_n) \in \mathbb{A}^{n+1} \setminus \{0\}$ we denote by $[v_0 : \ldots : v_n]$ its equivalence class.

(2) For $k \leq n$ we refer to the map
$$\begin{array}{rcl} \iota\colon \mathbb{A}\mathrm{P}^k & \to & \mathbb{A}\mathrm{P}^n \\ [v_0 : \ldots : v_k] & \mapsto & [v_0 : \ldots : v_k : 0 : \ldots : 0] \end{array}$$
as the standard embedding. We saw on page **??** that this map is indeed an embedding.

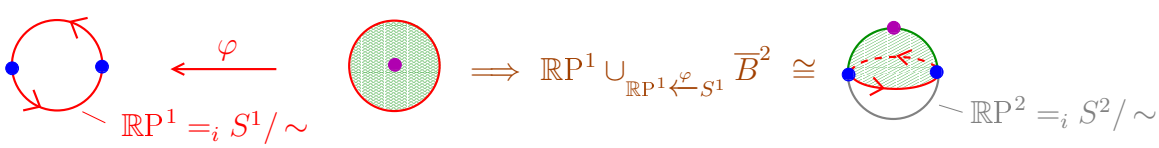
Lemma 29.2. (Projective Space–Cell Attachment Lemma) Let $n \in \mathbb{N}$.

(1) We consider the continuous map
$$\begin{array}{rcl} \varphi\colon S^{n-1} & \to & \mathbb{R}\mathrm{P}^{n-1} \\ (x_1, \ldots, x_n) & \mapsto & [x_1 : \ldots : x_n]. \end{array}$$
There exists a homeomorphism $\mathbb{R}\mathrm{P}^{n-1} \cup_{\mathbb{R}\mathrm{P}^{n-1} \xleftarrow{\varphi} S^{n-1}} \overline{B}^n \to \mathbb{R}\mathrm{P}^n$ such that the restriction to $\mathbb{R}\mathrm{P}^{n-1}$ is given by the standard embedding $\mathbb{R}\mathrm{P}^{n-1} \to \mathbb{R}\mathrm{P}^n$.

(2) We think of $\overline{B}^{2n}$ and S^{2n-1} as subspaces of $\mathbb{C}^n =_i \mathbb{R}^{2n}$. We consider the attaching map
$$\begin{array}{rcl} \varphi\colon S^{2n-1} & \to & \mathbb{C}\mathrm{P}^{n-1} \\ (z_1, \ldots, z_n) & \mapsto & [z_1 : \ldots : z_n]. \end{array}$$
There exists a homeomorphism $\mathbb{C}\mathrm{P}^{n-1} \cup_{\mathbb{C}\mathrm{P}^{n-1} \xleftarrow{\varphi} S^{2n-1}} \overline{B}^{2n} \to \mathbb{C}\mathrm{P}^n$ such that the restriction to $\mathbb{C}\mathrm{P}^{n-1}$ is given by the standard embedding $\mathbb{C}\mathrm{P}^{n-1} \to \mathbb{C}\mathrm{P}^n$.

Proof.

(1) We consider the map
$$\begin{array}{rcl} \Theta\colon \mathbb{R}\mathrm{P}^{n-1} \cup_\varphi \overline{B}^n & \to & \mathbb{R}\mathrm{P}^n \\ [P] & \mapsto & \begin{cases} [x_1 : \cdots : x_n : 0], & \text{if } P = [x_1 : \cdots : x_n] \in \mathbb{R}\mathrm{P}^{n-1}, \\ [P : \sqrt{1 - \|P\|^2}], & \text{if } P \in \overline{B}^n. \end{cases} \end{array}$$



It follows immediately from the definition of φ that the map Θ is well-defined. Furthermore it follows easily from the Topological Pushout Proposition 6.12 (1∗) that the map Θ is continuous. It is basically clear that the map Θ is a surjection and it is straightforward to verify that the map Θ is an injection. Using the Topological-Quotient Proposition 6.1 (1b), the Projective Spaces-Compact+Hausdorff Proposition 6.10 and the Compact-Hausdorff Proposition 2.10 (3) one can easily verify that the map is a homeomorphism. Finally it is clear that the restriction of the map Θ to $\mathbb{R}\mathrm{P}^{n-1}$ is the standard embedding.

(2) The proof for $\mathbb{C}\mathrm{P}^n$ is basically identical to the proof of (1). ■

Proposition 29.3. (Projective Space–π_1-Proposition)
(1) We have $\pi_1(\mathbb{R}\mathrm{P}^1) \cong \mathbb{Z}$ and for $n \geq 2$ we have $\pi_1(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}_2$.
(2) For any $n \in \mathbb{N}_0$ we have $\pi_1(\mathbb{C}\mathrm{P}^n) = 0$.

Proof.

(1) Since we already calculated the fundamental groups of $\mathbb{R}\mathrm{P}^n$ in the Real Projective Space–π_1-Proposition 15.3 we outsource this part as an exercise to the reader.
(2) Note that $\mathbb{C}\mathrm{P}^0$ consists of a single point. It follows from the Projective Space–Cell Attachment Lemma 29.2 that we obtain $\mathbb{C}\mathrm{P}^n$ from $\mathbb{C}\mathrm{P}^0$ by iteratively attaching cells of dimension $2, 4, \ldots, 2n$. It follows from the Cell Attachment-π_1-Proposition 29.1 that $\pi_1(\mathbb{C}\mathrm{P}^n) \cong \pi_1(\mathbb{C}\mathrm{P}^0) = 0$. ■

29.4. Attaching arbitrarily many cells.

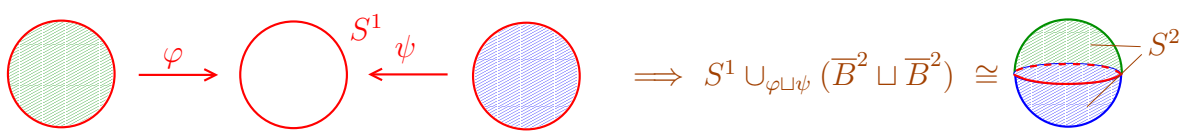
Definition. Let X be a topological space and let $\{\varphi\colon S^{n_i} \to X\}_{i\in I}$ be a family of continuous maps.
(1) We consider the topological space

$$X \cup_{X \leftarrow \bigsqcup_{i\in I} S^{n_i-1} \,:\, \bigsqcup_{i\in I} \varphi_i} \bigsqcup_{i\in I} \overline{B}^{n_i}.$$

As before we introduce the notion of an (open) cell and of an attaching map of a cell. We say that the above topological space is obtained from X by attaching cells along the attaching maps φ_i.
(2) It follows from the fact that the inclusions $S^{n-1} \to \overline{B}^n$ are closed embeddings together with the Topological Pushout-Maps Lemma 6.14 (4) that the natural map of X into the above topological space is a closed embedding. Often we will use this fact as an excuse to view X as a closed subset of the above topological space.

Example.

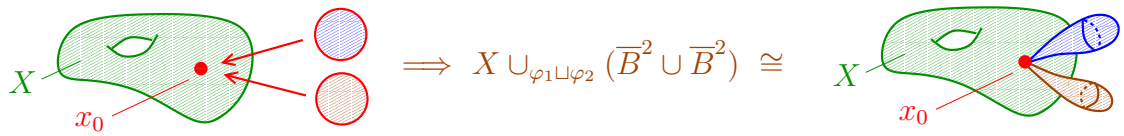
(1) Let $n \in \mathbb{N}_0$. We attach two n-cells to S^{n-1} where in both cases the attaching map is given by the identity map $\varphi := \psi := \mathrm{id}\colon S^{n-1} \to S^{n-1}$. Using the Compact-Hausdorff Proposition 2.10 (3) one can easily show that there exists a homeomorphism $S^{n-1} \cup_{\varphi\sqcup\psi} (\overline{B}^n \sqcup \overline{B}^n) \to S^n$ which on S^{n-1} is given by $x \mapsto (x, 0)$.



(2) Let (X, x_0) be a pointed topological space. Furthermore let I be a set and for each $i \in I$ let $n_i \in \mathbb{N}_0$. Given $i \in I$ we denote by $\varphi_i\colon S^{n_i-1} \to X$ the constant map that sends all points to x_0. Using the Ball Quotient-Sphere Lemma 6.5 one can show that there exists a homeomorphism

$$X \cup_{X \leftarrow \bigsqcup_{i\in I} S^{n_i-1} \,:\, \bigsqcup_{i\in I} \varphi_i} \bigsqcup_{i\in I} \overline{B}^{n_i} \xrightarrow{\cong} \underbrace{(X, x_0) \vee \bigvee_{i\in I} (S^{n_i}, N)}_{\substack{\text{wedge of pointed topological} \\ \text{spaces where } N \in S^{n_i} \text{ is the} \\ \text{North Pole}}}$$

that is the identity on X.



Proposition 29.4. (Cell Attachment–Hausdorff Proposition) Let X be a topological space and let $\{\varphi_i \colon S^{n_i} \to X\}_{i \in I}$ be a family of continuous maps. We consider the topological space

$$\widehat{X} \;:=\; X \cup_{X \leftarrow \bigsqcup_{i \in I} S^{n_i - 1} \,:\, \bigsqcup_{i \in I} \varphi_i} \bigsqcup_{i \in I} \overline{B}^{n_i}$$

that we obtain from X by attaching cells along the attaching maps φ_i.

(1) If U, V are disjoint open subset of X, then there exist disjoint open subsets $\widehat{U}$ and $\widehat{V}$ of $\widehat{X}$ with $X \cap \widehat{U} = U$ and $X \cap \widehat{V} = V$.

(2) If X is Hausdorff, then $\widehat{X}$ is also Hausdorff.

Proof. For each $i \in I$ we consider the natural map $\Phi_i \colon \overline{B}^n_i \to \widehat{X}$. It follows immediately from the Topological-Pushout Proposition 6.12 (0) and the Topological-Coproduct Proposition 5.8 (0) that given any subset $U \subset \widehat{X}$ the following statement holds:

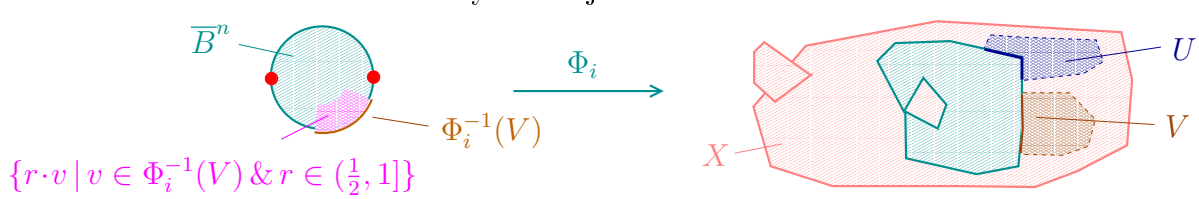
$$U \text{ is open in } \widehat{X} \iff \begin{array}{l} \text{(a) the intersection } U \cap X \text{ is open in } X \text{ and} \\ \text{(b) for all } i \in I \text{ the preimage } \Phi_i^{-1}(U) \text{ is open in } \overline{B}^n. \end{array}$$

After this preparation we turn to the actual proof of the two statements:

(1) Let U and V be two disjoint open subsets of X. We set

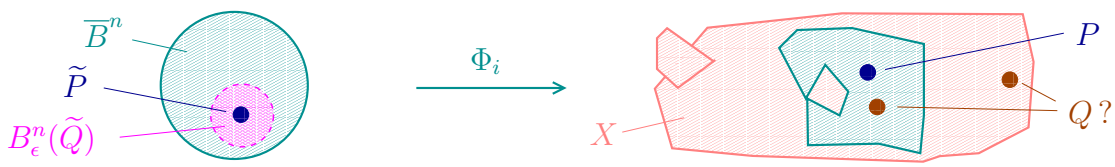
$$\begin{aligned} \widehat{U} \;&:=\; U \;\cup\; \bigcup_{i \in I} \Phi_i(\{r \cdot u \,|\, u \in \Phi_i^{-1}(U) \text{ and } r \in (\tfrac{1}{2}, 1]\}). \\ \widehat{V} \;&:=\; V \;\cup\; \bigcup_{i \in I} \Phi_i(\{r \cdot v \,|\, v \in \Phi_i^{-1}(V) \text{ and } r \in (\tfrac{1}{2}, 1]\}). \end{aligned}$$

It follows immediately from the above discussion that $\widehat{U}$ and $\widehat{V}$ are open subsets of $\widehat{X}$. Furthermore it is clear that they are disjoint.



(2) Let $P, Q \in \widehat{X}$ be two distinct points. If P and Q lie in X, then it follows from the hypothesis that X is Hausdorff and (1) that we can find disjoint open neighborhoods $\widehat{U}$ of P and $\widehat{V}$ of Q in $\widehat{X}$.

It remains to consider the case that at least one of P and Q lies in $\widehat{X} \setminus X$. Let us assume that $P \in \widehat{X} \setminus X$. This means that there exists an $i \in I$ and a $\widetilde{P} \in B^n$ such that $P = \Phi_i(\widetilde{P})$. Since $Q \neq P$ there exists an $\epsilon > 0$ such that $B^n_\epsilon(\widetilde{P}) \subset B^n$ and such that $Q \notin \Phi_i(B^n_\epsilon(\widetilde{P}))$. We set $U := \Phi_i(B^n_{\frac{\epsilon}{2}}(\widetilde{Q}))$ and $V := X \setminus \Phi_i(\overline{B}^n_{\frac{\epsilon}{2}}(\widetilde{Q}))$. It is clear that U and V are disjoint. It follows from the discussion at the beginning of the proof that U and V are open subsets of $\widehat{X}$. ■


$\overline{B}^n$
$\widetilde{P}$
$B^n_\epsilon(\widetilde{Q})$
Φ_i
P
$Q\,?$
X

30. CW-complexes: Definitions and basic properties

30.1. Definition of CW-complexes and examples.

Definition. A CW-complex X is a topological space X together with a filtration

$$\varnothing = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X$$

by subspaces which has the following properties:

(CW1) We have $X = \varinjlim X^n$, as described in the Topological-Direct Limit Proposition 25.4. In other words, as a set we have $X = \bigcup_{n\in\mathbb{N}_0} X^n$ and $U \subset X$ is open if and only if for each $n \in \mathbb{N}_0$ the intersection $U \cap X^n$ is open in X^n.

(CW2) For each $n \in \mathbb{N}_0$ there exists a set I, a family $\{\varphi_i \colon S^{n-1} \to X^{n-1}\}_{i\in I}$ of continuous maps and a homeomorphism

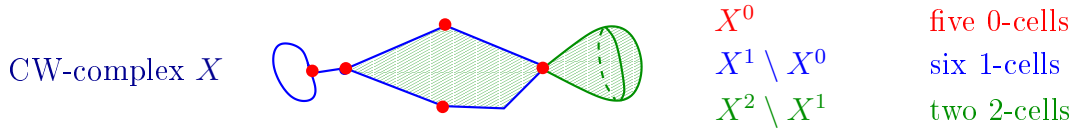
$$\Xi_n \colon X^{n-1} \cup_{X^{n-1} \leftarrow \bigsqcup_{i\in I} S^{n-1} \,:\, \bigsqcup_{i\in I} \varphi_i} \bigsqcup_{i\in I} \overline{B}^n \;\to\; X^n$$

which is the inclusion on X^{n-1}.

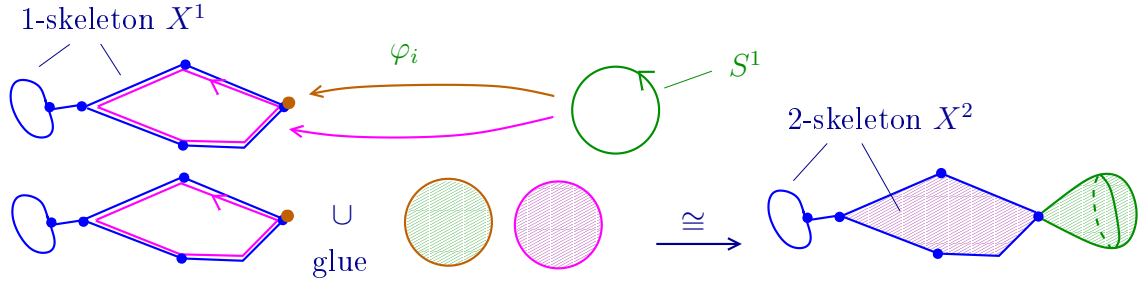
Definition. Using the notation from above we introduce the following language:

(1) Given $n \in \mathbb{N}_{\geq 0}$ we refer to X^n as the n-skeleton of X.
(2) The components of $X^n \setminus X^{n-1}$ are called the open n-cells of X.
(3) For each open n-cell σ of X we refer to its closure $\overline{\sigma}$ in X as an n-cell of X.
(4) We set
$$\dim(X) \;:=\; \sup\{n \in \mathbb{Z}_{\geq -1} \,|\, X^n \neq X^{n+1}\} \;\in\; \mathbb{Z}_{\geq -1} \cup \{\infty\}.$$
(5) We say the CW-complex is finite if it has only finitely many cells and we say it is countable if it has only countably many cells.
(6) A CW-structure on a topological space X is filtration as in (1).

Illustration. In the figure below we show a 2-dimensional CW-complex X with its filtration $X^0 \subset X^1 \subset X^2$.



In the next figure we indicate how we obtain the 2-skeleton from the 1-skeleton by gluing on 2-disks.



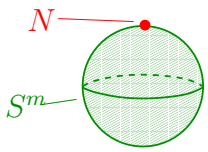
Examples.

(1) The one type of CW-complexes that is easy to understand are 0-dimensional CW-complexes. Similar to the discussion on page 250 it follows from the observation that $\overline{B}^0$ consists of a point and that $S^{-1} = \varnothing$ that a CW-complex of dimension ≤ 0 is essentially the same as a set of points with the discrete topology.

(2) Let $m \in \mathbb{N}_0$. As usual we denote by $N := (0, \ldots, 0, 1)$ the North Pole of S^m. We consider the filtration $\varnothing \subset \{N\} \subset S^m$. That this is a CW-structure can be seen as follows:

(CW1) This condition is vacuous since the m-skeleton equals S^m.

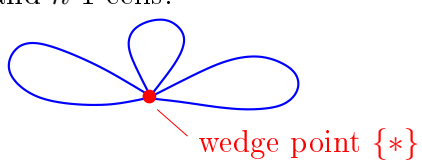
(CW2) The case $n = 0$ is clear since $\{N\}$ has the discrete topology. For $n = 1, \ldots, m-1$ there is nothing to show. Finally note that on page **??** we showed that we obtain S^m from $\{N\}$ by attaching an m-cell.



the filtration $\varnothing \subset \{N\} \subset S^m$ is a CW-structure with
one 0-cell
one m-cell

(3) Let $n \in \mathbb{N}_0$ and let $X = \bigvee_{i=1}^{n} S^1$ be the wedge of n circles with wedge point $*$. The discussion on page 251 implies that the filtration $\varnothing \subset \{*\} \subset \bigvee_{i=1}^{n} S^1$ is a CW-complex with one 0-cell and n 1-cells.

$X = S^1 \vee S^1 \vee S^1$

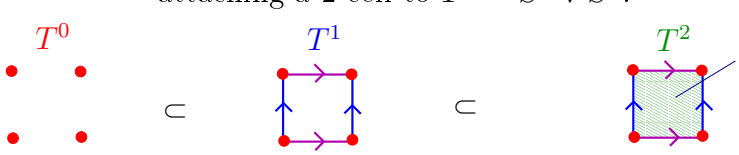


the filtration $\varnothing \subset \{*\} \subset S^1 \vee S^1 \vee S^1$
is a CW-complex with
one 0-cell
three 1-cells

(4) The figure below shows a filtration of the torus $T = ([0,1] \times [0,1]) \sim$ that is a CW-structure with one 0-cell, two 1-cells and one 2-cell. That this is a CW-structure can be seen as follows:

(CW1) This condition is vacuous since $T^2 = T$.

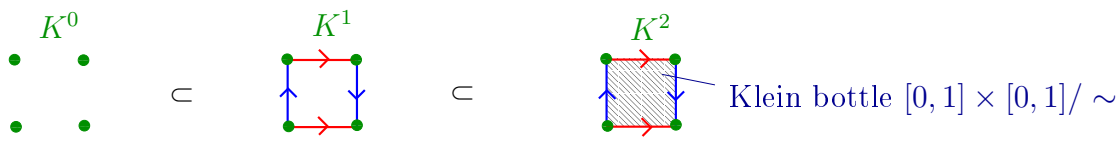
(CW2) The case $n = 0$ is clear since T^0 has the discrete topology. For $n = 1$ note that T^1 is clearly homeomorphic to the wedge of two circles along the unique point of T^0. Thus we obtain from the previous example that (CW2) is satisfied for $n = 1$. Finally note that on page 251 we showed that we obtain T from attaching a 2-cell to $T^1 \cong S^1 \vee S^1$.



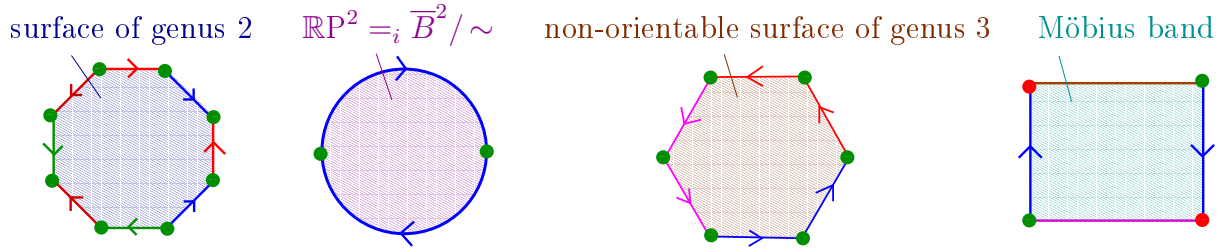
torus $T = ([0,1] \times [0,1])/\sim$ with
one 0-cell
two 1-cells
one 2-cell

(5) The figure below shows a filtration of the Klein bottle that is a CW-structure with one 0-cell, two 1-cells and one 2-cell.

(6) In the figure below we sketch CW-structures for the surface of genus 2, the projective plane $\mathbb{RP}^2 =_i \overline{B}^2/\sim$, the non-orientable surface of genus 3 and the Möbius band as CW-complexes. Generalizing these examples one can easily show, using the Surface



Classification Theorem 22.9, that every compact 2-dimensional topological manifold admits a CW-structure.



surface of genus 2	$\mathbb{R}\mathrm{P}^2 =_i \overline{B}^2/\sim$	non-orientable surface of genus 3	Möbius band
one 0-cell	one 0-cell	one 0-cell	two 0-cells
four 1-cells	one 1-cell	three 1-cells	three 1-cells
one 2-cell	one 2-cell	one 2-cell	one 2-cell

Examples.

(1) (a) Let us consider the topological space $\mathbb{R}$. We equip it with the filtration $\varnothing \subset \mathbb{Z} \subset \mathbb{R}$. We consider the family $\{\varphi_i \colon S^0 \to \mathbb{Z}\}_{i\in\mathbb{Z}}$ of maps given by $\varphi_i(\epsilon) = i + \frac{1+\epsilon}{2}$. Using the Cover-Open Subset Lemma 1.5 (2) one can verify that the map

$$\begin{aligned} \mathbb{Z} \cup_{\mathbb{Z} \leftarrow\!\shortmid \bigsqcup_{i\in\mathbb{Z}} S^0_i \,:\, \bigsqcup_{i\in\mathbb{Z}} \varphi_i} \bigsqcup_{i\in\mathbb{Z}} \overline{B}^1_i \;&\to\; \mathbb{R} \\ P \;&\mapsto\; \begin{cases} n, & \text{if } P = [n] \text{ for some } n \in \mathbb{Z} \\ i + \frac{1+t}{2}, & \text{if } P = [(t,i)] \text{ for some } t \in \overline{B}^1 = [-1,1] \text{ and } i \in \mathbb{Z} \end{cases} \end{aligned}$$

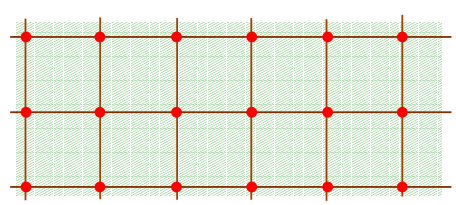
is a homeomorphism. Thus the filtration $\varnothing \subset \mathbb{Z} \subset \mathbb{R}$ is a CW-structure on $\mathbb{R}$.

(b) Using the Convex-to-Ball Proposition 2.12 one can easily generalize the above example and one can show that the subspaces

$$X^i \;:=\; \{(x_1,\ldots,x_n) \in \mathbb{R}^n \,|\, \text{at least } n-i \text{ coordinates lie in } \mathbb{Z}\} \text{ with } i = 0,\ldots,n$$

form a CW-structure on $\mathbb{R}^n$.

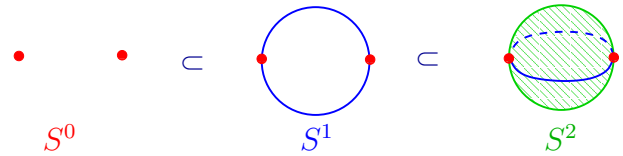
CW-structure on $\mathbb{R}$



CW-structure on $\mathbb{R}^2$

(2) Let I be a set and for each $i \in I$ let $n_i \in \mathbb{N}$. We consider the wedge $\bigvee_{i\in I} S^{n_i}_i$. Given $n \in \mathbb{N}_0$ we denote by X_n the wedge of all spheres of dimension $\leq n$. It follows from the discussion on page 257 that the filtration $\varnothing \subset X_0 \subset X_1 \subset \ldots$ is a CW-structure on $\bigvee_{i\in I} S^{n_i}_i$.

Convention. For $k < n$ we use in the following discussion the embedding $S^k \to S^n$ given by $x \mapsto (x, 0)$ to think of S^k as a subspace of S^n. Similarly we view $\mathbb{R}\mathrm{P}^k$ as a subspace of $\mathbb{R}\mathrm{P}^n$ and we view $\mathbb{C}\mathrm{P}^k$ as a subspace of $\mathbb{C}\mathrm{P}^n$.



Definition. We say that a CW-complex A is a subcomplex of a CW-complex X if $A \subset X$ and if the following two conditions are satisfied:

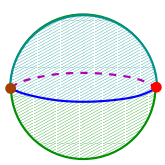
(1) For each $n \in \mathbb{N}_0$ we have $A^n = A \cap X^n$.
(2) Each open n-cell of A is also an open n-cell of X.

Examples.

(1) We consider the spheres S^n.
  (a) On page 261 we found a "tiny" CW-structure for S^n with just one 0-cell and one n-cell. But this CW-structure does not have any interesting subcomplexes. To fix this, recall that on page 257 we saw that we obtain S^{k+1} from S^k by attaching a single $(k+1)$-cell. It follows that a CW-structure for S^n is given by

$$\varnothing \subset S^0 \subset S^1 \subset S^2 \subset \ldots \subset S^n.$$

With this CW-structure we can view each S^k with $k < n$ as a subcomplex of S^n.



the filtration $\varnothing \subset S^0 \subset S^1 \subset S^2$ is a CW-structure with
two 0-cells
two 1-cells
two 2-cells

  (b) The new CW-structure on spheres gives us an immediate bonus: As on page 225 we consider the topological space

$$S^\infty := \left\{(x_1, x_2, \ldots) \in \mathbb{R}^\infty \,\middle|\, \sum_{i=1}^{\infty} x_i^2 = 1\right\} = \operatorname*{colim}_{\longrightarrow} S^{i-1}.$$

It follows immediately from (a) and the definitions that a CW-structure for S^∞ is given by

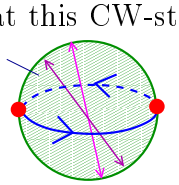
$$\varnothing \subset S^0 \subset S^1 \subset S^2 \subset \ldots \subset S^\infty.$$

(2) We consider the real projective spaces $\mathbb{R}\mathrm{P}^n$.
  (a) In the Projective Space–Cell Attachment Lemma 29.2 (1) we showed that we can obtain $\mathbb{R}\mathrm{P}^{k+1}$ from $\mathbb{R}\mathrm{P}^k$ by attaching a single $(k+1)$-cell. It follows that a CW-structure for $\mathbb{R}\mathrm{P}^n$ is given by

$$\varnothing \subset \mathbb{R}\mathrm{P}^0 \subset \mathbb{R}\mathrm{P}^1 \subset \mathbb{R}\mathrm{P}^2 \subset \ldots \subset \mathbb{R}\mathrm{P}^n$$

and that this CW-structure has precisely one cell in dimensions $0, 1, 2, \ldots, n$.

$\mathbb{R}\mathrm{P}^2 =_i S^2/\sim$



the filtration $\varnothing \subset \mathbb{R}\mathrm{P}^0 \subset \mathbb{R}\mathrm{P}^1 \subset \mathbb{R}\mathrm{P}^2$ is a CW-structure with
one 0-cell
one 1-cell
one 2-cell

(b) As on page 226 we consider $\mathbb{RP}^\infty = \underrightarrow{\operatorname{colim}}\, \mathbb{RP}^i$. As in (1b) we see that

$$\varnothing \subset \mathbb{RP}^0 \subset \mathbb{RP}^1 \subset \mathbb{RP}^2 \subset \ldots \subset \mathbb{RP}^\infty$$

is a CW-structure which has precisely one cell in each dimension.

(3) Finally we consider the complex projective spaces $\mathbb{CP}^n$.

(a) As in (2a) we see, this time using the Projective Space–Cell Attachment Lemma 29.2 (2), that a CW-structure for $\mathbb{CP}^n$ is given by

$$\varnothing \subset \mathbb{CP}^0 \subset \mathbb{CP}^1 \subset \mathbb{CP}^2 \subset \ldots \subset \mathbb{CP}^n$$

and that this CW-structure has precisely one cell in dimensions $0, 2, 4, \ldots, 2n$.

(b) As on page 226 we consider $\mathbb{CP}^\infty = \underrightarrow{\operatorname{colim}}\, \mathbb{CP}^i$ and we see that

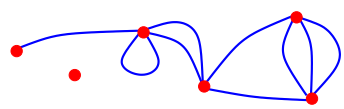
$$\varnothing \subset \mathbb{CP}^0 \subset \mathbb{CP}^1 \subset \mathbb{CP}^2 \subset \ldots \subset \mathbb{CP}^\infty$$

is a CW-structure which has precisely one cell in each even dimension.

Lemma 30.1. (Graph–CW-Complex Lemma)

(1) If $(X, G = (V, E, \theta), \Theta\colon |G| \to X)$ is an undirected topological graph then $\varnothing \subset V \subset X$ is a CW-structure.

(2) Let $\varnothing \subset X^0 \subset X^1 = X$ be a CW-complex of dimension ≤ 1. There exists an undirected abstract graph (V, E, θ) and a homeomorphism $f\colon |G| \to X$ such that $f(V) = X^0$.



vertices form the 0-skeleton

edges correspond to 1-cells

Sketch of proof.

(1) This statement follows basically immediately from the definitions.

(2) By definition of a CW-complex there exists a set C, a family $\{\varphi_c\colon S^0 \to X^0\}_{c\in C}$ of continuous maps and a homeomorphism

$$\Xi\colon X^0 \cup_{X^0 \leftarrow \bigsqcup_{c\in C} S^0_c \,:\, \bigsqcup_{c\in C} \varphi_c} \bigsqcup_{c\in C} \overline{B}^1_c \;\to\; X^1 = X$$

which is the inclusion on X^0. Given $c \in C$ we set $\theta(c) := \{\varphi_c(-1), \varphi_c(1)\} \subset \mathcal{P}(X^0)$. It is clear that $G := (X^0, C, \theta)$ is an undirected abstract graph. The map Ξ gives rise to a homeomorphism $f\colon |G| \to X$ with $f(V) = X^0$. ■

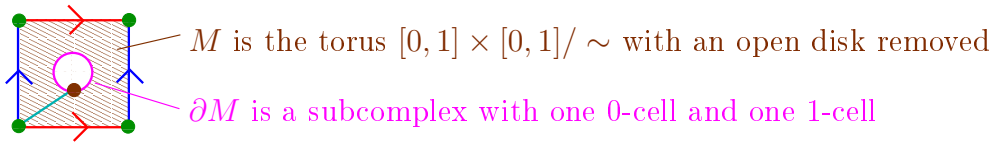
Theorem 30.2. (Smooth Manifold-CW-Theorem) Every compact connected n-dimensional smooth manifold M admits a finite CW-structure with the following properties:

(1) the maximal dimension of a cell is n,

(2) the boundary ∂M is a subcomplex.

Proof. This theorem is proved in [Fri23]. ■

Example. In the figure below we consider the smooth manifold M that is given by removing an open disk from a torus. We see that M has a CW-structure such that the boundary ∂M is a subcomplex.



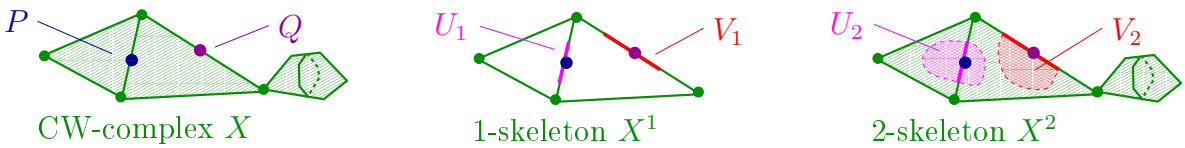
30.2. Attaching maps and characteristic maps.

Proposition 30.3. (CW-Complex–Hausdorff Proposition) Every CW-complex is Hausdorff.

Proof. Let X be a CW-complex and let P, Q be two distinct points in X. Recall that $X = \bigcup_{n \in \mathbb{N}_0} X^n$. Thus there exists an $n \in \mathbb{N}_0$ with $P, Q \in X^n$.

Since the empty topological space is Hausdorff we obtain from iteratively applying the Cell Attachment–Hausdorff Proposition 29.4 (2) that $X^0, \ldots, X^n$ are Hausdorff. In particular we see that there exist disjoint open neighborhoods U_n of $P \in X^n$ and V_n of $Q \in X^n$.

It follows from iteratively applying the Attachment–Hausdorff Proposition 29.4 (1) that for each $d \geq n$ we can find disjoint open neighborhoods U_d of $P \in X^d$ and V_d of $Q \in X^d$ such that for $d \geq n$ we have $U_d \subset U_{d+1}$ and $V_d \subset V_{d+1}$.



It remains to prove the following claim.

Claim. The sets $U := \bigcup_{d \geq n} U_d$ and $V := \bigcup_{d \geq n} V_d$ are disjoint open neighborhoods of P and Q in X.

Proof. Note that for each $d \geq n$ we have $(U \cap X^d) \cap (V \cap X^d) = U^d \cap V^d = \varnothing$. This shows that U and V are disjoint. It follows immediately from the Direct Limit–Open Subset Lemma 25.5 that U and V are open subsets of X. ■

Definition. Let X be a CW-complex

(1) Given $n \in \mathbb{N}_0$ we denote by $\mathrm{Cell}_n(X)$ the set of n-cells of X. Furthermore we denote by $\mathrm{Cell}(X) := \bigcup_{n \in \mathbb{N}_0} \mathrm{Cell}_n(X)$ the set of all cells of X.

(2) Let $c \in \mathrm{Cell}_n(X)$ be an n-cell of X.

  (a) An attaching map for c is a continuous map $\varphi \colon S^{n-1} \to X^{n-1}$ such that there exists a homeomorphism

$$X^{n-1} \cup_{X^{n-1} \hookleftarrow S^{n-1} \,:\varphi} \overline{B}^n \;\to\; \underbrace{X^{n-1} \cup c}_{\text{subspace of } X^n}$$

  which is the identity on X^{n-1}.

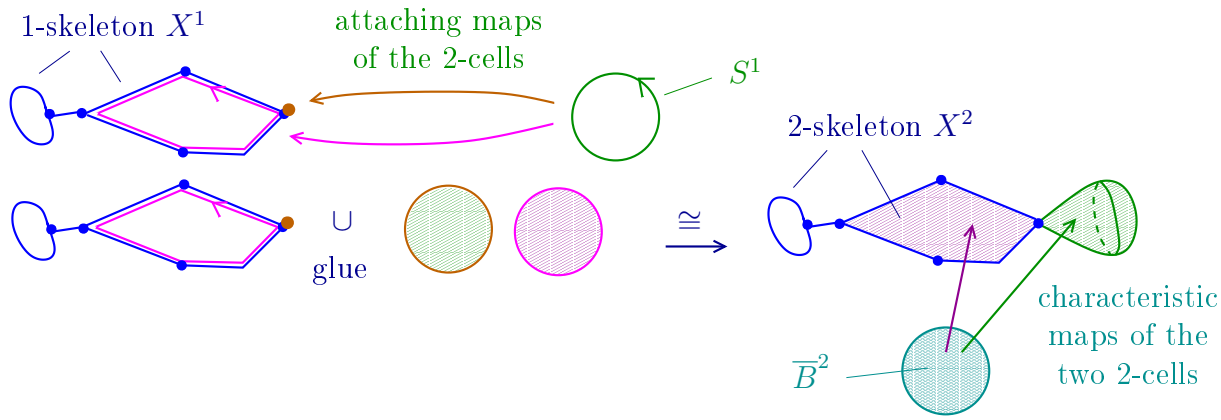
  (b) A characteristic map for c is a continuous map $\Phi \colon \overline{B}^n \to X$ such that the map

$$X^{n-1} \cup_{X^{n-1} \hookleftarrow S^{n-1} \,:\Phi|_{S^{n-1}}} \overline{B}^n \;\to\; X^{n-1} \cup c$$

induced by the identity on X^{n-1} and by the map Φ on $\overline{B}^n$ is well-defined and it is a homeomorphism.

It follows fairly easily from the definitions that a characteristic map, and thus also an attaching map, always exists.

(3) A full set of attaching maps for the cells of the CW-complex X is defined as a family of continuous maps $\{\varphi_c\colon S^{\dim(c)-1} \to X^{\dim(c)-1}\}_{c\in \mathrm{Cell}(X)}$ such that each φ_c is an attaching map for the cell c. Similarly we define a full set of characteristic maps for the n-cells of X.



Proposition 30.4. (Cell-Compact Proposition) Let X be a CW-complex.

(1) Let c be an n-cell of X. If $\Phi\colon \overline{B}^n \to X$ is a characteristic map for c, then $\Phi(\overline{B}^n) = c$.

(2) Every cell is compact.

Proof. Let c be an n-cell of X.

(1) Let $\Phi\colon \overline{B}^n \to X$ be a characteristic map for c. We start out with the following claim.

Claim. The image $\Phi(\overline{B}^n)$ is a closed subset of X.

Proof. Since characteristic maps are continuous and since $\overline{B}^n$ is compact we obtain from the Compact Image Lemma 2.8 that $\Phi(\overline{B}^n)$ is compact. By the CW-Complex–Hausdorff Proposition 30.3 we know that X is Hausdorff. Thus it follows from the Compact-Closed Lemma 1.9 (2) that $\Phi(\overline{B}^n)$ is a closed subset of X. ⊞

Note that we have the following inclusions

$$\Phi(\overline{B}^n) \underset{\substack{\uparrow \\ \text{by an elementary} \\ \text{argument}}}{\subset} \underbrace{\overline{\Phi(B^n)}}_{=c=\overline{f}} \underset{\substack{\uparrow \\ \text{definition of the closure}}}{=} \bigcap \text{closed subsets containing } \Phi(B^n) \underset{\substack{\uparrow \\ \text{by the claim}}}{\subset} \Phi(\overline{B}^n).$$

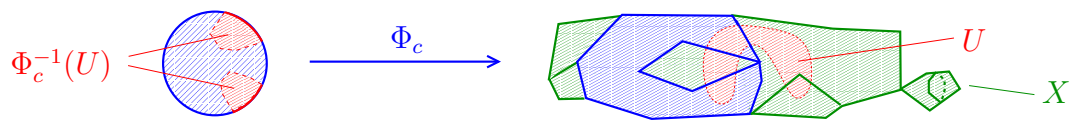
It follows that $\overline{\Phi(B^n)} = \Phi(\overline{B}^n)$. Next recall that by definition of an n-cell there exists an open n-cell $f \subset X$ with $\overline{f} = c$. It follows easily from the definition of a characteristic map that the restriction of Φ to B^n defines a homeomorphism $\Phi\colon B^n \to f$. It follows from the above that we have $c = \overline{f} = \overline{\Phi(B^n)} = \Phi(\overline{B}^n)$.

(2) This statement follows from (1) and the Compact Image Lemma 2.8. ■

Proposition 30.5. (Characteristic Maps Proposition) Let X be a CW-complex. Furthermore let $\{\Phi_c\colon \overline{B}^{\dim(c)} \to X^{\dim(c)}\}_{c\in\mathrm{Cell}(X)}$ be a full set of characteristic maps for the cells of X.

(1) Given any subset $U \subset X$ the following statements are equivalent:
   (a) U is open in X.
   (b) For every $c \in \mathrm{Cell}(X)$ the intersection $c \cap U$ is open in c.
   (c) For every $c \in \mathrm{Cell}(X)$ the preimage $\Phi_c^{-1}(U)$ is open in $\overline{B}^{\dim(c)}$.
   The same statements also hold if we replace "open" by "closed".

(2) Let $f\colon X \to Y$ be a map to a topological space Y. The following statements are equivalent:
   (a) f is continuous.
   (b) For every $c \in \mathrm{Cell}(X)$ the restriction of f to c is continuous.
   (c) For every $c \in \mathrm{Cell}(X)$ the map $f \circ \Phi_c\colon \overline{B}^{\dim(c)} \to Y$ is continuous.



Proof. Let X be a CW-complex. We write $C := \mathrm{Cell}(X)$ and given $n \in \mathbb{N}_0$ we write $C_n := \mathrm{Cell}_n(X)$. By definition of a CW-complex we know that for each $n \in \mathbb{N}_0$ there exists a family $\{\varphi_c\colon S^{n-1} \to X^{n-1}\}_{c\in C_n}$ of continuous maps and a homeomorphism

$$\Xi_n\colon X^{n-1} \cup_{X \leftarrow \bigsqcup_c S_c^{n-1} \,:\, \bigsqcup_c \varphi_c} \bigsqcup_{c\in C_n} \overline{B}_c^n \;\to\; X^n$$

which is the inclusion on X^{n-1}. We start out with the following technical claim:

Claim. For each $n \in \mathbb{N}_0$ the map

$$\Omega_n\colon X^{n-1} \cup_{X \leftarrow \bigsqcup_{c\in C} S_c^{n-1} \,:\, \bigsqcup_{c\in C} \Phi_c|_{S^{n-1}}} \bigsqcup_{c\in C} \overline{B}_c^n \;\to\; X^n$$

induced by the identity on X^{n-1} and the maps $\Phi_c\colon \overline{B}_c^n \to X^n$ is well-defined and it is a homeomorphism.

Proof. We consider the map

$$\underbrace{X^{n-1} \cup_{X \leftarrow \bigsqcup_{c\in C_n} S_c^{n-1} \,:\, \bigsqcup_{c\in C_n} \Phi_c|_{S^{n-1}}} \bigsqcup_{c\in C_n} \overline{B}_c^n}_{=:\widetilde{S}} \xrightarrow{\Omega_n} X^n \xrightarrow[\cong]{\Xi_n^{-1}} \underbrace{X^{n-1} \cup_{X \leftarrow \bigsqcup_{c\in C_n} S_c^{n-1} \,:\, \bigsqcup_{c\in C_n} \varphi_c} \bigsqcup_{c\in C_n} \overline{B}_c^n}_{=:S}.$$

Since Ξ_n is a homeomorphism it suffices to show that $\Xi_n^{-1} \circ \Omega_n\colon \widetilde{S} \to S$ is a homeomorphism.

First note that one can easily verify that $\Xi_n^{-1} \circ \Omega_n\colon \widetilde{S} \to S$ is a bijection. Since each Φ_c is continuous we obtain from the Topological-Pushout Proposition 6.12 (1b) and the Topological-Coproduct Proposition 5.8 (1b) that $\Xi_n^{-1} \circ \Omega_n\colon \widetilde{S} \to S$ is continuous.

Given $c \in C_n$ we denote by $\widetilde{c} \subset \widetilde{S}$ the image of $\overline{B}_c^n$ on the left hand side. Note that $\Xi_n^{-1} \circ \Omega_n\colon \widetilde{c} \to c$ is a continuous bijection. By the Compact Image Lemma 2.8 we know that $\widetilde{c}$ is compact and by the CW-Complex–Hausdorff Proposition 30.3 we know that c is Hausdorff. Therefore it follows from the Compact-Hausdorff Proposition 2.10 (3) that

$\Xi_n^{-1} \circ \Omega_n \colon \widetilde{c} \to c$ is a homeomorphism. We denote by $g_c \colon c \to \widetilde{c}$ its inverse. Furthermore we denote by $p_c \colon \overline{B}_c^n \to S$ the natural map. Since each $g_c \circ p_c$ is continuous we obtain from the Topological-Pushout Proposition 6.12 (1b) and the Topological-Coproduct Proposition 5.8 (1b) that $\Omega_n^{-1} \circ \Xi_n \colon S \to \widetilde{S}$ is continuous. ⊞

Now we turn to the actual proof of the statements of the proposition.

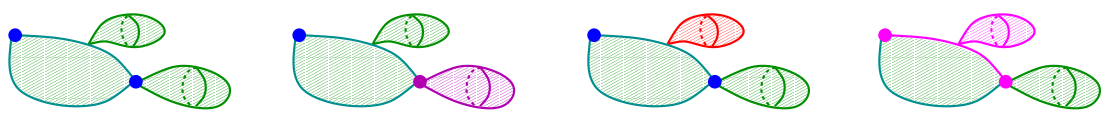
(1) Let $U \subset X$ be a subset. The (a)⇒(b)-direction follows from the definition of the subspace topology and the (b)⇒(c)-direction follows from the continuity of the attaching maps. It remains to deal with the (c)⇒(a)-direction. Thus we assume that U satisfies (c). It follows from the claim and the Topological-Pushout Proposition 6.12 (0) and the Topological-Coproduct Proposition 5.8 (0), applied iteratively for $n = 0, 1, 2, \ldots$, that each $X^n \cap U$ is open. But by definition of a CW-complex this means that U is open in X.

(2) This statement follows immediately from (1). ■

30.3. **Subcomplexes.** We recall the following definition from page 263.

Definition. We say that a CW-complex A is a subcomplex of a CW-complex X if $A \subset X$ and if the following two conditions are satisfied:

(1) For each $n \in \mathbb{N}_0$ we have $A^n = A \cap X^n$.
(2) Each open n-cell of A is also an open n-cell of X.



CW-complex X | subcomplex | not a subcomplex | subcomplex

Example. If X is a CW-complex, then any skeleton X^n, with the obvious filtration, is basically by definition a subcomplex of X.

Lemma 30.6. (Subcomplex–Closed Lemma) Each subcomplex of a CW-complex X is a closed subset of X.

Proof. We provide the somewhat technical proof in [Fri23]. ■

Proposition 30.7. (Subcomplex–Characterization Proposition) Let X be a CW-complex.

(1) Every subcomplex of X is the union of open cells of X.
(2) Let $\{\sigma_i\}_{i \in I}$ be a family of open cells of X. We set $A := \bigcup_{i \in I} \sigma_i$. The following statements are equivalent:
  (a) A is a subcomplex.
  (b) A is a closed subset of X.
  (c) For every $i \in I$ the closure $\overline{\sigma}_i$ of $\sigma_i \subset X$ is contained in A.

Proof. Also in this case we outsource the rather technical proof to [Fri23]. ■

Lemma 30.8. (Subcomplex–Intersection-Union Lemma) Let X be a CW-complex. The union and also the intersection of arbitrarily many subcomplexes of X is again a subcomplex of X.

Proof. Let X be a CW-complex and let $\{A_k\}_{k\in K}$ be a family of subcomplexes of X. For each $k \in K$ let $\{\sigma_i\}_{i\in I_K}$ be the corresponding family of open cells of X.

(1) It follows easily from the Subcomplex–Characterization Proposition 30.7 (1), together with the (a)⇔(b)-direction of the Subcomplex–Characterization Proposition 30.7 (2) that $\bigcup\limits_{k\in K} A_k = \bigcup\limits_{k\in K}\bigcup\limits_{i\in I_K} \sigma_i$ is a subcomplex of X.

(2) It follows from the Subcomplex–Characterization Proposition 30.7 (1), together with the (a)⇔(c)-direction of the Subcomplex–Characterization Proposition 30.7 (2) and the fact that the intersection of closed sets is always closed that $\bigcap\limits_{k\in K} A_k = \bigcup\limits_{i\in \bigcap\limits_{k\in K} I_K} \sigma_i$ is a subcomplex of X. ■

Lemma 30.9. (CW-Complex Pasting Lemma) Let X be a CW-complex and let $\{X_i\}_{i\in I}$ be a family of subcomplexes such that $\bigcup\limits_{i\in I} X_i = X$. The following three statements hold:

(1) (a) A subset $U \subset X$ is open if and only if each $X_i \cap U$ is an open subset of X_i.
(b) A subset $A \subset X$ is closed if and only if each $X_i \cap A$ is a closed subset of X_i.

(2) A map $f\colon X \to Y$ to a topological space Y is continuous if and only if each map $f|_{X_i}\colon X_i \to Y$ is continuous.

Proof.

(1) (a) Given any subset $U \subset X$ we have

$$U \subset X \text{ is open } \overset{\substack{\text{by the Characteristic Maps}\\ \text{Proposition 30.5 (1)}}}{\Leftrightarrow} \forall_{c\in \mathrm{Cell}(X)}\ c\cap U \text{ is open in } c \overset{\substack{\text{follows from } \cup X_i = X \text{ and the observation that}\\ \text{it is a consequence of the Subcomplex–Characterization}\\ \text{Proposition 30.7 (a)}\Rightarrow\text{(c) that every cell of some } X_i \text{ is a cell of } X}}{\Leftrightarrow} \forall_{i\in I}\ \forall_{c\in \mathrm{Cell}(X_i)}\ c\cap U \text{ is open in } c$$
$$\underset{\substack{\text{follows from the Characteristic Maps Proposition 30.5 (1) applied to the CW-complex } X_i}}{\Leftrightarrow} \forall_{i\in I}\ X_i \cap U \text{ is open in } X_i.$$

(b) The same logic as in (a) also applies for "closed" instead of "open".
This statement is an easy consequence of the Subcomplex–Characterization Proposition 30.7 (a)⇒(c) and the Characteristic Maps Proposition 30.5 (1).

(2) The "only if"-direction is an immediate consequence of the Basics-of-Continuity Lemma 2.1 (3). The "if"-direction follows easily from (1a). ■

30.4. The Finiteness Theorem for CW-complexes.

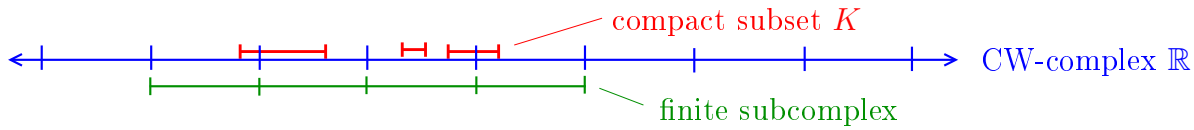
Theorem 30.10. (CW-Complex-Finiteness Theorem) Every compact (connected) subset of a CW-complex X is contained in a finite (connected) subcomplex.

Example.

(1) We consider the wedge $X := \bigvee\limits_{i\in\mathbb{N}_0} S^1_i$. As discussed on page 262, we can view X as a CW-complex with one 0-cell, given by the wedge point, and 1-cells indexed by $c \in \mathbb{N}_0$.

It follows from the CW-Complex-Finiteness Theorem 30.10 that each compact subset of X is contained in the union of finitely many S^1_i.

(2) We consider the real line $\mathbb{R}$ with the CW-structure $\varnothing \subset \mathbb{Z} \subset \mathbb{R}$. In this context the CW-Complex-Finiteness Theorem 30.10 gives us the familiar statement that any compact subset of $\mathbb{R}$ is bounded.



Corollary 30.11. (CW-Complex Compactness Corollary) A CW-complex is compact if and only if it is finite.

Proof. A finite CW-complex is by definition the union of finitely many cells. Thus it follows from the Cell-Compact Proposition 30.4 (2) that a finite CW-complex is compact.

Conversely let X be a compact CW-complex. It follows from the CW-Complex-Finiteness Theorem 30.10 that X is contained in a finite subcomplex Y of X. But from $X \subset Y \subset X$ it follows that $X = Y$, i.e. X is a finite CW-complex. ■

Lemma 30.12. Let X be a CW-complex.

(1) Let $A \subset X$ be a subset. If every open cell of X contains at most one point of A, then A is closed and discrete.

(2) If $C \subset X$ is a compact subset, then C is contained in a finite union of open cells.

(3) Every cell of X is contained in a finite subcomplex of X.

Proof of Lemma 30.12. We start out with the following claim.

Claim. For any CW-complex Y the statements (1), (2) and (3) of the lemma are equivalent.

Proof. Let Y be a CW-complex.

(1)⇒(2). Let $C \subset Y$ be a compact subset. For each open cell σ that intersects C we pick a single point in $A \cap \sigma$. We denote the resulting set of points by A. By (1) the set A is closed and discrete. Since A is a closed subset of the compact set C we obtain from the Compact-Closed Lemma 1.9 (1) that A is compact. Therefore A is in fact finite. But that means that C intersects only finitely many cells, i.e. C is contained in a finite union of open cells.

(2)⇒(3). This direction follows from the following stronger subclaim.

Subclaim. For every $n \in \mathbb{N}_0$ the following holds: If Y^{n-1} satisfies (2), then Y^n satisfies (3).

Proof. We prove the claim by induction on $n \in \mathbb{N}_0$. For $n = 0$ there is nothing really to show. Now suppose that the statement holds for some $n \in \mathbb{N}_0$. Let $C_\sigma \subset Y$ be an $(n+1)$-cell. We denote by U_σ the corresponding open cell. Note that by the Cell-Compact Proposition 30.4 (2) we know that C_σ is compact. Furthermore note that it follows from the discussion on page **??** that U_σ is an open subset of Y^{n+1}. In particular we see that U_σ is an open subset of C_σ. Finally note that it follows from these observations and the Compact-Closed Lemma 1.9 that $C_\sigma \setminus U_\sigma$ is a compact subset of the n-skeleton Y^n.

Since we assume that (2) holds for n it follows that the subset $C_\sigma \setminus U_\sigma$ is contained in a finite union of open cells of dimension $\leq n$. By induction each of these open cells is contained

in a finite subcomplex of A. By the Subcomplex–Intersection-Union Lemma 30.8 the union of these finitely many finite subcomplexes forms a subcomplex of Y. This subcomplex is clearly finite and it contains C_σ. □

(3)⇒(1). We start out with the following subclaim.

Subclaim. If $B \subset Y$ is a subset such that no two points lie in the same open cell, then B is a closed subset.

Proof. Let $B \subset Y$ be a subset such that no two points lie in the same open cell. We need to show that B is a closed subset of X. By the Characteristic Maps Proposition 30.5 it suffices to show that the intersection of B with each cell c is closed. Thus let c be a cell of X. By (3) we know that c is contained in a finite subcomplex Y'. Since B has at most one point in common with any open cell we see that the intersection $B \cap Y'$ must be finite. Thus the intersection $B \cap c$ is finite. It follows from the CW-Complex–Hausdorff Proposition 30.3 that $B \cap c$ is a closed subset of c. □

Now let $A \subset Y$ be a subset such that no two points lie in the same open cell. By the subclaim we know that A is a closed subset of Y. We still need to show that A is discrete. Note that for any point $x \in A$ the subset set $A \setminus \{x\}$ also satisfies the hypothesis of the subclaim. Thus $A \setminus \{x\}$ is a closed subset of X. Hence $\{x\}$ is open in A, so A is discrete. ⊞

Finally we turn to the actual proof of the lemma. Let X be a CW-complex. We need to show that for X all three statements hold. First note that Statement (1) holds for X^0 by definition of the topology on X^0. Now suppose that for some $n \in \mathbb{N}_0$ we have (1) for the n-skeleton X^n By claim we know that (2) and (3) also hold for X^n. By the subclaim in the proof of (2)⇒(3) we know that (3) actually holds for X^{n+1}. Thus we also have (1) for X^{n+1}. Summarizing we have all three statements for X^n for all $n \in \mathbb{N}_0$. But any cell lies in some X^n, and so we know (3) for X itself, and by the claim this implies that (1) and (2) also hold for X. ■

Proof of the CW-Complex-Finiteness Theorem 30.10. Let X be a CW-complex and let $C \subset X$ be a compact subset. By Lemma 30.12 (2) the subset C is contained in a union of finitely many open cells. By Lemma 30.12 (3) each of these open cells is contained in a finite subcomplex. The union of these finitely many finite subcomplexes is a finite subcomplex Z which contains C.

If C is connected, then we take the component of Z that contains C. ■

30.5. Fundamental groups of CW-complexes.

Lemma 30.13. (CW-Skeleton-π_0-Lemma) Let X be a CW-complex.

(1) The inclusion induced map $\pi_0(X^0) \to \pi_0(X)$ is a surjection.
(2) For every $k \geq 1$ the inclusion induced map $\pi_0(X^1) \to \pi_0(X)$ is a bijection.

Proof. We leave the proof as an exercise to the reader. ■

Example. We equip S^∞, $\mathbb{R}\mathrm{P}^\infty$ and $\mathbb{C}\mathrm{P}^\infty$ with the usual CW-structures. Their 1-skeleta are $S^1, \mathbb{R}\mathrm{P}^1$ and $\mathbb{C}\mathrm{P}^1$. Since these are path-connected we obtain from the CW-Skeleton-π_0-Lemma 30.13 that S^∞, $\mathbb{R}\mathrm{P}^\infty$ and $\mathbb{C}\mathrm{P}^\infty$ are path-connected.

Proposition 30.14. (CW-Skeleton-π_1-Proposition) Let X be a connected CW-complex and let x_0 be a point in the 0-skeleton of X.

(1) The inclusion induced map $\pi_1(X^1, x_0) \to \pi_1(X, x_0)$ is an epimorphism.
(2) For any $k \geq 2$ the inclusion induced map $\pi_1(X^k, x_0) \to \pi_1(X, x_0)$ is an isomorphism.

Example. It follows from the CW-Skeleton-π_1-Proposition 30.14 together with some of our earlier calculations that

$$\begin{array}{lclclcl} \pi_1(S^\infty) & \cong & \pi_1(\text{2-skeleton of } S^\infty) & \cong & \pi_1(S^2) & = & 0, \\ \pi_1(\mathbb{R}\mathrm{P}^\infty) & \cong & \pi_1(\text{2-skeleton of } \mathbb{R}\mathrm{P}^\infty) & \cong & \pi_1(\mathbb{R}\mathrm{P}^2) & = & \mathbb{Z}_2, \\ \pi_1(\mathbb{C}\mathrm{P}^\infty) & \cong & \pi_1(\text{2-skeleton of } \mathbb{C}\mathrm{P}^\infty) & \cong & \pi_1(\mathbb{C}\mathrm{P}^1) & = & 0. \end{array}$$

Proof.

(1) Let $\gamma\colon [0,1] \to X$ be a loop in (X, x_0). It follows from the Compact Image Lemma 2.8 and the CW-Complex-Finiteness Theorem 30.10 that there exists a finite connected subcomplex A of X with $\gamma([0,1]) \subset A$. We consider the following commutative diagram

$$\begin{tikzcd} \pi_1(A^1, x_0) \arrow[r] \arrow[d, two heads] & \pi_1(X^1, x_0) \arrow[d] \\ \pi_1(A, x_0) \arrow[r] & \pi_1(X, x_0). \end{tikzcd}$$

Since A is a finite CW-complex we see that A is obtained from its 1-skeleton A^1 by attaching finitely many cells of dimension ≥ 2. We obtain from iteratively applying the Cell Attachment-π_1-Proposition 29.1 that the left vertical map is an epimorphism. By choice of A we see that $[\gamma]$ lies in the image of the bottom horizontal map. Since the diagram commutes we see that $[\gamma]$ lies in the image of the right vertical map.

(2) The proof of this statement is a variation on the argument of (1) and it follows from the same logic as in the proof of the Direct Limit-π_1-Proposition 26.1. We leave it to the reader to fill in the details. ■

Proposition 30.15. (CW-π_1-Finite Presentation–Proposition) Let X be a connected CW-complex and let $x_0 \in X$.

(1) If there are only finitely many cells in dimensions 0 and 1, then $\pi_1(X, x_0)$ is finitely generated.
(2) If there are only finitely many cells in dimensions 0, 1 and 2, then $\pi_1(X, x_0)$ is finitely presented.

Proof. This proposition follows easily from the CW-Skeleton-π_1-Proposition 30.14 together with the Cell Attachment-π_1-Proposition 29.1. ■

30.6. **CW-complexes and direct limits.** The following propositions give a convenient variation on the Exhaustion-π_1-Proposition 26.2.

Proposition 30.16. (CW-Direct Limit-π_n-Proposition) Let X be a CW-complex and let $X_1 \subset X_2 \subset X_3 \subset \dots$ be a sequence of subcomplexes such that $X = \bigcup_{i \in \mathbb{N}} X_i$.

(1) We have $X = \varinjlim X_i$.

(2) For any $n \in \mathbb{N}_0$ and any $w \in X_1$ we have a natural isomorphism

$$\varinjlim \pi_n(X_i, w) \xrightarrow{\cong} \pi_n\big(\underbrace{\varinjlim X_i}_{=X}, w\big).$$

Sketch of proof.

(1) This statement follows immediately from the characterization of the direct limit $\varinjlim X_i$ in the Topological-Direct Limit Proposition 25.4 together with the CW-Complex Pasting Lemma 30.9.

(2) The proof of this proposition is basically identical to the proof of the Exhaustion-π_1-Proposition 26.2. We just need to replace the Compact Subspace–Exhaustion Lemma 20.11 by the CW-Complex-Finiteness Theorem 30.10. ■

Examples.

(1) For ever $n \in \mathbb{N}_0$ we have

$$\pi_n(S^\infty) \underset{\substack{\uparrow \\ \text{by the Direct Limit-}\pi_n \\ \text{Proposition 30.16}}}{\cong} \varinjlim \pi_n(S^i) \underset{\substack{\uparrow \\ \text{since by the Sphere-Low-}\pi_n \text{ Proposition 28.8} \\ \text{we have } \pi_n(S^i) = 0 \text{ for } i > n}}{=} \varinjlim (\,? \to \ldots \to ? \to \overset{\substack{\pi_n(S^{n+1}) \\ \downarrow}}{0} \to 0 \to \ldots) \underset{\substack{\uparrow \\ \text{by the CW-Direct Limit} \\ \text{Stabilization Lemma 25.1}}}{\cong} 0.$$

(2) Recall that by the Infinite Projective Space-Sphere Quotient Lemma 25.8 we know that the group $\{\pm 1\}$ acts continuously and discretely on S^∞ and that $S^\infty/\{\pm 1\}$ is homeomorphic to $\mathbb{R}\mathrm{P}^\infty$. Therefore it follows from (1) and the Action-π_1–Theorem 15.1 that $\pi_1(\mathbb{R}\mathrm{P}^\infty) \cong \pi_1(S^\infty/\{\pm 1\}) \cong \mathbb{Z}_2$. Furthermore it follows from (1), the Action–Covering Proposition 14.5 and the Covering-π_n-Isomorphism Proposition 28.10 that for every $n \in \mathbb{N}_{\geq 2}$ we have $\pi_n(\mathbb{R}\mathrm{P}^\infty) \cong \pi_n(S^\infty) = 0$.

31. Homotopies on CW-complexes

31.1. The topology of the product of a CW-complex with an interval.

Proposition 31.1. (CW×Intervall Proposition) Let X be a CW-complex.

(1) The following two topologies on $X \times [0,1]$ agree:
  (a) The product topology on the product $X \times [0,1]$.
  (b) The topology on $X \times [0,1]$ viewed as the direct limit $\varinjlim \underbrace{(X^k \times [0,1])}_{\text{product topology}}$.

(2) Let $f\colon X \times [0,1] \to Y$ be a map to a topological space Y. The following statements are equivalent:
  (a) The map f is continuous where $X \times [0,1]$ is equipped with the product topology.
  (b) For every $k \in \mathbb{N}_0$ the restriction of f to $X^k \times [0,1]$ is continuous.

Proof. First we show that any subset of $X \times [0,1]$ that is open with respect to (a) is also open with respect to (b). Thus let $U \subset X \times [0,1]$ be a subset that is open with respect to the product topology on $X \times [0,1]$. We need to show that U is open in $\varinjlim (X^k \times [0,1])$. By definition of the direct limit topology this means that we need to show that for each $k \in \mathbb{N}_0$ the intersection $U \cap (X^k \times [0,1])$ is open in $X^k \times [0,1]$ where we equip $X^k \times [0,1]$ with the product topology. By the Product Topology-Basics Lemma 5.3 (1a) we know that the subspace topology is compatible with taking products. In our context this implies that the product topology on $X^k \times [0,1]$ conveniently enough agrees with the subspace topology coming from $X \times [0,1]$. It follows from this discussion that $U \cap (X^k \times [0,1])$ is indeed open in $X^k \times [0,1]$.

The converse, that every subset of $X \times [0,1]$ that is open with respect to (a) is also open with respect to (b) is much more difficult and technical. It is not a particularly interesting task to read the proof, but it is good to know that this direction is non-trivial and that it requires some thought. In this spirit we outsource the proof to [Fri23]. ■

Lemma 31.2. (Infinite Sphere–Contractibility Lemma) The infinite-dimensional sphere S^∞ is contractible.

Proof. We want to show that S^∞ is contractible. It suffices to show that there exists a homotopy H from id_{S^∞} to a constant map. Our goal is to find such H. We set

$$X \;:=\; \{(0, x_2, x_3, \dots) \in \mathbb{R}^\infty \mid x_2, x_3, \dots \in \mathbb{R}\}.$$

Next we consider the maps

$$\begin{aligned} F\colon \mathbb{R}^\infty \times [0,1] &\to \mathbb{R}^\infty \\ (P,t) &\mapsto \underbrace{P\cdot(1-t) + (0,P)\cdot t}_{\neq 0 \text{ if } P \neq 0} \end{aligned} \quad \text{and} \quad \begin{aligned} G\colon X \times [0,1] &\to \mathbb{R}^\infty \\ (Q,t) &\mapsto \underbrace{Q\cdot(1-t) + (1,0,\dots)\cdot t}_{\neq 0 \text{ if } Q \neq 0}. \end{aligned}$$

Note that F_1 takes values in X and note that G_1 is a constant map. For non-zero $P \in \mathbb{R}^\infty$ and $P \in X$ we define $\widetilde{F}(P,t) \in S^\infty$ and $\widetilde{G}(P,t) \in S^\infty$ to be the normalizations of $F(P,t)$

and $G(P,t)$ to length one. Next we consider the map

$$\begin{array}{rcl} H\colon S^\infty \times [0,1] & \to & S^\infty \\ (P,t) & \mapsto & \begin{cases} \widetilde{F}(P,2t), & \text{if } t \in [0,\frac{1}{2}], \\ \widetilde{G}(\underbrace{\widetilde{F}(P,1)}_{\in X}, 2t-1), & \text{if } t \in [\frac{1}{2},1]. \end{cases} \end{array}$$

Clearly $H_0 = \mathrm{id}$ and it is straightforward to see that H_1 is a constant map. It is tempting to conclude that we are done. But in fact we still need to show the following claim.

Claim. The map H is continuous.

Proof. The proof proceeds in several steps:

(1) On page 263 we saw that we can equip S^∞ with a CW-structure where each i-skeleton is given by $S^i \subset S^\infty$.

(2) By (1) and by the CW×Intervall Proposition 31.1 (2) it suffices to that show for each $n \in \mathbb{N}$ the map

$$\begin{array}{rcl} H\colon S^n \times [0,1] & \to & S^\infty \\ (x,t) & \mapsto & H((x,0),t) \end{array}$$

is continuous.

(3) Let $n \in \mathbb{N}_0$. Note that $H(S^n \times [0,1]) \subset S^{n+1}$. Thus we obtain the following commutative diagram

$$\begin{tikzcd} S^n \times [0,1] \arrow[r, "H"] \arrow[dr, "H"'] & S^\infty \\ & S^{n+1} \arrow[u, hook] \end{tikzcd} .$$

By the Topological-Direct Limit Proposition 25.4 (1a) we know that the vertical map is a continuous. It follows from (2) and this discussion that it suffices to show that for each $n \in \mathbb{N}_0$ the diagonal map is a continuous.

(4) By (3) and by the Homotopy Combination Lemma 8.2 (1) it suffices to show that for each $n \in \mathbb{N}_0$ the restriction of $H\colon S^n \times [0,1] \to S^{n+1}$ to the subspaces $S^n \times [0,\frac{1}{2}]$ and $S^n \times [\frac{1}{2},1]$ are continuous. But now we are in familiar territory and the continuity is a consequence of elementary arguments. ■

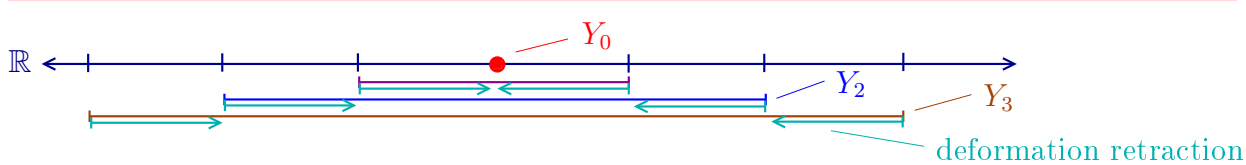
31.2. Trees.

Proposition 31.3. (Iterated Deformation Retract Proposition) Let X be a CW-complex and let $Y_0 \subset Y_1 \subset Y_2 \subset \dots$ be a sequence of subcomplexes of X such that the following two conditions are satisfied:

(1) we have $X = \bigcup_{k\in\mathbb{N}} Y_k$,

(2) each Y_k is a deformation retract of Y_{k+1}.

Then Y_0 is a deformation retract of X.

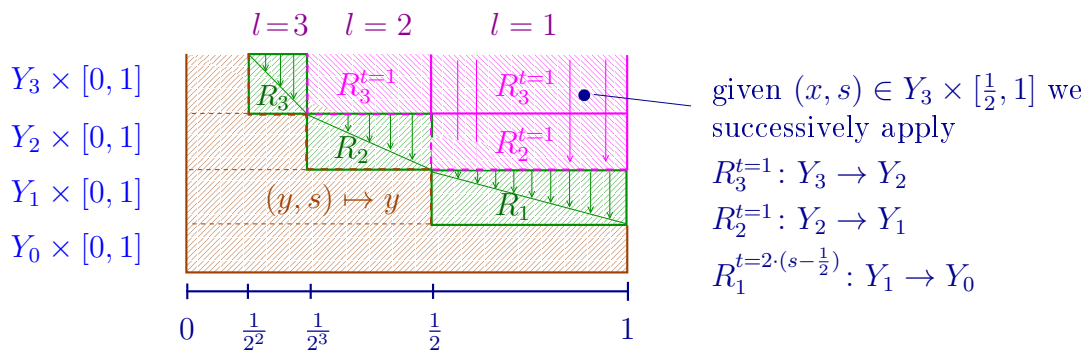


Proof. By hypothesis there exists for each $k \in \mathbb{N}_0$ a deformation retraction

$$R_k \colon Y_k \times [0,1] \;\to\; Y_k$$

from Y_k to Y_{k-1}. We consider the map

$$\begin{aligned} F \colon X \times [0,1] \;&\to\; X \\ (x,s) \;&\mapsto\; \begin{cases} x, & \text{if } x \in Y_k \text{ and } s \in [0, \frac{1}{2^k}], \\ \underbrace{(R_l^{t=2^l\cdot(s-\frac{1}{2^l})} \circ R_{l+1}^{t=1} \circ \ldots \circ R_k^{t=1})(x)}_{\in Y_{l-1}}, & \text{if } x \in Y_k \text{ and } s \in [\frac{1}{2^l}, \frac{1}{2^{l-1}}] \text{ with } 1 \le l \le k. \end{cases} \end{aligned}$$



One can easily verify that this map is well-defined. It follows from the Homotopy Combination Lemma 8.2 (1) that the restriction of F to each $Y_k \times [0,1]$ is continuous. Thus it follows from the CW×Intervall Proposition 31.1 (2) that $F \colon X \times [0,1] \to X$ is continuous. This map is the desired deformation retraction from X to Y_0. ■

Definition. A simply connected CW-complex of dimension ≤ 1 is called a tree.

Proposition 31.4. (Tree–Contractible Proposition) Let T be a non-empty connected CW-complex of dimension ≤ 1. The following statements are equivalent:

(1) T is a tree.
(2) Each 0-cell is a deformation retract of T.
(3) T is contractible.

Proof. Let T be a non-empty connected CW-complex of dimension ≤ 1. The (2)⇒(3)-direction follows from the Deformation Retract–Homotopy Equivalence Lemma 9.3 and the (3)⇒(1)-direction follows from the Homotopy-π_1-Proposition 13.4.

Thus at this stage it remains to prove the (1)⇒(2)-direction. Let $v \in T$ be a 0-cell. By the Iterated Deformation Retract Proposition 31.3 it suffices to prove the following claim:

Claim. There exists a sequence $Y_0 \subset Y_1 \subset Y_2 \subset \ldots$ of subcomplexes of T with the following properties:

(1) We have $\bigcup_{i \in \mathbb{N}_0} Y_i = T$.
(2) Each Y_i is a deformation retract of Y_{i+1}.

Proof. We construct a sequence $\{v\} = Y_0 \subset Y_1 \subset Y_2 \subset \ldots$ of subcomplexes of T as follows:

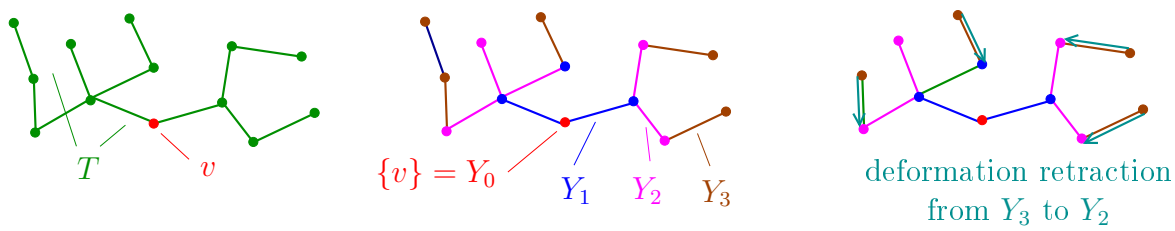
(a) We set $Y_0 := \{v\}$.

(b) Given Y_i we define Y_{i+1} to be the union of Y_i with all 1-cells which contain at least one 0-cell of Y_i.

We make the following observations:

(i) Note that every 1-cell of any CW-complex defines a subcomplex of T. Thus it follows from the Subcomplex–Intersection-Union Lemma 30.8 that Y_{i+1} is a subcomplex of T.

(ii) We set $Y := \bigcup_{i\in\mathbb{N}} Y_i$. We claim that $T = Y$. This can be seen as follows. First note that, basically by construction of Y, we have for each characteristic map $\Phi\colon \overline{B}^n \to T$, $n = 0, 1$ that $\Phi^{-1}(Y) = \varnothing$ or $\Phi^{-1}(Y) = \overline{B}^n$. Since T is 1-dimensional it follows from the Characteristic Maps Proposition 30.5 (1) that Y is a subset of T that is closed and that is open. Since T is connected and since Y is non-empty we see that $Y = T$.



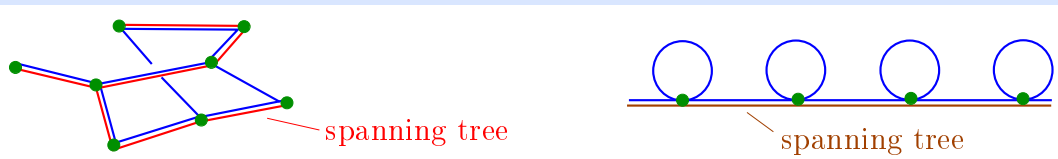
It remains to show that each Y_i is a deformation retract of Y_{i+1}. As we will see, this is almost obvious, except that we need to worry a little bit about continuity issues. Let V be the set of 0-cells in Y_{i+1} that are not contained in Y_i. Note that for each $v \in V$ the corresponding 1-cell contains two 0-cells, namely v and one 0-cell w in Y_i. Finally we consider the map

$$F\colon \Big(\overbrace{\Big(Y_i \sqcup \bigsqcup_{v\in V} [0,1]_v\Big)/v \sim (0,v)}^{=Y_{i+1}}\Big) \times [0,1] \;\to\; Y_{i+1} = \Big(Y_i \sqcup \bigsqcup_{v\in V} [0,1]_v\Big)/v \sim (0,v)$$
$$([P],t) \;\mapsto\; \begin{cases} P, & \text{if } P \in Y_i, \\ [(x\cdot(1-t),v)], & \text{if } P = (x,v) \in [0,1]_v. \end{cases}$$

It follows almost immediately from the Homotopy Pushout Lemma from [Fri23] that the map F is continuous. It is now basically clear that F defines a deformation retraction from Y_{i+1} to Y_i. ■

31.3. The fundamental group of 1-dimensional CW-complexes.

Definition. Let X be a 1-dimensional CW-complex. A spanning tree is a subcomplex T of X that is a tree and that contains all 0-cells of X.



Theorem 31.5. (1-Complex-π_1-Theorem) Let X be a connected non-empty 1-dimensional CW-complex X.

(1) X admits a spanning tree.

Let T be a spanning tree for X and let I be the set of 1-cells of X that are not contained in T.

(2) (a) X is homotopy equivalent to $\bigvee_{i\in I} S^1$.

(b) Given any $x_0 \in X$ the group $\pi_1(X, x_0)$ is isomorphic to the free group on the set I.

(3) If the CW-complex X is finite, then we have $\#I = 1 - \chi(X)$, where the Euler characteristic $\chi(X)$ of X is defined as $\chi(X) := \#\{\text{0-cells}\} - \#\{\text{1-cells}\}$.

Proposition 31.6. (1-Complex-mod-Tree Proposition) Let X be a 1-dimensional CW-complex. For every subtree T of X the projection $X \to X/T$ is a homotopy equivalence.

Proof of the 1-Complex-mod-Tree Proposition 31.6. The proof of the proposition is basically identical to the proof of the Graph-mod-Tree Proposition 17.5. We just need to replace the Finite Tree–Contractible Proposition 17.4 by the Tree–Contractible Proposition 31.4. ■

Proof of the 1-Complex-π_1-Theorem 31.5. Let X be a connected non-empty 1-dimensional CW-complex X.

(1) It follows almost immediately from the Spanning Tree–Existence Proposition 7.6 together with the correspondence given by the Graph–CW-Complex Lemma 30.1 that X admits a spanning tree.

Let T be a spanning tree for X and let I be the set of 1-cells of X that are not contained in T.

(2) (a) By the 1-Complex-mod-Tree Proposition 31.6 we know that the projection $X \to X/T$ is a homotopy equivalence. It follows from the Abstract Graph–Quotient Lemma 7.9 that X/T admits the structure of a 1-dimensional CW-complex with a single 0-cell and one 1-cell for each $i \in I$. It follows from the discussion on page 156 that X/T is homeomorphic to the wedge $\bigvee_{i\in I} S^1$.

(b) This statement follows from (a) together with with the Homotopy-π_1-Proposition 13.4 and the Wedge-of-Circles–π_1-Lemma 20.7.

(3) This statement follows from the Graph-Homotopy Type Corollary 17.6 together with the Graph–CW-Complex Lemma 30.1. ■

31.4. Local properties of CW-complexes.

Proposition 31.7. (CW-Complex–Local Properties Proposition)

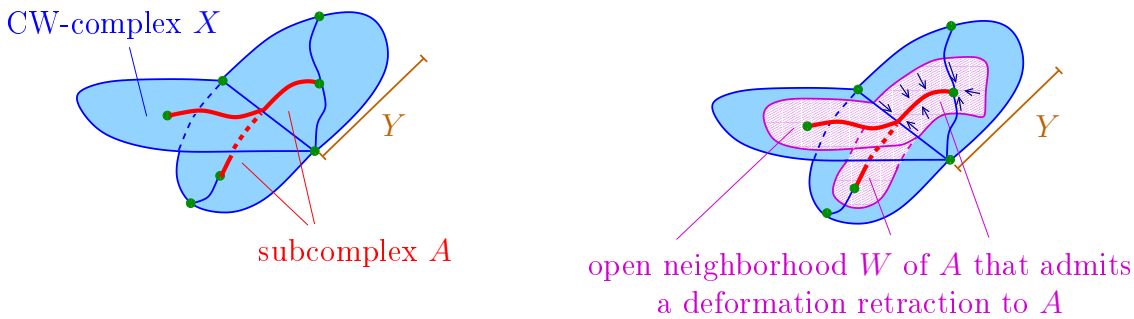
(1) Let X be a CW-complex, let $x \in X$ and let U be an open neighborhood of x. There exists an open neighborhood V of x that is contained in U and that admits a deformation retraction to $\{x\}$.

(2) CW-complexes are locally contractible.

Proof. The proof of this proposition is technically fairly demanding and it also rests on the CW×Intervall Proposition 31.1 (2). Since the proof, despite or because of the difficulty, is not very interesting to read we hide it in [Fri23]. ■

Proposition 31.8. (Subcomplex–Neighborhood Proposition) Let X be a CW-complex.

(1) For each subcomplex A of X there exists an open neighborhood W of A and a deformation retraction $F\colon W \times [0,1] \to W$ from W to A.
(2) One can choose the deformation retraction such that for each subcomplex Y of X the map F restricts to a deformation retraction $F\colon (Y \cap W) \times [0,1] \to Y \cap W$ from $Y \cap W$ to $Y \cap A$.



Proof. The proof of this proposition is similar to the proof of the CW-Complex–Local Properties Proposition 31.7. Again we refer to [Fri23] for details. ■

32. Coverings of CW-complexes

32.1. Coverings of CW-complexes.

Proposition 32.1. (CW-Complex Covering Proposition) Let $p\colon \widetilde{X} \to X$ be a connected covering of a non-empty CW-complex X.

(1) The filtration

$$\varnothing = p^{-1}(X^{-1}) \subset p^{-1}(X^0) \subset p^{-1}(X^1) \subset \ldots \subset \widetilde{X}$$

defines a CW-structure for $\widetilde{X}$ which has the property that for any $k \in \mathbb{N}_0$ we have

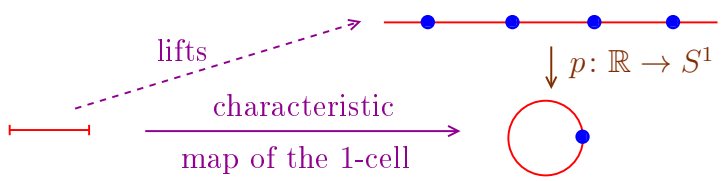
$$\#k\text{-cells of } \widetilde{X} \;=\; \left[\widetilde{X} : X\right] \cdot \#k\text{-cells of } X.$$

(2) (a) Each characteristic map $\Phi\colon \overline{B}^n \to X^n$ admits precisely $[\widetilde{X} : X]$ lifts.

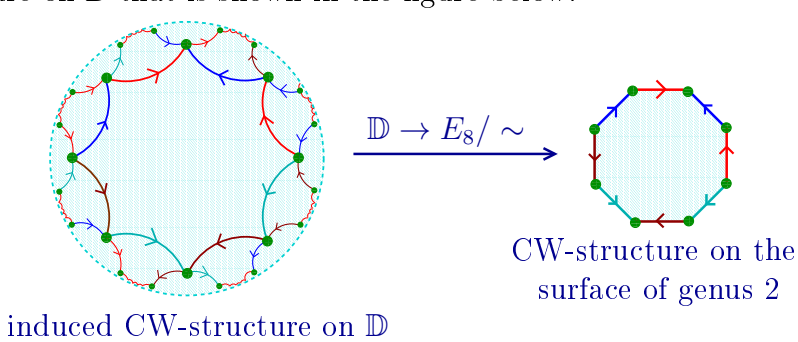
(b) Let $\{\Phi_c\colon \overline{B}^{\dim(c)} \to X\}_{c\in \mathrm{Cell}(X)}$ be a full set of characteristic maps for X. The set of all lifts of all these characteristic maps is a full set of characteristic maps for $\widetilde{X}$.

Examples.

(1) We equip S^1 with the CW-structure $\varnothing \subset \{1\} \subset S^1$ consisting of one 0-cell and one 1-cell. We denote by $p\colon \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi i t)$ the universal covering of S^1. It follows from the CW-Complex Covering Proposition 32.1, that a CW-structure on $\mathbb{R}$ is given by $\varnothing \subset p^{-1}(\{1\}) \subset S^1$, i.e. by $\varnothing \subset \mathbb{Z} \subset \mathbb{R}$.



(2) We consider $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| < 1\}$. Recall that Hadamard's Theorem 27.8 says that there exists a covering map $\mathbb{D} \to H_8/\sim \;\cong E_8/\sim$. If we equip $E_8/\sim$ with the "usual" CW-structure then we obtain from the CW-Complex Covering Proposition 32.1 the CW-structure on $\mathbb{D}$ that is shown in the figure below.



Proof. We start out with an observation. Let $\Phi\colon \overline{B}^n \to X$ be any continuous map. Since each $\overline{B}^n$ is simply connected and locally path-connected we obtain from the Map Lifting Criterion 27.1 and the Lift Uniqueness Proposition 14.7 that for each $y \in p^{-1}(\Phi(0))$ there exists a unique lift $\widetilde{\Phi}_y\colon \overline{B}^n \to \widetilde{X}$ of $\Phi\colon \overline{B}^n \to X$ with $\widetilde{\Phi}_y(0) = y$. Note that this observation already implies (2a).

We turn to the proof of (1) and (2b). We introduce the following notation:

- For $n \in \mathbb{N}_0$ we set $\widetilde{X}^n := p^{-1}(X^n)$.
- Let $\{\Phi_c \colon \overline{B}^{\dim(c)} \to X\}_{c \in \mathrm{Cell}(X)}$ be a full set of characteristic maps for the CW-complex X. By the above observation we obtain for each $c \in \mathrm{Cell}(X)$ the family of continuous maps $\{\widetilde{\Phi}_{c,y} \colon \overline{B}^{\dim(c)} \to \widetilde{X}\}_{y \in p^{-1}(\Phi_c(0))}$.

Statements (1) and (2b) follow from the following two claims.

Claim 1. A set $U \subset \widetilde{X}$ is open if and only if for each $n \in \mathbb{N}_0$ the intersection $\widetilde{X}^n \cap U$ is open in $\widetilde{X}^n$.

Claim 2. Let $n \in \mathbb{N}_0$. We consider the map

$$\Theta \colon \widehat{X}^n := \widetilde{X}^{n-1} \cup \bigcup_{c \in \mathrm{Cell}_n(X)} \bigcup_{y \in p^{-1}(\Phi_c(0))} \overline{B}^n_{c,y} \;\to\; \widetilde{X}^n$$

that is induced by the identity on $\widetilde{X}^{n-1}$ and by the maps $\widetilde{\Phi}_{c,y} \colon \overline{B}^n \to \widetilde{X}^n$. This map is a homeomorphism.

Proof of Claim 1. Since $p \colon \widetilde{X} \to X$ is a covering there exists an open cover $\{\widetilde{W}_i\}_{i \in I}$ of $\widetilde{X}$ such that for each $i \in I$ the image $p(\widetilde{W}_i) =: W_i$ is an open subset of X and that the map $\widetilde{p}_i := p|_{\widetilde{W}_i} \colon \widetilde{W}_i \to W_i$ is a homeomorphism. Now we see that

$$\begin{aligned}
& \forall_{n \in \mathbb{N}_0} \; \widetilde{X}^n \cap U \text{ is open in } \widetilde{X}^n \\
\iff & \forall_{n \in \mathbb{N}_0} \; \forall_{i \in I} \; \widetilde{W}_i \cap (\widetilde{X}^n \cap U) \text{ is open in } \widetilde{W}_i \cap \widetilde{X}^n && \text{(Cover-Open Subset Lemma 1.5)} \\
\iff & \forall_{n \in \mathbb{N}_0} \; \forall_{i \in I} \; p_i(\widetilde{W}_i \cap (\widetilde{X}^n \cap U)) \text{ is open in } p_i(\widetilde{W}_i \cap \widetilde{X}^n) && \text{(since each } p_i \text{ is a homeomorphism)} \\
\iff & \forall_{i \in I} \; \forall_{n \in \mathbb{N}_0} \; (W_i \cap p_i(U)) \cap X^n \text{ is open in } W_i \cap X^n && \text{(since } \widetilde{X}^n = p^{-1}(X^n) \text{ and } p(\widetilde{W}_i) = W_i) \\
\iff & \forall_{i \in I} \; \forall_{n \in \mathbb{N}_0} \; (W_i \cap p_i(U)) \cap X^n \text{ is open in } X^n && \text{(since } W_i \text{ is open in } X) \\
\iff & \forall_{i \in I} \; W_i \cap p_i(U) \text{ is open in } W_i && \text{(since } X \text{ is a CW-complex)} \\
\iff & \forall_{i \in I} \; \widetilde{W}_i \cap U \text{ is open in } \widetilde{W}_i && \text{(since } p_i \text{ is a homeomorphism)} \\
\iff & U \text{ is open in } \widetilde{X} && \text{(Cover-Open Subset Lemma 1.5).}
\end{aligned}$$

⊞

Proof of Claim 2. First note that one can easily verify that Θ is a bijection. Also note that it follows from the Topological-Pushout Proposition 6.12 (0) that the map Θ is continuous. To show that it is actually a homeomorphism we have to work a little harder.

To do so we consider the map

$$\widehat{p} \colon \widehat{X}^n = \widetilde{X}^{n-1} \cup \bigcup_{c \in \mathrm{Cell}_n(X)} \bigcup_{y \in p^{-1}(\Phi_c(0))} \overline{B}^n_{c,y} \;\to\; X^n$$

that is induced by the projection $p \colon \widetilde{X}^{n-1} \to X^{n-1}$ and the maps $\overline{B}^n_{c,y} \to \overline{B}^n_c \xrightarrow{\Phi_c} X^n$. It follows again from the Topological-Pushout Proposition 6.12 (0) that $\widehat{p}$ is continuous. Also note that p satisfies $p \circ \Theta = \widehat{p} \colon \widetilde{X}^n \to X^n$. It is not hard to show that $\widehat{p} \colon \widehat{X}^n \to \widetilde{X}^n$ is a covering. (We leave it to the reader to write down the details.) We now see that p and $\widehat{p}$ are

both local homeomorphisms. It follows from $p \circ \Theta = \widehat{p}$ that Θ is a local homeomorphism. Since Θ is a bijection it follows that Θ is a homeomorphism. ■

Lemma 32.2. (1-Complex-π_n-Lemma) Let X be a connected 1-dimensional CW-complex and let $x_0 \in X$ be a base point. Then $\pi_n(X, x_0) = 0$ for every $n \geq 2$.

Proof. Let X be a connected 1-dimensional CW-complex, let $x_0 \in X$ and let $n \geq 2$. By the CW-Complex–Local Properties Proposition 31.7 we know that X is locally simply connected. It follows from the Covering Existence Theorem 27.2 that there exists a universal covering $p\colon \widetilde{X} \to X$, i.e. a covering such that $\widetilde{X}$ is simply connected. We pick $\widetilde{x}_0 \in \widetilde{X}$ with $p(\widetilde{x}_0) = x_0$. We perform the following calculation:

$$\begin{array}{ccccccc}
 & \text{by the Covering-}\pi_n\text{-Isomorphism Proposition 28.10, since } n \geq 2 & & & \text{by the CW-Complex Covering Proposition 32.1} & & \\
 & \overset{\cong}{\longleftarrow} & & & & & \\
\pi_n(X, x_0) & & \pi_n(\widetilde{X}, \widetilde{x}_0) & = & \pi_n(\text{simply connected 1-dimensional CW-complex}) & & \\
 & & & = & \pi_n(\text{contractible topological space}) & = & 0. \\
 & & & \text{by the Tree–Contractible Proposition 31.4} & & \text{by the Homotopy-}\pi_n\text{-Proposition 28.6} &
\end{array}$$

■

32.2. Subgroups of finitely presented groups.

Proposition 32.3. (Finite-Index Subgroup–Presentation Proposition) Every finite-index subgroup of a finitely presented group is finitely presented.

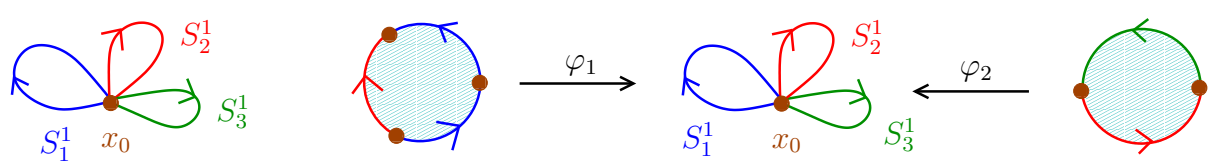
Construction. Let $\langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_m \rangle$ be a finite presentation. For $i = 1, \ldots, k$ let S^1_i be a copy of S^1. For $j = 1, \ldots, m$ we do the following:

(1) We write the relation r_j as a reduced word $g_{s_1}^{\epsilon_1} \cdot \ldots \cdot g_{s_n}^{\epsilon_n}$ with $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$.

(2) We denote by $\varphi_j\colon S^1 =_i [0,1]/0 \sim 1 \to \bigvee_{i=1}^{k} S^1_i$ the path which on the interval $[\frac{i-1}{n}, \frac{i}{n}]$ travels through $S^1_{s_i}$ at constant speed in the direction determined by ϵ_i.

We denote by Y the CW-complex that is given by attaching m 2-cells to $\bigvee_{i=1}^{k} S^1_i$ along the attaching maps $\varphi_1, \ldots, \varphi_m$. We refer to Y as the CW-complex associated to the presentation.

2-dimensional CW-complex for the presentation
$\langle g_1, g_2, g_3 \,|\, g_1 \cdot g_2^{-1} \cdot g_1, g_3 \cdot g_2 \rangle$



CW-complex is given by attaching two 2-cells to $S^1_1 \vee S^1_2 \vee S^1_3$

Lemma 32.4. (Presentation–CW Lemma) Let $\langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_m \rangle$ be a finite presentation for a group π. If Y is the associated CW-complex, then $\pi_1(Y) \cong \pi$.

Proof of the Presentation–CW Lemma 32.4. Let $\pi = \langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_m \rangle$ be a finitely presented group. We use the notation introduced in the above construction of the CW-complex associated to π. We denote by X the wedge of the k circles and we denote by

x_0 the wedge point. By the Wedge-of-Circles–π_1-Lemma 20.7 we can make the identification $\pi_1(X, x_0) = \langle g_1, \dots, g_k \rangle$. This identification has the property that for $j = 1, \dots, m$ the corresponding loop $\varphi_j \colon S^1 \to X$ represents $r_j \in \pi_1(X, x_0) = \langle g_1, \dots, g_k \rangle$. We obtain that

$$\begin{array}{rcl} \pi_1(Y, x_0) & \overset{\substack{\text{Cell Attachment-}\pi_1\text{-Proposition 29.1, applied } m \text{ times} \\ \downarrow}}{\cong} & \langle g_1, \dots, g_k \rangle / \langle\langle [\varphi_1] \rangle\rangle \dots / \langle\langle [\varphi_m] \rangle\rangle \overset{\substack{\text{since } [\varphi_j] = r_j \\ \downarrow}}{=} \langle g_1, \dots, g_k \rangle / \langle\langle r_1 \rangle\rangle \dots / \langle\langle r_m \rangle\rangle \\ & = & \langle g_1, \dots, g_k \rangle / \langle\langle r_1, \dots, r_m \rangle\rangle \underset{\substack{\uparrow \\ \text{by definition}}}{=} \langle g_1, \dots, g_k \,|\, r_1, \dots, r_m \rangle. \end{array}$$

follows from the Quotient Presentation–Lemma 21.6 ■

Proof of the Finite-Index Subgroup–Presentation Proposition 32.3. Suppose we are given a finitely presented group $\pi = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_m \rangle$. By the Presentation–CW Lemma 32.4 there exists a finite connected 2-dimensional CW-complex Y with $\pi_1(Y) \cong \pi$. Now let Γ be a finite-index subgroup of π. By the CW-Complex–Local Properties Proposition 31.7 (2) together with the Covering Existence Theorem 27.2 there exists a finite covering $p \colon \widetilde{Y} \to Y$ such that $\pi_1(\widetilde{Y}) \cong \Gamma$. By the CW-Complex Covering Proposition 32.1 (1) the 2-complex $\widetilde{Y}$ has finitely many 1-cells and 2-cells. Thus it follows from the CW-π_1-Finite Presentation–Proposition 30.15 that $\Gamma \cong \pi_1(\widetilde{Y})$ is a finitely presented group. ■

32.3. Subgroups of free groups.

Proposition 32.5. (Free Subgroup Proposition)
(1) Every subgroup of a free group (of any rank) is again a free group.
(2) If F is finitely generated and if G is a finite-index subgroup of F, then

$$\operatorname{rank}(G) - 1 \;=\; [F : G] \cdot (\operatorname{rank}(F) - 1).$$

Proof. Let $F = \langle S \rangle$ be a free group on a set S and let G be a subgroup of F.

(1) We consider the 1-dimensional CW-complex $X = \bigvee_{s \in S} S^1_s$. By the 1-Complex-$\pi_1$-Theorem 31.5 we have $\pi_1(X, P) \cong F$. By the CW-Complex–Local Properties Proposition 31.7 (2) and the Covering Existence Theorem 27.2 there exists a covering $p \colon (Y, Q) \to (X, P)$ of degree $[F : G]$ such that $p_*(\pi_1(Y, Q)) = G$.

Recall that by the Covering-π_1-Monomorphism Proposition 14.11 the induced map $p_* \colon \pi_1(Y, Q) \to \pi_1(X, P)$ is injective. Thus we see that $\pi_1(Y, Q)$ itself is isomorphic to G. By the CW-Complex Covering Proposition 32.1 we can equip Y with the structure of a 1-dimensional CW-complex. We deduce from the 1-Complex-π_1-Theorem 31.5 that $G \cong \pi_1(Y, Q)$ is a free group.

(2) Now we suppose that F is a free group of finite rank and we suppose that G is a finite-index subgroup of F. Then

$$\begin{array}{rcl} \operatorname{rank}(G) & \overset{\substack{\text{by the 1-Complex-}\pi_1\text{-Theorem 31.5} \\ \downarrow}}{=} & \#\{\text{1-cells of } Y\} - \#\{\text{0-cells of } Y\} + 1 \\ & \underset{\substack{\uparrow \\ \text{by the CW-Complex Covering Proposition 32.1}}}{=} & [F : G] \cdot (\underbrace{\#\{\text{1-cells of } X\}}_{=\operatorname{rank}(F)} - \underbrace{\#\{\text{0-cells of } X\}}_{=1}) + 1 \;=\; [F : G] \cdot (\operatorname{rank}(F) - 1) + 1. \end{array}$$

■

33. Constructions of CW-complexes

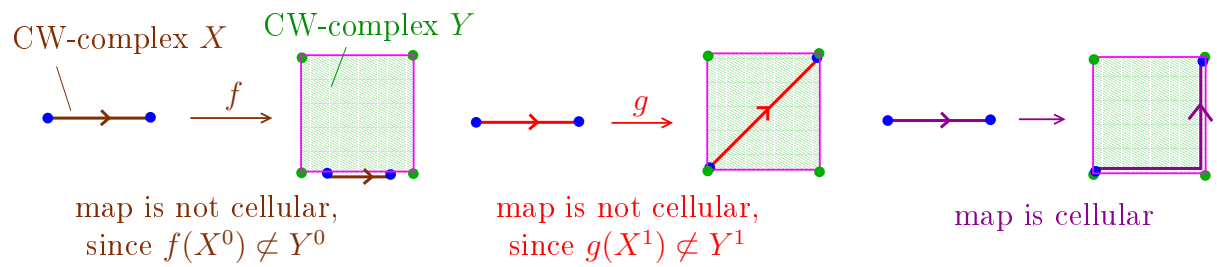
33.1. The two categories of CW-complexes.

Definition. The full category of CW-complexes is the category $\mathcal{FullCW}$ that is given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{FullCW}) &:= \text{all CW-complexes,} \\ \mathrm{Mor}(X,Y) &:= \text{all } \textit{continuous} \text{ maps from } X \text{ to } Y. \end{aligned}$$

Definition. A map $f\colon X \to Y$ between two CW-complexes is called cellular if f is continuous and if for each $n \in \mathbb{N}_0$ we have $f(X^n) \subset Y^n$.

Example. In the figure below we consider maps from the CW-complex $X = [0,1]$ to the CW-complex $Y = [0,1] \times [0,1]$, where each is equipped with the "obvious" CW-structure. The first two maps are not cellular, the one to the right is cellular.



Definition. The category of CW-complexes is the category $\mathcal{CW}$ that is given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{CW}) &:= \text{all CW-complexes,} \\ \mathrm{Mor}(X,Y) &:= \text{all } \textit{cellular} \text{ maps from } X \text{ to } Y. \end{aligned}$$

33.2. Constructions of CW-structures: Pushouts.

Lemma 33.1. (CW-Disjoint Union Proposition) Let $\{X_i\}_{i\in I}$ be a family of CW-complexes. The disjoint union $\bigsqcup_{i\in I} X_i$ admits a natural CW-structure where each n-skeleton is given by $\bigsqcup_{i\in I} X_i^n$.

(a) The natural maps $X_j \to \bigsqcup_{i\in I} X_i$ are cellular.

(b) If Z is a CW-complex and if $f_i\colon X_i \to Z$, $i \in I$ are continuous (cellular) maps, then the corresponding map $\bigsqcup_{i\in I} f_i\colon \bigsqcup_{i\in I} X_i \to Z$ is also continuous (cellular).

Proof. The proof is elementary, and the more one says, the more confusing it gets. ■

We continue with the following much more interesting proposition.

Proposition 33.2. (CW-Gluing Proposition) Let X and Y be CW-complexes, let A be a subcomplex of X and let $f\colon A \to Y$ be a cellular map. As on page 91 we consider the following topological space:

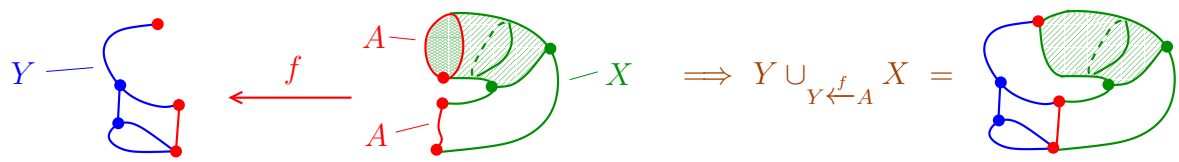
$$Y \cup_{Y \xleftarrow{f} A} X \;:=\; Y \cup_f X \;:=\; (Y \sqcup X)/\!\sim \qquad \text{where } f(a) \sim a \text{ for } a \in A$$

By the Topological-Pushout Proposition 6.12 the following diagram is a pushout diagram in the category $\mathcal{T}op$ of topological spaces:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "f"] & X \arrow[d, "x \mapsto [x]", "p"'] \\ Y \arrow[r, "y \mapsto [y]", "q"'] & Y \cup_f X. \arrow[ul, phantom, "\lrcorner"] \end{tikzcd}$$

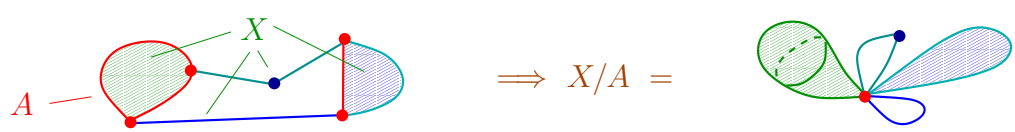
(1) The subspaces $(Y \cup_f X)^n := q(Y^n) \cup p(X^n)$ define a CW-structure on $Y \cup_f X$.

(2) If $\{\Phi_c \colon \overline{B}^{\dim(c)} \to X\}_{c \in \mathrm{Cell}(X)}$ and $\{\Phi_d \colon \overline{B}^{\dim(d)} \to Y\}_{d \in \mathrm{Cell}(Y)}$ are full sets of characteristic maps of X and Y, then $\{p \circ \Phi_c \colon \overline{B}^{\dim(c)} \to Y \cup_f X\}_{c \in \mathrm{Cell}(X) \setminus \mathrm{Cell}(A)}$ together with $\{q \circ \Phi_d \colon \overline{B}^{\dim(d)} \to Y \cup_f X\}_{d \in \mathrm{Cell}(Y)}$ is a full set of characteristic maps of $Y \cup_f X$.



Proof. The proof is somewhat lengthy, but at the end of the day there are no real technical difficulties. Thus we outsource the proof to [Fri23]. ■

Lemma 33.3. (CW-Quotient Lemma) Let X be a CW-complex and let A be a non-empty subcomplex. We denote by $p \colon X \to X/A$ the natural projection. The subsets $p(X^n)$, $n \in \mathbb{N}_0$, define a CW-structure on X/A.



Proof. On page **??** we pointed out that we have the following pushout diagram:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "x \mapsto [x]"] & X \arrow[d] \\ \{*\} \arrow[r] & X/A. \arrow[ul, phantom, "\lrcorner"] \end{tikzcd}$$

The lemma is thus an almost immediate consequence of the CW-Gluing Proposition 33.2. ■

Lemma 33.4. (CW-Wedge Lemma) Let $\{X_i\}_{i \in I}$ be a family of CW-complexes and suppose that for each $i \in I$ we are given a point $x_i \in X_i^0$. The corresponding wedge $\bigvee_{i \in I} (X_i, x_i)$, admits a natural CW-complex structure.

Proof. It follows from the CW-Disjoint Union Proposition 33.1 that $X := \bigsqcup_{i \in I} X_i$ has a natural CW-structure such that $A := \bigsqcup_{i \in I} \{x_i\}$ is a subcomplex of X. It follows from the CW-Quotient Lemma 33.3 that $X/A = \bigvee_{i \in I} (X_i, x_i)$ inherits a natural CW-structure. ■

Lemma 33.5. (Skeleton-Quotient Lemma) Let X be a CW-complex and let $n \in \mathbb{N}$. Let $\{\Phi_c \colon \overline{B}^{n+1} \to X^n\}_{c \in \mathrm{Cell}(X)}$ be a full set of characteristic maps of the $(n+1)$-cells of X. The map

$$\begin{aligned} \bigvee_{c \in \mathrm{Cell}(X)} \overbrace{(\overline{B}_c^{n+1}/S_c^n, [S_c^n])}^{\substack{\text{the pointed topological space } (\overline{B}^{n+1}/S^n, [S^n]) \text{ is} \\ \text{by the Ball Quotient-Sphere Lemma 6.5} \\ \text{homeomorphic to } (S^{n+1}, N)}} &\to X^{n+1}/X^n \\ [x] &\mapsto [\Phi_c(x)] \quad \text{if } x \in \overline{B}_c^{n+1} \end{aligned}$$

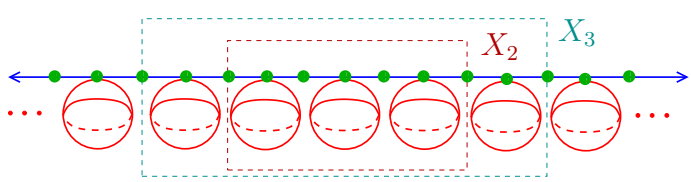
is a homeomorphism.

Proof. It is basically clear that the given map $f \colon \bigvee_{c \in \mathrm{Cell}(X)} (\overline{B}_c^{n+1}/S_c^n, [S_c^n]) \to X^{n+1}/X^n$ is a bijection. Next note that it follows immediately from the Twice Quotient Lemma 6.3, the Wedge Proposition 17.1 (1b) and the Topological-Quotient Proposition 6.1 (1) that f is continuous. To upgrade this bijective continuous map to a homeomorphism we have to work a little harder.

To do so note that by the Ball Quotient-Sphere Lemma 6.5 the pointed topological space $(\overline{B}^{n+1}/S^n, [S^n])$ is homeomorphic to (S^{n+1}, N). It follows from the discussion on page 261 that we can view $\overline{B}^{n+1}/S^n$ as a CW-complex where the point $[S^n]$ is the unique 0-cell. Therefore we can use the CW-Wedge Lemma 33.4 and the CW-Quotient Lemma 33.3 to view both sides of the Skeleton-Quotient Lemma 33.5 as CW-complexes. The map f gives a bijection between the cells of the two sides. Using the Compact-Hausdorff Proposition 2.10 (3) one can show that the restriction of f^{-1} to each cell of X^{n+1}/X^n is continuous. It follows from the Characteristic Maps Proposition 30.5 that f^{-1} is indeed continuous. ■

Lemma 33.6. (Direct Limit–CW Lemma) Let $\{f_i \colon X_i \to X_{i+1}\}_{i \in \mathbb{N}_0}$ be a sequence of cellular embeddings between CW-complexes. The direct limits $\varinjlim X_i^n$ of the skeleta of the X_i define a CW-structure on $\varinjlim X_i$.

Example. The Direct Limit–CW Lemma 33.6 allows us to view the line with infinitely many spheres attached as a CW-complex.



the topological space
$X = \bigcup_{i \in \mathbb{N}_0} X_i = \varinjlim X_i$

Proof. As so often, it is hard to tell whether a proof is elementary or not. This proof turns out is not that elementary, but it is certainly not very enlightening. Thus we hide the proof in [Fri23]. ■

33.3. Products of CW-complexes.

Definition. Let $\mathcal{C}$ be a category and let $X, Y \in \mathrm{Ob}(\mathcal{C})$ be two objects. A product of X and Y in the category $\mathcal{C}$ is some object $P \in \mathrm{Ob}(\mathcal{C})$ together with two morphisms $p \colon P \to X$ and $q \colon P \to Y$ such that the following property is satisfied: Whenever we are given two

morphisms $f\colon Z \to X$ and $g\colon Z \to Y$ there exists a unique morphism $\Phi\colon Z \to P$ such that $f = p \circ \Phi$ and $g = q \circ \Phi$. The situation is summarized in the following diagram

$$\begin{tikzcd} & & X \\ Z \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] \arrow[r, dashed, "\exists!\Phi"] & P \arrow[ur, "p"] \arrow[dr, "q"'] & \\ & & Y. \end{tikzcd}$$

By the Product Uniqueness Lemma 4.1 we know that if the product exists, then it is unique in a suitable sense.

Proposition 5.1. (Topological-Product Proposition) Let X and Y be two topological spaces. We consider the topology given by

$$\mathcal{TP} := \left\{ U \subset X \times Y \;\middle|\; \begin{array}{l} \text{for all } (x,y) \in U \text{ there exist open neighborhoods} \\ A \subset X \text{ of } x \text{ and } B \subset Y \text{ of } Y \text{ with } A \times B \subset U \end{array} \right\}.$$

(1) (a) The natural projections $X \times Y \to X$ and $X \times Y \to Y$ are continuous.
(b) Given a topological space Z and given continuous maps $f\colon Z \to X$ and $g\colon Z \to Y$ the map $(f,g)\colon Z \to (X \times Y, \mathcal{TP})$ that is given by $z \mapsto (f(z), g(z))$ is continuous.
(2) The topological space $X \times Y$ with the natural projections is a product of X and Y in the category $\mathcal{Top}$ of topological spaces.

Definition. Let X and Y be two CW-complexes.
(1) Given a p-cell c for X and given a q-cell d for Y we equip c and d with the subspace topology and we equip $c \times d$ with the product topology.
(2) We denote by $\mathcal{CWP}$ the topology on $X \times Y$ that is defined by the condition that $U \subset X \times Y$ is open if and only if each intersection $(c \times d) \cap U \subset c \times d$ is open.

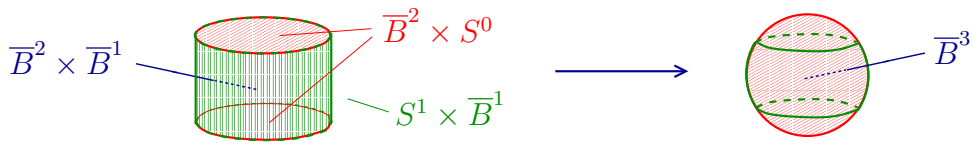
Proposition 33.7. (CW-Category–Product Proposition) Let X and Y be two CW-complexes.
(0) (a) A CW-structure for the topological space $(X \times Y, \mathcal{CWP})$ is given by the filtration

$$(X \times Y)^n := \bigcup_{\substack{p,q \in \mathbb{N}_0 \text{ with} \\ p+q=n}} (X^p \times Y^q) \qquad \text{with } n \in \mathbb{N}_0.$$

(b) The n-cells of $X \times Y$ are precisely the sets of the form $c \times d$ where c is a p-cell of X and d is an $(n-p)$-cell of Y.
(1) (a) The natural projections $X \times Y \to X$ and $X \times Y \to Y$ are cellular, in particular they are continuous with respect to $\mathcal{CWP}$.
(b) If Z is a CW-complex and if $f\colon Z \to X$ and $g\colon Z \to Y$ are continuous maps, then the map $Z \to (X \times Y, \mathcal{CWP})$ that is given by $z \mapsto (f(z), g(z))$ is also continuous.
(2) The CW-complex $(X \times Y, \mathcal{CWP})$ with the natural projections is a product of the CW-complexes X and Y in the full category $\mathcal{FullCW}$ of CW-complexes.

Lemma 33.8. (Ball-times-Ball Lemma) Let $p, q \in \mathbb{N}_0$. There exists a homeomorphism $\overline{B}^p \times \overline{B}^q \to \overline{B}^{p+q}$ that restricts to a homeomorphism $(\overline{B}^p \times S^{q-1}) \cup (S^{p-1} \times \overline{B}^q) \to S^{p+q-1}$.

Proof of the Ball-times-Ball Lemma 33.8. This lemma is proved in [Fri23]. ■



Proof of the CW–Product Proposition 33.7. Let X and Y be two CW-complexes.

(0) We start out with the following claim.

Claim 0. Let $p, q \in \mathbb{N}_0$. We view the skeleta X^p and Y^q as CW-complexes in their own right. The $\mathcal{CWP}$-topology on $X^p \times Y^q$ agrees with the subspace topology coming from $(X \times Y, \mathcal{CWP})$

Proof. We consider the inclusion map $f\colon (X^p \times Y^q, \mathcal{CWP}) \to (X \times Y, \mathcal{CWP})$. Note that it follows from the Subcomplex–Characterization Proposition 30.7 that every cell of X^p is a cell of X and that every cell of Y^q is a cell of Y. It follows basically immediately from this observation and the definition of the CW-product topology that the map $f\colon (X^p \times Y^q, \mathcal{CWP}) \to (X \times Y, \mathcal{CWP})$ is continuous. By the Open-Injective Lemma 2.9 it remains to show that f is a closed map. Let A be a subset that is closed in $(X^p \times Y^q, \mathcal{CWP})$. We need to show that A is closed in $(X \times Y, \mathcal{CWP})$. By definition of $(X \times Y, \mathcal{CWP})$ and by an elementary argument it suffices to show that for each cell c of X and each cell d of Y the intersection $(c \times d) \cap A$ is closed in $c \times d$.

Note that it follows from the Cell-Compact Proposition 30.4, the Subcomplex–Closed Lemma 30.6 and the CW-Complex-Finiteness Theorem 30.10 that there exist finitely many cells $x_1, \dots, x_r$ of X^p such that $c \cap X^p \subset x_1 \cup \dots \cup x_r$ and that there exist finitely many cells $y_1, \dots, y_s$ of X^p such that $d \cap Y^q \subset y_1 \cup \dots \cup y_s$. We now see that

$$\begin{aligned}
(c \times d) \cap A &\overset{\substack{\text{since } A \subset X^p \times Y^q \\ \downarrow}}{=} (X^p \times Y^q) \cap ((c \times d) \cap A) = ((c \cap X^p) \times (d \cap Y^q)) \cap A \\
&= ((c \cap X^p) \times (d \cap Y^q)) \cap ((x_1 \cup \dots \cup x_r) \times (y_1 \cup \dots \cup y_s)) \cap A \\
&= \underbrace{((c \cap X^p) \times (d \cap Y^q))}_{\substack{\text{closed in } c \times d \text{ since it follows from} \\ \text{the Subcomplex–Closed Lemma 30.6} \\ \text{that } X^p \subset X \text{ and } Y^q \subset Y \text{ are closed}}} \cap \left(\bigcup_{i=1}^{r} \bigcup_{j=1}^{s} \underbrace{((x_i \times y_j) \cap A)}_{\substack{\text{by hypothesis closed} \\ \text{in } x_i \times y_j\text{, thus compact,} \\ \text{thus closed in } c \times d}} \right).
\end{aligned}$$

We have thus shown that $(c \times d) \cap A$ is indeed a closed subset of $c \times d$. ⊞

Now we turn to the actual proof that the filtration $(X \times Y)^n := \bigcup_{p+q=n} (X^p \times Y^q)$ defines a CW-structure on $(X \times Y, \mathcal{CWP})$. The following two claims show that this filtration satisfies the two conditions (CW1) and (CW2) from page 260.

Claim 1. A subset $U \subset (X \times Y, \mathcal{CWP})$ is open if and only if for each $n \in \mathbb{N}_0$ the intersection $(X \times Y)^n \cap U$ is open in $(X \times Y)^n$.

Proof. Let $U \subset X \times Y$. Then:

$$\begin{aligned} U \text{ is open in } \mathcal{CWP} &\iff \underset{p\in\mathbb{N}_0}{\forall}\ \underset{q\in\mathbb{N}_0}{\forall}\ \underset{c\in\mathrm{Cell}_p(X)}{\forall}\ \underset{d\in\mathrm{Cell}_q(Y)}{\forall}\ (c\times d)\cap U \text{ is open in } c\times d \\ &\underset{\substack{\uparrow \\ \text{follows from Claim 0}}}{\iff} \underset{p\in\mathbb{N}_0}{\forall}\ \underset{q\in\mathbb{N}_0}{\forall}\ (X^p\times Y^q)\cap U \text{ is open in } X^p\times Y^q \\ &\underset{\substack{\uparrow \\ \text{follows from the Pasting Proposition 2.4 (2), since } (X\times Y)^n \text{ is the union} \\ \text{of the finitely many closed subsets } X^p\times Y^q\text{, where } p+q=n}}{\iff} \underset{n\in\mathbb{N}_0}{\forall}\ (X\times Y)^n\cap U \text{ is open in } (X\times Y)^n \end{aligned}$$

⊞

Claim 2. For each $n \in \mathbb{N}_0$ there exists a set I, a family $\{\varphi_i\colon S^{n-1} \to (X\times Y)^{n-1}\}_{i\in I}$ of continuous maps and a homeomorphism

$$(X\times Y)^{n-1} \cup_{X\times Y \leftarrow \bigsqcup_{i\in I} S_i^{n-1} \,:\, \bigsqcup_{i\in I}\varphi_i} \bigsqcup_{i\in I} \overline{B}_i^n \ \to\ (X\times Y)^n$$

which is the inclusion on $(X\times Y)^{n-1}$.

Proof. Let $\{\Phi_c\colon \overline{B}^{\dim(c)} \to X\}_{c\in\mathrm{Cell}(X)}$ and $\{\Psi_d\colon \overline{B}^{\dim(d)} \to Y\}_{d\in\mathrm{Cell}(Y)}$ be full sets of characteristic maps of the CW-complexes X and Y. Note that for each $p,q\in\mathbb{N}_0$ the restriction of the map $\Phi_c\times\Psi_d\colon \overline{B}^p\times\overline{B}^q \to X\times Y$ to $(\overline{B}^p\times S^{q-1})\cup(S^{p-1}\times\overline{B}^q)$ takes values in $(X^p\times Y^{q-1})\cup(X^{p-1}\cup Y^q)\subset(X\times Y)^{n-1}$. Thus we can consider the topological space

$$(X\times Y)^{n-1} \ \cup_\sim \bigsqcup_{\substack{p,q\in\mathbb{N}_0 \text{ with} \\ p+q=n}} \ \bigsqcup_{c\in\mathrm{Cell}_p(X)} \ \bigsqcup_{d\in\mathrm{Cell}_q(Y)} (\overline{B}^p\times\overline{B}^q)$$

where given c,d we use the restrictions of $\Phi_p\times\Psi_q$ to $(\overline{B}^p\times S^{q-1})\cup(S^{p-1}\times\overline{B}^q)$ to glue $\overline{B}^p\times\overline{B}^q$ to $(X\times Y)^{n-1}$. Next we consider the map

$$(X\times Y)^{n-1} \ \cup_\sim \bigsqcup_{\substack{p,q\in\mathbb{N}_0 \text{ with} \\ p+q=n}} \ \bigsqcup_{c\in\mathrm{Cell}_p(X)} \ \bigsqcup_{d\in\mathrm{Cell}_q(Y)} (\overline{B}^p\times\overline{B}^q) \ \to\ (X\times Y)^n$$

that is given by the inclusion on $(X\times Y)^{n-1}$ and that is given by all the product maps $\Phi_c\times\Psi_d\colon \overline{B}^p\times\overline{B}^q \to X\times Y$. It follows from the Topological Pushout Proposition 6.12 (0∗) that this map is continuous. It is clearly a bijection. Using the $\mathcal{CWP}$ topology on $(X\times Y)^n$ it is also not difficult to show that the map is closed. It follows from the Open-Injective Lemma 2.9 that the above map is a homeomorphism.

The claim now follows from the above discussion and the observation that the Ball-times-Ball Lemma 33.8 allows us to replace $\overline{B}^p\times\overline{B}^q$ by $\overline{B}^{p+q}=\overline{B}^n$ and it allows us to replace $(\overline{B}^p\times S^{q-1})\cup(S^{p-1}\times\overline{B}^q)$ by $S^{p+q-1}=S^{n-1}$. ⊞

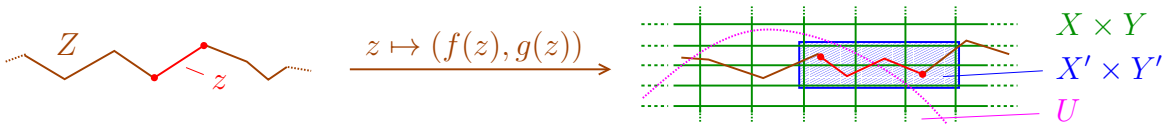
It remains to show that the n-cells of $X\times Y$ are precisely the sets of the form $c\times d$ where c is a p-cell of X and d is an $(n-p)$-cell of Y. But this follows easily from Claim 2.

(1) (a) We consider the natural projection $p\colon (X\times Y,\mathcal{CWP})\to X$. We need to show that this map is continuous. Note that it suffices to show that for each cell $c\subset X$ and each cell $d\subset Y$ the map $c\times d\to X\times Y\to X$ is continuous. But by the Product Topology-Basics Lemma 5.3 (1a) and the Topological-Product Proposition 5.1 (1a) we know that the maps $c\times d\to(X\times Y,\mathcal{TP})$ and $(X\times Y,\mathcal{TP})\to X$ are continuous.

Thus the composition is also continuous. The same argument of course also shows that the natural projection $p\colon X \times Y \to Y$ is continuous.

(b) Let Z be a CW-complex and let $f\colon Z \to X$ and $g\colon Z \to Y$ be two continuous maps. We need to show that the map $Z \to (X \times Y, \mathcal{CWP})$ that is given by $z \mapsto (f(z), g(z))$ is also continuous. Let $U \subset (X \times Y, \mathcal{CWP})$ be an open subset. We need to show that $(f,g)^{-1}(U)$ is an open subset of the CW-complex Z. By the Characteristic Maps Proposition 30.5 it remains to show that for each cell z of Z the intersection $z \cap (f,g)^{-1}(U)$ is open in z.

It follows from the Cell-Compact Proposition 30.4, the Compact Image Lemma 2.8 and the CW-Complex-Finiteness Theorem 30.10 that there exist finite subcomplexes X' of X and Y' of Y with $f(z) \subset X'$ and $g(z) \subset Y'$.



Claim. The intersection $(X' \times Y') \cap U$ is an open subset of $(X' \times Y', \mathcal{TP})$.

Proof. First note that we have the equality

$$(X' \times Y') \cap U \;=\; \bigcup_{c \in \mathrm{Cell}(X')} \bigcup_{d \in \mathrm{Cell}(Y')} (c \times d) \cap U.$$

Since U is open in $\mathcal{CWP}$ we see that each $(c \times d) \cap U$ is open in $c \times d$. Note that each $c \times d$ is a closed subset of $X' \times Y'$. It follows from these observations, and the fact that the finite CW-complexes X' and Y' have only finitely many cells, together with the Pasting Proposition 2.4 (2) that $(X' \times Y') \cap U$ is indeed an open subset of $X' \times Y'$. ⊞

Next note that

$$z \cap (f,g)^{-1}(U) \;=\; z \cap (f,g)^{-1}((X' \times Y') \cap U).$$

By the elementary Basics-of-Continuity Lemma 2.1 the map $(f,g)\colon Z \to X' \times Y'$ is also continuous. Thus it follows from the above equality and the claim, that $z \cap (f,g)^{-1}(U)$ is indeed an open subset of z.

(2) As in the case of the Topological-Product Proposition 5.1 this statement follows almost immediately from (1). ■

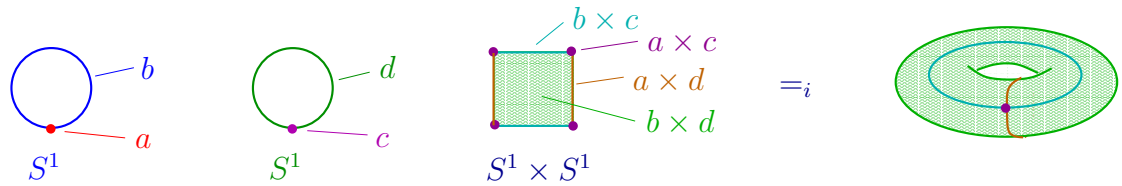
Proposition 33.9. (CW-Complex Product Proposition) Let X and Y be two CW-complexes.

(1) The identity map $(X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$ is continuous, in other words, we have the inclusion $\mathcal{TP} \subset \mathcal{CWP}$.

(2) Suppose we are in one of the following two settings:
   (a) X and X are both finite CW-complexes, or
   (b) $Y = [0,1]$ where Y is equipped with the usual CW-structure with two 0-cells and one 1-cell,

   then the identity map $(X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$ is a homeomorphism, in other words, we have $\mathcal{CWP} = \mathcal{TP}$. Thus the CW-product structure from the CW-Category–Product Proposition 33.7 is a CW-structure on $(X \times Y, \mathcal{TP})$.

Example. If we equip $S^1 =_i [0,1]/0 \sim 1$ with the usual CW-structure with one 0-cell and one 1-cell, then the product CW-structure on the torus $S^1 \times S^1 =_i ([0,1] \times [0,1])/\sim$ is precisely the CW-structure that we had already encountered on page 261. We refer to the figure below for an illustration.



Proof. Let X and Y be two CW-complexes.

(1) In the Product Topology-Basics Lemma 5.3 (1a) we showed that the product topology is compatible with taking the subspace topology. In our context this means that for cells $c \subset X$ and $d \subset Y$ the product topology on $c \times d$ agrees with the subspace topology on $c \times d \subset (X \times Y, \mathcal{TP})$. It follows immediately from this discussion that the identity map $(X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$ is continuous.

(2) (a) Suppose that X and X are both finite CW-complexes. It follows from the CW-Category–Product Proposition 33.7 (0b) that the CW-complex $X \times Y$ has only finitely many cells. It thus follows from the CW-Complex Compactness Corollary 30.11 that $(X \times Y, \mathcal{CWP})$ is compact. Furthermore note that it follows from the CW-Complex–Hausdorff Proposition 30.3 and the Product Topology-Properties Proposition 5.4 that $(X \times Y, \mathcal{TP})$ is Hausdorff. It follows from these facts, the Compact-Hausdorff Proposition 2.10 (3) and Statement (1) that $\mathrm{id}\colon (X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$ is indeed a homeomorphism.

(b) Suppose we have $Y = [0,1]$ with the obvious CW-structure. By (1) we know that $\mathcal{TP} \subset \mathcal{CWP}$. It remains to show that $\mathcal{CWP} \subset \mathcal{TP}$. Thus let $U \in \mathcal{CWP}$. Let $\{\Phi_c\colon \overline{B}^{\dim(c)} \to X\}_{c \in \mathrm{Cell}(X)}$ be a full set of characteristic maps for the CW-complex X. Since $U \in \mathcal{CWP}$ we know that for each $c \in \mathrm{Cell}(X)$ the intersection $(c \times [0,1]) \cap U$ is open in $c \times [0,1]$. Next note that it follows from the Topological-Product Proposition 5.1 (1) that the map $\Phi_c \times \mathrm{id}_{[0,1]}\colon \overline{B}^{\dim(c)} \times [0,1] \to X \times [0,1]$ is continuous. Thus we see that for each $c \in \mathrm{Cell}(X)$ the preimage $(\Phi_c \times \mathrm{id}_{[0,1]})^{-1}(U)$ is open in $\overline{B}^{\dim(c)} \times [0,1]$. It follows from this discussion together with the (c)⇒(a)-direction of the CW×Intervall Proposition 31.1 (1) that $U \in \mathcal{TP}$. ■

Example. The formulation of the CW-Complex Product Proposition 33.9 already suggests that the conclusion does not hold without some assumptions on X and Y. In [Fri23] we show that if X is the wedge of uncountably many intervals and if Y is the wedge of countably many intervals, then the topologies $\mathcal{CWP}$ and $\mathcal{TP}$ disagree on $X \times Y$.

Part IV

Singular Homology

34. The idea behind singular homology

This chatty chapter can be found in the long version of the lecture notes. It does not contain any results that we will use later on. The definition and the theory of singular homology was developed over several decades in the first half of the 20th century. In this chapter we will try to outline how one might have come up with the idea behind singular homology. The stress in this chapter is on explaining ideas and examples, without getting bogged down with technical discussions. We will not make use of any of the results and definitions of this chapter. We will use this feature as an excuse for taking a few liberties here and there.

So let's start with our story. Historically mathematicians found, and still find it, convenient to work with simplicial complexes. Loosely speaking a simplicial complex is a subset of $\mathbb{R}^n$ that is built out of simplices. Thus we first should tell the reader what we mean by a simplex:

Definition.

(1) We say $k+1$ points in $\mathbb{R}^n$ are in general position if they are not contained in a $(k-1)$-dimensional affine hyperplane of $\mathbb{R}^n$.

(2) Given any finite subset $S \subset \mathbb{R}^n$ we denote by

$$\mathrm{Hull}(S) \ := \ \Big\{ \sum_{x \in S} \lambda_x \cdot x \ \Big| \ \text{each } \lambda_x \in \mathbb{R}_{\geq 0} \text{ and } \sum_{x \in S} \lambda_x = 1 \Big\}$$

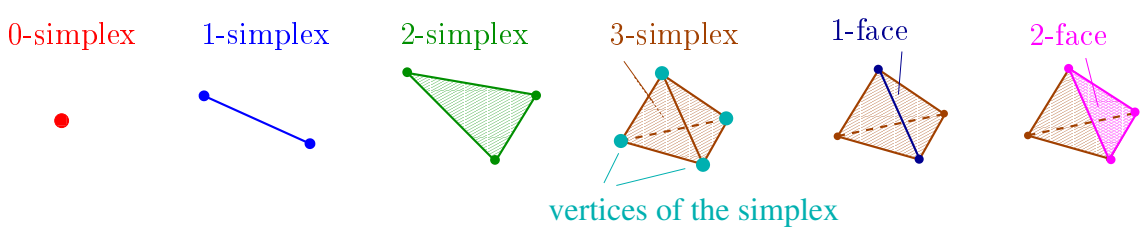
the convex hull of S.

(3) A k-simplex in $\mathbb{R}^n$ is the convex hull of $k+1$ points of $\mathbb{R}^n$ that are in general position.

(4) Let $\sigma \subset \mathbb{R}^n$ be a k-simplex.

(a) There exists a unique subset $V \subset \mathbb{R}^n$ with $k+1$ elements and with $\sigma = \mathrm{Hull}(V)$. We refer to V as the vertex set of σ.

(b) Let V be the vertex set of σ. For any subset $T \subset V$ of with $(l+1)$ elements we refer to $\mathrm{Hull}(T)$ as an l-face of $\sigma = \mathrm{Hull}(V)$.



Definition. A simplicial complex in $\mathbb{R}^n$ is a finite family $X = \{\sigma_i\}_{i=1,\dots,k}$ of simplices in $\mathbb{R}^n$ with the following properties:

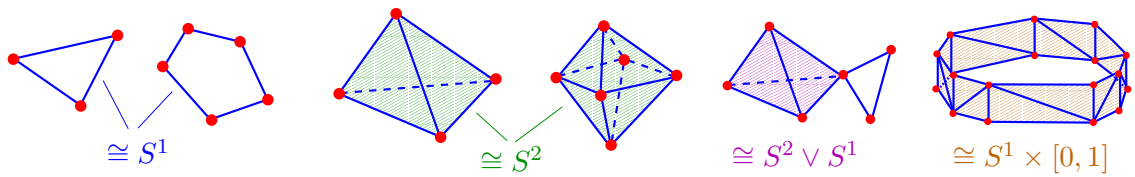
(1) Any face of a simplex is also contained in the family.

(2) The intersection of any two simplices is either empty or it is a single simplex, which is a face of both simplices.

Given a simplicial complex $X = \{\sigma_i\}_{i=1,\dots,k}$ as above we denote by X also the subspace $\sigma_1 \cup \dots \cup \sigma_k \subset \mathbb{R}^n$.

Example.

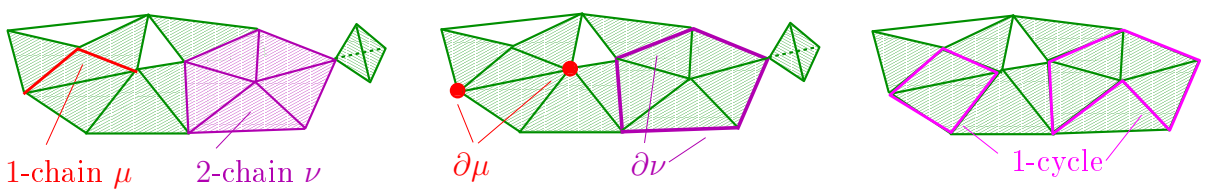
(1) In the figure below we see many examples of simplicial complexes. In particular we see that many topological spaces that we like, are actually homeomorphic to a simplicial complex.



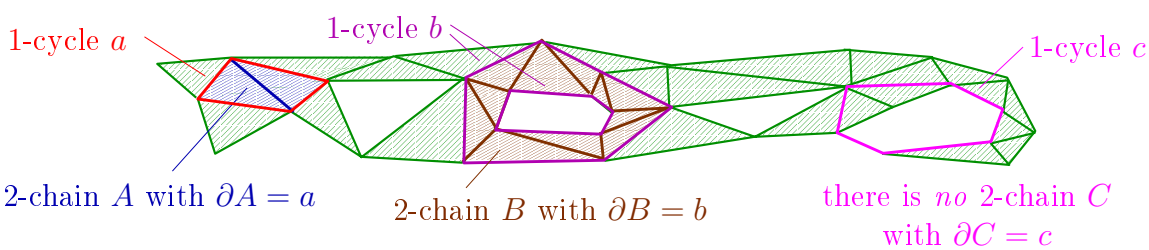
(2) In [Fri23] it is shown that every compact smooth manifold is homeomorphic to a simplicial complex and in [Fri23] it is also shown that every finite CW-complex is homotopy equivalent to a simplicial complex.

Definition. Let X be a simplicial complex.

(1) An n-chain in X is a collection of n-simplices of X, in other words, it is a subset of the set of n-simplices of X.
(2) The boundary $\partial\sigma$ of an n-chain σ is the union of all $(n-1)$-simplices that are the faces of an *odd* number of n-simplices of the n-chain.
(3) We say an n-chain is an n-cycle if its boundary is the empty set of $(n-1)$-simplices.



In the next figure we see that cycles come in two different flavors. Some n-cycles are the boundary of an $(n+1)$-chain and are thus not very interesting. On the other hand there are n-cycles which are not the boundary of an $(n+1)$-chain, and these non-bounding cycles seem to detect something interesting, like the presence of a "hole".



The idea now is to "measure" how many n-cycles are not the boundary of an $(n+1)$-cycle by introducing some algebra. This leads us to the following definition.

Definition. Let X be a simplicial complex.

(1) We define the n-th simplicial chain group of X as

$$\mathrm{C}_n^{\mathrm{simp}}(X) \;:=\; \text{set of all } n\text{-chains} \;\underset{\uparrow}{=}\; \mathbb{Z}_2^{\text{set of } n\text{-simplices of } X}.$$

for any set W we have $\mathcal{P}(W) = \mathbb{Z}_2^W$, we apply this logic to the set of all n-simplices of X

In other words the n-th chain group $\mathrm{C}_n^{\mathrm{simp}}(X)$ of X is defined as the free $\mathbb{Z}_2$-vector space on the set of all n-simplices of X. More informally speaking, $\mathrm{C}_n^{\mathrm{simp}}(X)$ is the set of all finite $\mathbb{Z}_2$-linear combinations of n-simplices of X.

(2) We define the boundary map

$$\begin{array}{rcl} \partial_n \colon \mathrm{C}_n^{\mathrm{simp}}(X) & \to & \mathrm{C}_{n-1}^{\mathrm{simp}}(X) \\ \sum\limits_{i=1}^{r} a_i \cdot \sigma_i & \mapsto & \sum\limits_{i=1}^{r} a_i \cdot \underbrace{\partial\sigma_i}_{\uparrow} \end{array}$$

we view $\partial\sigma_i$ as a linear combination of $(n-1)$-simplices

(3) We refer to the sequence

$$\ldots \xrightarrow{\partial_3} \mathrm{C}_2^{\mathrm{simp}}(X) \xrightarrow{\partial_2} \mathrm{C}_1^{\mathrm{simp}}(X) \xrightarrow{\partial_1} \mathrm{C}_0^{\mathrm{simp}}(X) \to 0$$

as the simplicial chain complex of X.

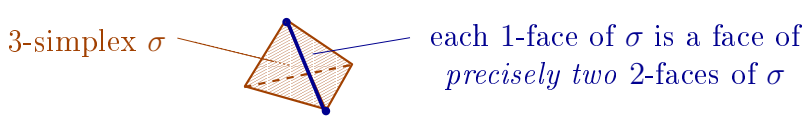
Lemma 34.1. (Simplicial–Double Boundary Lemma) Let X be a simplicial complex. For every $n \in \mathbb{N}_0$ the composition $\partial_n \circ \partial_{n+1} \colon \mathrm{C}_{n+1}^{\mathrm{simp}}(X) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(X)$ of two consecutive boundary maps is the zero map.

Sketch of proof. Since $\mathrm{C}_{n+1}^{\mathrm{simp}}(X)$ is a vector space with basis given by the $(n+1)$-simplices of X it suffices to prove that for each $(n+1)$-simplex of X we have $\partial_n(\partial_{n+1}(\sigma)) = 0$. Thus let σ be an $(n+1)$-simplex with vertex set $V = \{x_0, \ldots, x_{n+2}\}$. We compute

$$\begin{aligned} \partial_n(\partial_{n+1}(\sigma)) &= \partial_n\Big(\sum_{i=0}^{n+2} \mathrm{Hull}(\{x_0, \ldots, \widehat{x_i}, \ldots, x_{n+2}\}\Big) \\ &= \sum_{i=0}^{n+2} \sum_{\substack{j=0 \\ i \neq j}}^{n+2} \mathrm{Hull}(\{x_0, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, x_{n+2}\}) \underset{\uparrow}{=} 0. \end{aligned}$$

since each $(n-1)$-face appears twice and since we work with $\mathbb{Z}_2$-coefficients

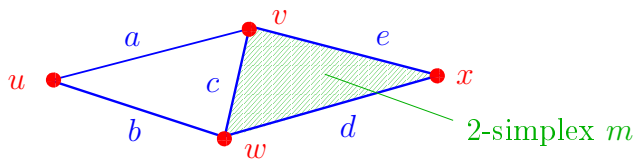
Another way of stating this argument is to say that each $(n-1)$-face of σ is the face of precisely two n-faces of σ. ■



Definition. Let X be a simplicial complex. Furthermore let $n \in \mathbb{N}_0$. It follows immediately from the above Simplicial–Double Boundary Lemma 34.1 that we have the inclusion $\mathrm{im}(\partial_{n+1} \colon \mathrm{C}_{n+1}^{\mathrm{simp}}(X) \to \mathrm{C}_n^{\mathrm{simp}}(X)) \subset \ker(\partial_n \colon \mathrm{C}_n^{\mathrm{simp}}(X) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(X))$. Thus it makes sense to define the n-th simplicial homology group as follows:

$$\mathrm{H}_n^{\mathrm{simp}}(X) := \frac{\ker(\partial_n \colon \mathrm{C}_n^{\mathrm{simp}}(X) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(X))}{\mathrm{im}(\partial_{n+1} \colon \mathrm{C}_{n+1}^{\mathrm{simp}}(X) \to \mathrm{C}_n^{\mathrm{simp}}(X))}.$$

Example. We consider the simplicial complex X that is shown below. It has four 0-simplices u, v, w, x, five 1-simplices a, b, c, d, e and one 2-simplex m.



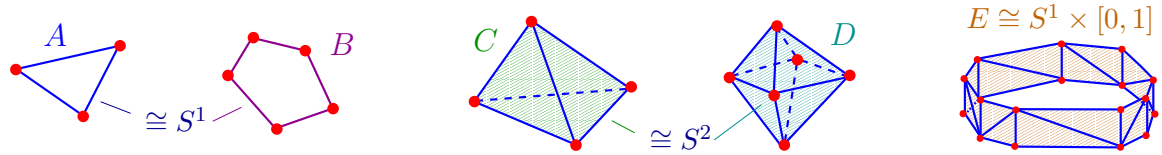
One can easily verify that the corresponding simplicial chain complex is given as follows:

$$0 \to \underbrace{\mathbb{Z}_2 \cdot m}_{=\mathrm{C}_2^{\mathrm{simp}}(X)} \xrightarrow{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}} \underbrace{\begin{matrix} \mathbb{Z}_2 \cdot a \\ \mathbb{Z}_2 \cdot b \\ \mathbb{Z}_2 \cdot c \\ \mathbb{Z}_2 \cdot d \\ \mathbb{Z}_2 \cdot e \end{matrix}}_{=\mathrm{C}_1^{\mathrm{simp}}(X)} \xrightarrow{\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}} \underbrace{\begin{matrix} \mathbb{Z}_2 \cdot u \\ \mathbb{Z}_2 \cdot v \\ \mathbb{Z}_2 \cdot w \\ \mathbb{Z}_2 \cdot x \end{matrix}}_{=\mathrm{C}_0^{\mathrm{simp}}(X)} \to 0.$$

An elementary, albeit rather painful calculation shows that

$$\mathrm{H}_k^{\mathrm{simp}}(X) \cong \begin{cases} \mathbb{Z}_2, & \text{if } k = 0, \\ \mathbb{Z}_2, & \text{if } k = 1, \\ 0, & \text{if } k \geq 2. \end{cases}$$

The previous example should convince the reader that simplicial homology groups can be computed, with enough patience, for any given simplicial complex. This is a good moment to return to some our previous examples. With sufficient energy one obtains the following simplicial homology groups:



$\mathrm{H}_0^{\mathrm{simp}}(A) \cong \mathbb{Z}_2$	$\mathrm{H}_0^{\mathrm{simp}}(B) \cong \mathbb{Z}_2$	$\mathrm{H}_0^{\mathrm{simp}}(C) \cong \mathbb{Z}_2$	$\mathrm{H}_0^{\mathrm{simp}}(D) \cong \mathbb{Z}_2$	$\mathrm{H}_0^{\mathrm{simp}}(E) \cong \mathbb{Z}_2$
$\mathrm{H}_1^{\mathrm{simp}}(A) \cong \mathbb{Z}_2$	$\mathrm{H}_1^{\mathrm{simp}}(B) \cong \mathbb{Z}_2$	$\mathrm{H}_1^{\mathrm{simp}}(C) = 0$	$\mathrm{H}_1^{\mathrm{simp}}(D) = 0$	$\mathrm{H}_1^{\mathrm{simp}}(E) \cong \mathbb{Z}_2$
$\mathrm{H}_2^{\mathrm{simp}}(A) = 0$	$\mathrm{H}_2^{\mathrm{simp}}(B) = 0$	$\mathrm{H}_2^{\mathrm{simp}}(C) \cong \mathbb{Z}_2$	$\mathrm{H}_2^{\mathrm{simp}}(D) \cong \mathbb{Z}_2$	$\mathrm{H}_2^{\mathrm{simp}}(E) = 0$
		$\mathrm{H}_3^{\mathrm{simp}}(C) = 0$	$\mathrm{H}_3^{\mathrm{simp}}(D) = 0$	

Using these examples we can already make a couple of educated guesses:

- The simplicial complexes A and B are homeomorphic and the corresponding simplicial homology groups $\mathrm{H}_k^{\mathrm{simp}}(A)$ and $\mathrm{H}_k^{\mathrm{simp}}(B)$ are isomorphic. We see the same behavior for the simplicial complexes C and D which are also homeomorphic. This suggests that the isomorphism type of simplicial homology might only depend on the homeomorphism type of the simplicial complex.
- If we look at the simplicial homology groups of the simplicial complexes that are homeomorphic to S^1 and that are homeomorphic to S^2 we might spot a pattern, namely one could make the guess that for any $n \geq 1$ and any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k^{\mathrm{simp}}(\text{simplicial complex homeomorphic to } S^n) \cong \begin{cases} \mathbb{Z}_2, & \text{if } k = 0, n, \\ 0, & \text{otherwise.} \end{cases}$$

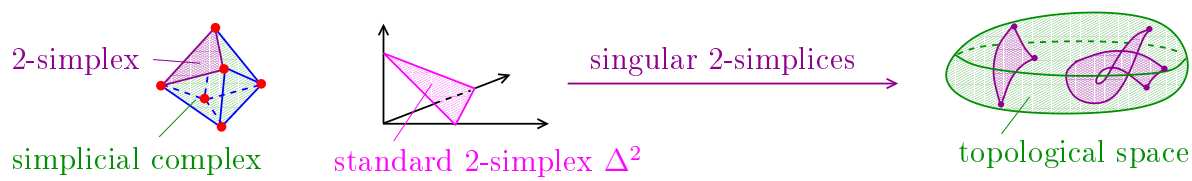
- The simplicial complexes $A \cong S^1$ and $E \cong S^1 \times [0,1]$ are homotopy equivalent. We notice that the simplicial homology groups are isomorphic. This suggests, on admittedly rather tenuous evidence, that the isomorphism type of simplicial homology might actually only depend on the homotopy type of the simplicial complex.

All this sounds very promising. But there are serious issues with the concept of simplicial homology:

- First of all it seems it seems rather restrictive to work just with simplicial complexes. Even though many topological spaces are homeomorphic to simplicial complexes, already for simple topological spaces, like the torus $S^1 \times S^1$, it is quite painful and unnatural to represent them as a simplicial complex.
- For simplicial homology to be really useful as a topological tool one would need to show that simplicial homology groups depend only on the homeomorphism type (or better homotopy type) of the simplicial complex. The longer one thinks about it, the harder it seems to prove such a statement.
- When we studied fundamental groups we saw that in applications it was essential that fundamental groups are covariantly functorial, i.e. that continuous maps induce maps on fundamental groups. Given a continuous map between two simplicial complexes there is no reasonable way how such a map will induce a map on the corresponding simplicial homology groups.

Because of all these issues we have to go back to the drawing board and come up with a better concept. The key idea is as follows:

- In *simplicial* homology $\mathrm{H}_k^{\mathrm{simp}}(X)$ we consider only simplicial complexes X and we study the simplices contained in them.
- In *singular* homology $\mathrm{H}_k^{\mathrm{simp}}(X)$ we allow any topological space X and this time we take the drastic step of considering "singular simplices", namely we study all continuous maps of the standard simplices Δ^n to X, regardless of whether or not the maps are injective.



This new concept will eventually solve all of the technical problems. For example we will see that, almost by definition, singular homology groups are covariantly functorial under continuous maps and thus homeomorphism invariants. Furthermore it will also not be particularly difficult to show that singular homology groups depend only on the homotopy type of the topological space. The problem with singular homology is that in our attempt to solve the technical issues of simplicial homology we threw computability out of the window. We will have to work quite hard to calculate singular homology groups. But eventually we will see that singular homology encompasses simplicial homology and that it is the best of two worlds: it has very nice theoretical properties and highly computable.

35. The singular homology groups of a topological space

35.1. Algebraic chain complexes.

Definition.

(1) An (algebraic) chain complex (C_*, ∂_*) is a sequence

$$\ldots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots C_1 \xrightarrow{\partial_1} C_0 \to 0$$

of homomorphisms between abelian groups such that the composition of any two consecutive boundary maps is the zero map, i.e. for every $i \in \mathbb{N}$ we have $\partial_{i-1} \circ \partial_i = 0$. For $n \in \mathbb{Z}_{<0}$ we always set $C_n := 0$.

(2) Let (C_*, ∂_*) be a chain complex and let $n \in \mathbb{N}_0$. By definition of a chain complex we have $\mathrm{im}(\partial_{n+1} \colon C_{n+1} \to C_n) \subset \ker(\partial_n \colon C_n \to C_{n-1})$. Thus it makes sense to define the n-th homology group of the chain complex as follows:

$$\mathrm{H}_n(C) \ := \ \frac{\ker(\partial_n \colon C_n \to C_{n-1})}{\mathrm{im}(\partial_{n+1} \colon C_{n+1} \to C_n)}.$$

(3) (a) We refer to elements in the kernel of $\partial_n \colon C_n \to C_{n-1}$ as cycles.
(b) We say that two elements $c, d \in C_n$ are homologous if there exists an $e \in C_{n+1}$ with $c - d = \partial_{n+1} e$.
(c) We say that a cycle $z \in C_n$ is null-homologous if z is homologous to the zero element of C_n, i.e. if $[z] = 0 \in \mathrm{H}_n(C)$.

Definition. A chain map $f \colon C_* \to D_*$ between two chain complexes consists of a family $\{f_n \colon C_n \to D_n\}_{n \in \mathbb{N}_0}$ of homomorphisms such that for each $n \in \mathbb{N}_0$ the following equality holds:

$$f_{n-1} \circ \partial_n \ = \ \partial_n \circ f_n.$$

Definition. We refer to category $\mathcal{ChCplx}$ that is given by

$$\mathrm{Ob}(\mathcal{ChCplx}) \ := \ \text{all chain complexes,}$$

and with morphisms

$$\mathrm{Mor}(C_*, D_*) \ := \ \text{all chain maps from } C_* \text{ to } D_*.$$

with the obvious composition of morphisms as the category of chain complexes.

Lemma 35.1. (Chain Complex-H_*-Functor Lemma)

(1) If $f \colon C_* \to D_*$ is a chain map between chain complexes C_* and D_*, then the map

$$\begin{aligned} f_* \colon \mathrm{H}_n(C_*) &\to \mathrm{H}_n(D_*) \\ [c] &\mapsto [f_n(c)] \end{aligned}$$

is a well-defined homomorphism.

(2) Given $n \in \mathbb{N}_0$ the maps

$$\begin{aligned} C_* &\mapsto \mathrm{H}_n(C_*) \\ (f \colon C_* \to D_*) &\mapsto (f_* \colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(D_*)) \end{aligned}$$

define a covariant functor from the category $\mathcal{ChCplx}$ of chain complexes to the category $\mathcal{AbGroup}$ of abelian groups.

Definition. In the setting of the Chain Complex-H_*-Functor Lemma 35.1 (1) we refer to the corresponding map $f_* \colon \mathrm{H}_n(C) \to \mathrm{H}_n(D)$ as the induced map on homology.

Proof.

(1) Let $c \in C_n$ be a cycle. First we have to show that $f_n(c)$ is again a cycle. Indeed we have

$$\partial_n(f_n(c)) \underset{\substack{\uparrow \\ \text{definition of a chain map}}}{=} f_{n-1}(\partial_n c) \underset{\substack{\uparrow \\ \text{since } c \text{ is a cycle}}}{=} f_n(0) \underset{\substack{\uparrow \\ \text{since } f_n \text{ is a} \\ \text{homomorphism}}}{=} 0.$$

Now let $c, d \in C_n$ be two cycles which represent the same element in $\mathrm{H}_n(C)$. We have to show that $[f_n(c)] = [f_n(d)]$. Indeed we have

$$[f_n(c)] - [f_n(d)] = [f_n(c-d)] \underset{\substack{\uparrow \\ \text{since } c,d \text{ are homologous there} \\ \text{exists an } e \in C_{n+1} \text{ with } \partial_{n+1}e = c-d}}{=} [f_n(\partial_{n+1}(e))] \underset{\substack{\uparrow \\ \text{by the definition} \\ \text{of a chain map}}}{=} [\partial_{n+1}(f_{n+1}(e))] \underset{\substack{\uparrow \\ \text{by the definition} \\ \text{of homology}}}{=} 0 \in \mathrm{H}_n(D).$$

This shows that the map $f_*\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(D_*)$ is well-defined. It follows easily from the definitions that this map is actually a homomorphism.

(2) The statement follows easily from (1) and the definitions. ■

Lemma 35.2. (Direct Sum-H_*-Lemma) Let $\{C_a\}_{a\in A}$ be a family of chain complexes. For every $n \in \mathbb{N}_0$ the natural maps $C_a \to \bigoplus_{a\in A} C_a$ induce an isomorphism

$$\bigoplus_{a\in A} \mathrm{H}_n(C_a) \xrightarrow{\cong} \mathrm{H}_n\Big(\bigoplus_{a\in A} C_a\Big).$$

Proof. The proof is a standard exercise in homological algebra, which, like every other author, we leave to the reader. ■

35.2. The singular chain complex.

Definition. Let $n \in \mathbb{N}_0$.

(1) Given k points $P_1, \dots, P_k$ in $\mathbb{R}^n$ such that $P_2 - P_1, \dots, P_k - P_1$ are linearly independent we refer to the convex hull of $P_1, \dots, P_k$ as a k-simplex.

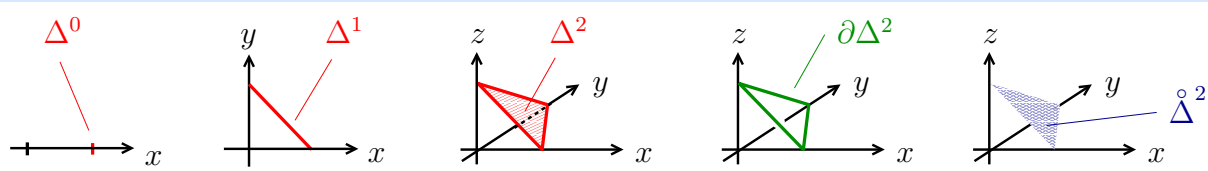
(2) Given $n \in \mathbb{N}_0$ we refer to the convex hull of the standard basis vectors $e_1, \dots, e_{n+1}$ of $\mathbb{R}^{n+1}$, i.e. to

$$\Delta^n := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0 + \dots + x_n = 1 \text{ and } x_i \geq 0 \text{ for all } i = 0, \dots, n\}$$

as the standard n-simplex.

(3) Given the standard n-simplex Δ^n we define its boundary as

$$\partial\Delta^n := \bigcup_{i=0}^{n} \{(t_0, \dots, t_n) \in \Delta^n \mid t_i = 0\}.$$



Lemma 35.3. (Δ^n-Homeomorphism Lemma) Let $n \in \mathbb{N}$.

(1) We set

$$\Delta_n := \Big\{(x_1, \dots, x_n) \in [0,1]^n \,\Big|\, \sum_{i=1}^{n} x_i \in [0,1]\Big\}$$

and refer to it as the planar n-simplex. The maps

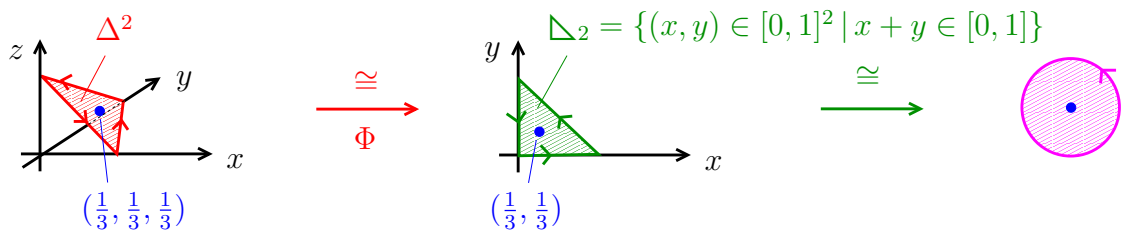
$$\begin{array}{rcl} \Phi\colon \Delta^n & \to & \triangle_n \\ (t_0,\dots,t_n) & \mapsto & (t_0,\dots,t_{n-1}) \end{array} \quad \text{and} \quad \begin{array}{rcl} \Psi\colon \triangle_n & \to & \Delta^n \\ (x_1,\dots,x_n) & \mapsto & \Big(x_1,\dots,x_n,1-\sum\limits_{i=1}^n x_i\Big) \end{array}$$

are homeomorphisms that are inverses of one another.

(2) The map

$$\Delta^n \xrightarrow{\Phi} \triangle_n \xrightarrow[\text{applied to } \triangle_n \text{ and } Q=(\frac{1}{n+1},\dots,\frac{1}{n+1})]{\text{the homeomorphism } \Xi \text{ obtained from the Convex-to-Ball Proposition 2.12 (2)}} \overline{B}^n$$

is a homeomorphism that restricts to a homeomorphism from $\partial\Delta^n$ to the sphere S^{n-1}.



Proof.

(1) Note that Φ extends "by the same formula" to a map $\mathbb{R}^{n+1}\to\mathbb{R}^n$ which is evidently continuous. Hence the restriction to Δ^n is also continuous. Now that this is settled we see that the remaining statements are basically trivial.

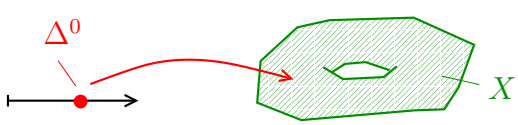
(2) The statement follows easily from (1), the observation that $\triangle_n$ is compact (which can be shown easily using the Heine-Borel Theorem 1.11) and the Convex-to-Ball Proposition 2.12 (2). ■

Definition. Let X be a topological space and let $n\in\mathbb{N}_0$. A singular n-simplex is a continuous map $\sigma\colon\Delta^n\to X$.

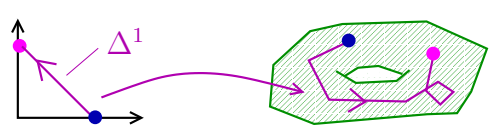
Example. Let X be a topological space.

(1) A singular 0-simplex is a continuous map from Δ^0 to X. Note that $\Delta^0=\{1\}\subset\mathbb{R}$. Thus a map $\Delta^0=\{1\}\to X$ is uniquely determined by a point in X. Given a point $P\in X$ we frequently denote by P also the singular 0-simplex $\Delta^0\to X$ that is given by sending the single point in Δ^0 to P.

(2) A singular 1-simplex is a continuous map $\Delta^1\to X$. Since the map $[0,1]\to\Delta^1$ given by $t\mapsto(1-t,t)$ is a homeomorphism we see that a singular 1-simplex is essentially the same as a path in X.



a singular 0-simplex in X is essentially the same as a point in X



a singular 1-simplex in X is essentially the same as a path in X

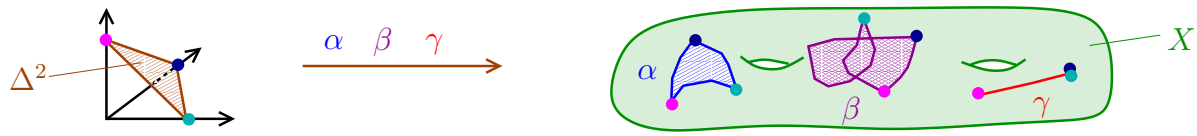
Definition. Let X be a topological space and let $n \in \mathbb{N}_0$.

(1) We define the n-th singular chain group as follows:

$$\mathrm{C}_n(X) \;:=\; \text{free abelian group generated by the set of singular } n\text{-simplices in } X.$$

For $n \in \mathbb{Z}_{<0}$ we set $\mathrm{C}_n(X) := 0$.

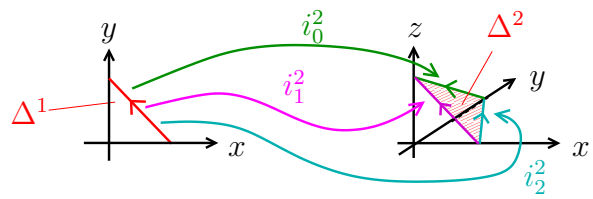
(2) We refer to an element in $\mathrm{C}_n(X)$ as a singular n-chain in X.



$-2\alpha + \beta + 3\gamma$ is a singular 2-chain

Definition. Let $n \in \mathbb{N}_0$. For $j \in \{0, \ldots, n\}$ we define the j-th face map as follows:

$$\begin{array}{rcl} i_j^n \colon \Delta^{n-1} & \to & \Delta^n \\ (t_0, \ldots, t_{n-1}) & \mapsto & (t_0, \ldots, t_{j-1}, 0, t_j, \ldots, t_{n-1}). \end{array}$$



Definition. Let X be a topological space and let $\sigma \colon \Delta^n \to X$ be a singular n-simplex. The boundary $\partial_n \sigma \in \mathrm{C}_{n-1}(X)$ of σ is defined as

$$\partial_n \sigma \;:=\; \sum_{j=0}^{n} (-1)^j \cdot \underbrace{\sigma \circ i_j^n}_{\text{map } \Delta^{n-1} \to X}.$$

By the Free Abelian Group–Existence Lemma 18.4 the map $\sigma \mapsto \partial_n \sigma$ extends uniquely to a linear map

$$\begin{array}{rcl} \partial_n \colon \mathrm{C}_n(X) & \to & \mathrm{C}_{n-1}(X) \\ \sum_{r=1}^{k} a_r \cdot \sigma_r & \mapsto & \sum_{r=1}^{k} a_r \cdot \partial \sigma_r, \end{array}$$

which we refer to as the n-th boundary map. If it is clear with what dimension we are working with, then we just write ∂ instead of ∂_n.

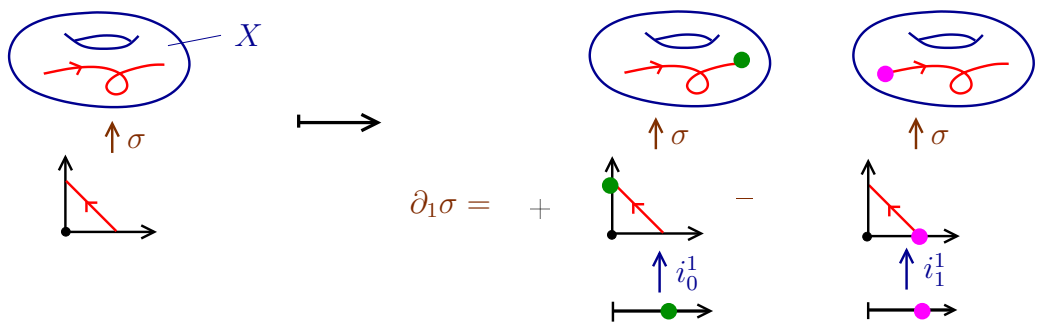
Examples.

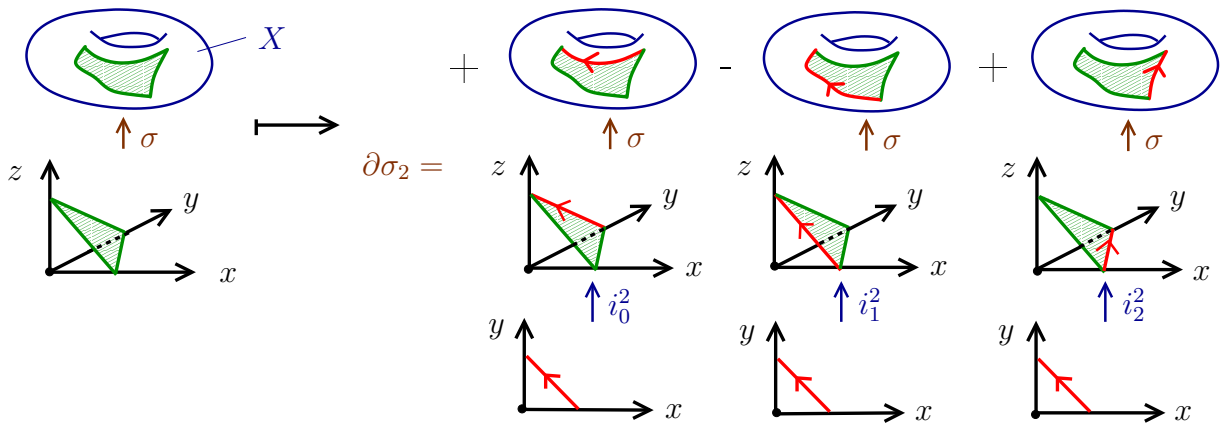
(1) For any singular 1-simplex $\sigma \colon \Delta^1 \to X$ we have

$$\partial_1 \sigma \;=\; (-1)^0 \cdot \sigma \circ i_0^1 + (-1)^1 \cdot \sigma \circ i_1^1 \;=\; \sigma(0,1) - \sigma(1,0) \;\in\; \mathrm{C}_0(X).$$

Recall that a a singular 1-simplex can be viewed as a map σ from the "interval" Δ^1 to X. The boundary of σ is thus the "endpoint" $\sigma(0,1)$ "minus" the "starting point" $\sigma(1,0)$.

(2) In the figure below we sketch the boundary of a singular 2-simplex.





Proposition 35.4. (Double Boundary Proposition) Let X be a topological space. For each $n \in \mathbb{N}_0$ the composition

$$\partial_{n-1} \circ \partial_n \colon \mathrm{C}_n(X) \;\to\; \mathrm{C}_{n-2}(X)$$

is the zero map.

Proof. Let $n \in \mathbb{N}_0$. It suffices to show that for every singular n-simplex $\sigma \colon \Delta^n \to X$ we have $\partial_{n-1}(\partial_n(\sigma)) = 0$. Thus let $\sigma \colon \Delta^n \to X$ be a singular n-simplex. We calculate that

$$\begin{aligned}
\partial_{n-1}(\partial_n(\sigma)) &\overset{\substack{\text{definition of } \partial_n \\ \downarrow}}{=} \partial_{n-1}\left(\sum_{j=0}^{n} (-1)^j \cdot \sigma \circ i_j^n\right) \overset{\substack{\text{definition of } \partial_{n-1} \\ \downarrow}}{=} \sum_{k=0}^{n-1} (-1)^k \cdot \sum_{j=0}^{n} (-1)^j \cdot (\sigma \circ i_j^n) \circ i_k^{n-1} \\
&= \sum_{0=k<j\leq n} (-1)^{k+j} \cdot \sigma \circ (i_j^n \circ i_k^{n-1}) + \sum_{0=j\leq k\leq n-1} (-1)^{k+j} \cdot \sigma \circ (i_j^n \circ i_k^{n-1}) \;=:\; (*).
\end{aligned}$$

Now we want to show that each expression $\sigma \circ (i_j^n \circ i_k^{n-1})$ appears precisely twice, but with opposite signs. From the definitions it follows easily that for $j \leq k$ we have

$$i_j^n \circ i_k^{n-1} \;=\; i_{k+1}^n \circ i_j^{n-1}.$$

Now we continue with the above calculation:

$$
\begin{aligned}
(*) \quad &\underset{\substack{\uparrow \\ \text{above calculation of } i_j^n \circ i_k^{n-1}}}{=} \sum_{0=k<j\leq n} (-1)^{k+j} \cdot \sigma \circ (i_j^n \circ i_k^{n-1}) + \sum_{0=j\leq k\leq n-1} (-1)^{k+j} \cdot \sigma \circ (i_{k+1}^n \circ i_j^{n-1}) \\
&\underset{\substack{\uparrow \\ \text{substitution } l=k+1}}{=} \sum_{0=k<j\leq n} (-1)^{k+j} \cdot \sigma \circ (i_j^n \circ i_k^{n-1}) + \sum_{0=j<l\leq n} (-1)^{l+j-1} \cdot \sigma \circ (i_l^n \circ i_j^{n-1}) \underset{\substack{\uparrow \\ \text{all summands left and right cancel}}}{=} 0.
\end{aligned}
$$

■

Definition. Let $f\colon X \to Y$ be a continuous map between topological spaces. For each $n \in \mathbb{N}_0$ we refer to

$$
\begin{aligned}
f_*\colon \mathrm{C}_n(X) &\to \mathrm{C}_n(Y) \\
\sum_{i=1}^k a_i \cdot \underbrace{\sigma_i}_{\substack{\uparrow \\ \text{map } \Delta^n \to X}} &\mapsto \sum_{i=1}^k a_i \cdot (\underbrace{f \circ \sigma_i}_{\substack{\uparrow \\ \text{map } \Delta^n \to Y}}).
\end{aligned}
$$

as the induced chain map.

Lemma 35.5. (C_*–Functor Lemma)

(1) Let $f\colon X \to Y$ be a continuous map between two topological spaces. The induced maps $f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$ form a chain map.

(2) The maps

$$\text{topological space } X \mapsto (\mathrm{C}_*(X), \partial_*)$$

and

$$(f\colon X \to Y) \mapsto (f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y))$$

define a covariant functor from the category $\mathcal{Top}$ of topological spaces to the category $\mathcal{ChCplx}$ of chain complexes.

Proof.

(1) Let $f\colon X \to Y$ be a continuous map between topological spaces. Given any singular n-simplex $\sigma\colon \Delta^n \to X$ we have

$$
f_*(\partial\sigma) \underset{\substack{\uparrow \\ \text{definition of } \partial}}{=} f_*\Big(\sum_{j=0}^n (-1)^j \cdot \sigma \circ i_j^n\Big) \underset{\substack{\uparrow \\ \text{definition of } f_*}}{=} \sum_{j=0}^n (-1)^j \cdot f \circ (\sigma \circ i_j^n) = \sum_{j=0}^n (-1)^j \cdot (\overbrace{f \circ \sigma}^{=f_*\sigma}) \circ i_j^n \underset{\substack{\uparrow \\ \text{definition of } \partial}}{=} \partial(f_*\sigma).
$$

Since the singular n-simplices generate $\mathrm{C}_n(X)$ we obtain the desired equality for all singular n-chains. By definition this means that $f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$ is a chain map.

(2) It follows immediately from the definitions that for two continuous maps $f\colon X \to Y$ and $g\colon Y \to Z$ we have $(g \circ f)_* = g_* \circ f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Z)$. With this observation the second statement is an almost immediate consequence of the first statement. ■

Lemma 35.6. (Path-Component–C_*–Lemma) Let X be a topological space with path-components $\{X_a\}_{a\in A}$. Given $a \in A$ we denote by $i_a\colon X_a \to X$ the inclusion map. The map

$$\bigoplus_{a\in A} i_a\colon \bigoplus_{a\in A} \mathrm{C}_*(X_a) \to \mathrm{C}_*(X)$$

is an isomorphism of chain complexes.

Proof. Evidently the standard simplex Δ^n is path-connected. Thus the image of a continuous map $\sigma\colon \Delta^n \to X$ is contained in a path-component X_a. This shows that

$$\bigsqcup_{a\in A} \text{set of singular } n\text{-simplices in } X_a \quad = \quad \text{set of singular } n\text{-simplices in } X.$$

It follows that the map

$$\begin{aligned} \bigoplus_{a\in A} i_a\colon \bigoplus_{a\in A} \mathrm{C}_n(X_a) &\to \mathrm{C}_n(X) \\ \Big(\sum_{i=1}^k a_i\cdot(\sigma\colon \Delta^n \to X_{a_i})\Big) &\mapsto \Big(\sum_{i=1}^k a_i\cdot(\sigma\colon \Delta^n \to X_{a_i})\Big) \end{aligned}$$

is an isomorphism of free abelian groups. It follows immediately from the definitions that this a chain map. ■

35.3. Definition of the singular homology groups.

Definition. Let X be a topological space and let $n \in \mathbb{N}_0$. We define the n-th singular homology group of X as the n-homology group of the singular chain complex. In other words, we set

$$\mathrm{H}_n(X) \;:=\; \mathrm{H}_n(\mathrm{C}_*(X)) \;=\; \frac{\ker(\partial_n\colon \mathrm{C}_n(X)\to \mathrm{C}_{n-1}(X))}{\mathrm{im}(\partial_{n+1}\colon \mathrm{C}_{n+1}(X)\to \mathrm{C}_n(X))}.$$

Throughout the lecture notes we will mostly shorten the name "singular homology" to "homology".

Example. There is only one example that we can do immediately: for the empty topological space $\varnothing$ we have for all $n \in \mathbb{N}_0$ that $\mathrm{C}_n(\varnothing) = 0$. It follows that for all $n \in \mathbb{N}_0$ we have $\mathrm{H}_n(\varnothing) = 0$.

Definition. Let X be a topological space.

(1) We say $c \in \mathrm{C}_n(X)$ is a singular n-cycle, or short n-cycle, if $\partial c = 0$.
(2) We say two singular n-chains c and d are homologous if there exists a singular $(n+1)$-chain $e \in \mathrm{C}_{n+1}(X)$ with $\partial e = c - d$.
(3) An n-cycle c is called null-homologous, if $[c] = 0 \in \mathrm{H}_n(X)$.

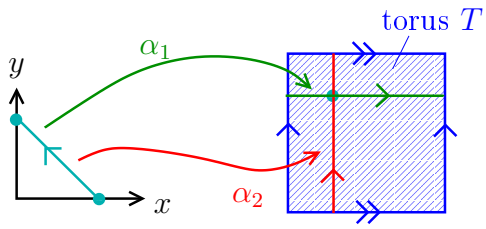
Example. In the figure below we show the torus $T = ([0,1]\times[0,1])/\sim$ together with two singular 1-simplices α_1 and α_2 and together with two singular 2-simplices μ and ν. It follows easily from the definitions that α_1 and α_2 are 1-cycles. It is a good exercise to verify that $\mu - \nu$ is a 2-cycle. With a little bit of practice it is fairly easy to write down cycles, but usually it is very hard to show that a cycle represents a non-trivial element in homology. For example, with our present knowledge it is impossible though to determine whether the cycles $\alpha_1, \alpha_2, \mu - \nu$ represent non-trivial homology classes.

Lemma 35.7. (H_n–Functor Lemma)

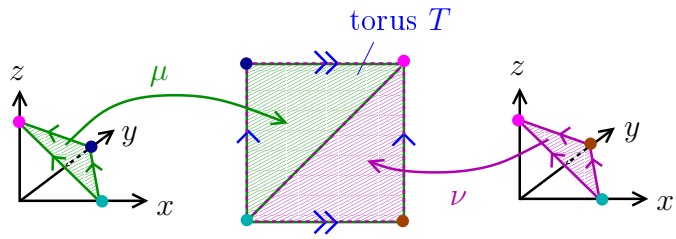
(1) Let $f\colon X \to Y$ be a continuous map between topological spaces. The map

$$\begin{aligned} f_*\colon \mathrm{H}_n(X) &\to \mathrm{H}_n(Y) \\ \Big[\sum_{i=1}^k a_i\cdot\sigma_i\Big] &\mapsto \Big[\sum_{i=1}^k a_i\cdot(f\circ\sigma_i)\Big] \end{aligned}$$

is a well-defined homomorphism.



α_1 and α_2 are 1-cycles



$\mu - \nu$ is a 2-cycle

(2) For each $n \in \mathbb{N}_0$ the map

$$X \mapsto \mathrm{H}_n(X)$$

and the map

$$(f\colon X \to Y) \mapsto \left(\begin{array}{ccc} f_*\colon \mathrm{H}_n(X) & \to & \mathrm{H}_n(Y) \\ \left[\sum\limits_{i=1}^{k} a_i \cdot \sigma_i\right] & \mapsto & \left[\sum\limits_{i=1}^{k} a_i \cdot (f \circ \sigma_i)\right] \end{array} \right)$$

define a covariant functor from the category of topological spaces to the category of abelian groups.

Proof.

(1) This statement follows from the C_*–Functor Lemma 35.5 (1) together with the Chain Complex-H_*-Functor Lemma 35.1 (1).

(2) This statement follows from the combination of (1), the C_*–Functor Lemma 35.5 (2) and the Chain Complex-H_*-Functor Lemma 35.1 (2). ■

Lemma 35.8. (Path-Component–H_*–Lemma) Let X be a topological space with path-components $\{X_a\}_{a\in A}$. Given $a \in A$ we denote by $i_a\colon X_a \to X$ the inclusion map. For each $n \in \mathbb{N}_0$ the map

$$\bigoplus_{a\in A} i_{a*}\colon \bigoplus_{a\in A} \mathrm{H}_n(X_a) \to \mathrm{H}_n(X)$$

is an isomorphism.

Proof. This lemma is an immediate consequence of the Path-Component–C_*–Lemma 35.6 and the Direct Sum-H_*-Lemma 35.2. ■

35.4. First calculations of homology groups.

Definition. Given a topological space X we refer to the homomorphism

$$\begin{array}{ccc} \epsilon_X\colon \mathrm{C}_0(X) & \to & \mathbb{Z} \\ \sum\limits_{i=1}^{k} a_i \cdot \sigma_i & \mapsto & \sum\limits_{i=1}^{k} a_i \end{array}$$

as the augmentation homomorphism.

Lemma 35.9. (Augmentation Lemma) Let X be a topological space.

(1) The map $\epsilon_X \circ \partial_1\colon \mathrm{C}_1(X) \to \mathbb{Z}$ is the zero map.

(2) The map

$$\begin{array}{ccc} \epsilon_X\colon \mathrm{H}_0(X) & \to & \mathbb{Z} \\ [\sigma] & \mapsto & \epsilon_X(\sigma) \end{array}$$

is well-defined. In the following we refer to this map also as the augmentation map.

(3) Given any continuous map $f\colon X \to Y$ between topological spaces the following diagram commutes

$$\begin{tikzcd} \mathrm{H}_0(X) \arrow[rr, "f_*"] \arrow[dr, "\epsilon_X"'] & & \mathrm{H}_0(Y). \arrow[dl, "\epsilon_Y"] \\ & \mathbb{Z}. & \end{tikzcd}$$

Proof. Let X be a topological space.

(1) Since the abelian group $\mathrm{C}_1(X)$ is generated by singular 1-simplices it suffices to prove that for each singular 1-simplex $\sigma\colon \Delta^1 \to X$ we have $(\epsilon_X \circ \partial_1)(\sigma) = 0$. But this is the case since

$$\epsilon_X(\partial_1(\sigma)) \underset{\substack{\uparrow \\ \text{see the discussion on page 302}}}{=} \epsilon_X(\sigma(0,1) - \sigma(1,0)) \underset{\substack{\uparrow \\ \text{by definition of } \epsilon_X}}{=} 1 - 1 = 0.$$

(2) If we have $\sigma, \tau \in \mathrm{C}_0(X)$ with $[\sigma] = [\tau] \in \mathrm{H}_0(X)$, then by definition of $\mathrm{H}_0(X)$ there exists a $\mu \in \mathrm{C}_1(X)$ with $\partial\mu = \sigma - \tau$. We now see that

$$\epsilon_X(\sigma) = \epsilon_X(\tau - \partial\mu) = \epsilon_X(\tau) - \underbrace{\epsilon_X(\partial\mu)}_{=0 \text{ by } (1)} = \epsilon_X(\tau).$$

(3) This statement follows easily from the definitions. ■

Proposition 35.10. (H_0-Proposition)

(1) Let X be a path-connected non-empty topological space. The augmentation map

$$\begin{aligned} \epsilon_X\colon \mathrm{H}_0(X) &\to \mathbb{Z} \end{aligned}$$

is a natural isomorphism. In fact given any point $P \in X$ the inverse is given by the map

$$\begin{aligned} \iota\colon \mathbb{Z} &\to \mathrm{H}_0(X) \\ n &\mapsto n \cdot [P]. \end{aligned}$$

(2) Let $f\colon X \to Y$ be a continuous map between two topological spaces.
   (a) If X and Y are path-connected non-empty topological spaces, then the induced map $f_*\colon \mathrm{H}_0(X) \to \mathrm{H}_0(Y)$ is an isomorphism.
   (b) If $X = Y$ is a non-empty path-connected topological space, then the induced $f_*\colon \mathrm{H}_0(X) \to \mathrm{H}_0(X)$ is in fact the identity map.

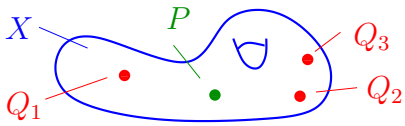
Proof.

(1) Let X be a path-connected non-empty topological space and let P be a point in X (which exists, since X is non-empty). It is clear that $\epsilon_X \circ \iota = \mathrm{id}_{\mathbb{Z}}$, in particular the homomorphism ϵ_X is surjective. It remains to show that ϵ_X is injective.

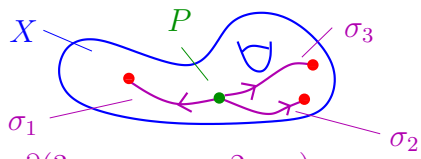
Thus let $c \in \mathrm{C}_0(X)$ with $\epsilon_X(c) = 0$. By definition of $\mathrm{C}_0(X)$ we can find points $Q_1, \dots, Q_k$ in X and $a_1, \dots, a_k \in \mathbb{Z}$ such that $c = \sum_{i=1}^{k} a_i \cdot Q_i$. Since X is path-connected we can find, for $i = 1, \dots, k$, a singular 1-simplex σ_i with $\sigma_i((0,1)) = Q_i$ and $\sigma_i((1,0)) = P_i$. Then we have

$$\partial\Big(\sum_{i=1}^{k} a_i \cdot \sigma_i\Big) = \sum_{i=1}^{k} a_i \cdot \partial\sigma_i \underset{\substack{\uparrow \\ \text{see the discussion on page 302}}}{=} \sum_{i=1}^{k} a_i \cdot (Q_i - P) = \sum_{i=1}^{k} a_i \cdot Q_i - \underbrace{\Big(\sum_{i=1}^{k} a_i\Big)}_{=\epsilon_X(c)=0} \cdot P = \underbrace{\sum_{i=1}^{k} a_i \cdot Q_i}_{=c}.$$

This implies that $[c] = 0 \in \mathrm{H}_0(X)$. Thus we have shown that ϵ_X is injective.



$c = 3 \cdot Q_1 - Q_2 - 2 \cdot Q_3$



$\partial(3 \cdot \sigma_1 - \sigma_2 - 2 \cdot \sigma_3) = c$

(2) (a) Let $f\colon X \to Y$ be a continuous map between two topological spaces. By the Augmentation Lemma 35.9 (3) we have the commutative diagram:

$$\begin{tikzcd} \mathrm{H}_0(X) \arrow[rr, "f_*"] \arrow[dr, "\epsilon_X"'] & & \mathrm{H}_0(Y). \arrow[dl, "\epsilon_Y"] \\ & \mathbb{Z}. & \end{tikzcd}$$

If X and Y are path-connected non-empty topological spaces, then we know by (1) that the diagonal maps are isomorphisms. Thus the horizontal map is also an isomorphism.

(b) If $X = Y$ is a path-connected non-empty topological space, then in the above diagram the two diagonal maps are the same isomorphism, thus the horizontal map is the identity. ■

Lemma 35.11. (Singleton-H$_*$-Lemma) If X is a singleton, i.e. if X is a topological space which consists of a single point, then

$$\mathrm{H}_n(X) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

Proof. Let $X = \{\star\}$ be a topological space consisting of a single point. Given $n \in \mathbb{N}_0$ let $\sigma_n\colon \Delta^n \to \{\star\}$ be the unique map. We calculate that

$$\partial_n(\sigma_n) \underset{\substack{\uparrow \\ \text{see the definition} \\ \text{of } \partial_n \text{ on page 302}}}{=} \sum_{j=0}^{n} \cdot (-1)^j \cdot \underbrace{\sigma_n \circ i_j^n}_{=\sigma_{n-1}} = \sigma_{n-1} \cdot \sum_{j=0}^{n} \cdot (-1)^j = \begin{cases} \sigma_{n-1}, & \text{if } n \text{ even}, \\ 0, & \text{if } n \text{ odd}. \end{cases}$$

We see that the singular chain complex is of the form

$$\ldots \mathbb{Z} \cdot \sigma_4 \xrightarrow{\cong} \mathbb{Z} \cdot \sigma_3 \xrightarrow{=0} \mathbb{Z} \cdot \sigma_2 \xrightarrow{\cong} \mathbb{Z} \cdot \sigma_1 \xrightarrow{=0} \mathbb{Z} \cdot \sigma_0 \to 0.$$

The lemma follows almost immediately from this observation. ■

Example. One of our goals is to calculate the homology groups of the spheres. Now we can at least calculate the homology groups of the 0-dimensional sphere $S^0 = \{\pm 1\}$. More precisely, by the Singleton-H$_*$-Lemma 35.11 and the Path-Component–H$_*$–Lemma 35.8 we have

$$\mathrm{H}_n(S^0) \cong \begin{cases} \mathbb{Z}^2, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

Later we will cleverly reduce the computation of the homology groups of all other spheres to this calculation.

path-connected topological spaces that one can do "by hand". For all other topological spaces we will need to work significantly harder, and this task will keep us busy for many chapters.

35.5. Singular 1-chains.

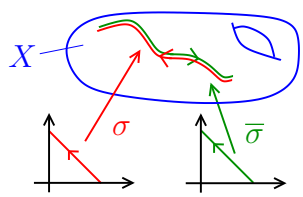
Notation. Let X be a topological space. Given a singular 1-simplex $\sigma\colon \Delta^1 \to X$ we denote by $\overline{\sigma}$ the singular 1-simplex that is given by $\overline{\sigma}(t, 1-t) = \sigma(1-t, t)$.

Lemma 35.12. (Singular 1-Simplex Lemma) Let X be a topological space and let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex.

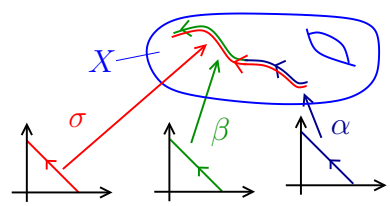
(1) σ is a cycle if and only if $\sigma((0,1)) = \sigma((1,0))$.
(2) The singular 1-chains $-\sigma$ and $\overline{\sigma}$ are homologous.
(3) Let $s \in (0,1)$. We consider the singular 1-simplices

$$\begin{array}{rcl} \alpha\colon \Delta^1 & \to & X \\ (t, 1-t) & \mapsto & \sigma(st, 1-st) \end{array} \quad \text{and} \quad \begin{array}{rcl} \beta\colon \Delta^1 & \to & X \\ (t, 1-t) & \mapsto & \sigma(s+t(1-s), 1-(s+t(1-s))). \end{array}$$

The singular 1-chains σ and $\alpha + \beta$ are homologous.



σ and $-\overline{\sigma}$ are homologous



σ and $\alpha + \beta$ are homologous

Proof. The first statement follows from the discussion of the boundary of a singular 1-simplex on page 302. We leave the proof of the remaining two statements as an exercise to the reader. ■

36. Homology and homotopies

36.1. Chain homotopies.

Definition. Let $f, g\colon C_* \to D_*$ be chain maps between two chain complexes (C_*, ∂_*) and (D_*, ∂_*). A chain homotopy $P = \{P_n\}_{n\in\mathbb{N}_0}$ between f and g is defined as a family $\{P_n\colon C_n \to D_{n+1}\}_{n\in\mathbb{N}_0}$ of homomorphisms such that for each $n \in \mathbb{N}_0$ we have

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; f_n - g_n\colon C_n \to D_n.$$

If there exists a chain homotopy between chain maps f and g, then we say that f and g are chain homotopic and we write $f \simeq g$.

Lemma 36.1. (Chain Homotopy–H_*-Lemma) Let $f, g\colon C_* \to D_*$ be two chain maps between chain complexes. If f and g are chain homotopic, then $f_* = g_*\colon \mathrm{H}_n(C) \to \mathrm{H}_n(D)$ for all $n \in \mathbb{N}_0$.

Proof. Let $P = \{P_n\}_{n\in\mathbb{N}_0}$ be a chain homotopy between f and g. Then for any cycle $z \in C_n$ we get the equalities

$$\begin{aligned} f(z) - g(z) \;=\; (f-g)(z) \;&=\; (\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n)(z) \\ &=\; (\partial_{n+1} \circ P_n)(z) + P_{n-1}(\underbrace{\partial_n(z)}_{=0}) \;=\; \partial(P_n(z)), \end{aligned}$$

i.e. $f(z)$ and $g(z)$ are homologous. ■

Lemma 36.2. (Chain Homotopy-Lemma) Let C_* and D_* be two chain complexes.

(1) Chain homotopy defines an equivalence relation on the set of all chain maps from C_* to D_*.

(2) Let $f, f'\colon C_* \to D_*$ and $g, g'\colon D_* \to E_*$ be chain maps. Then

$$f \simeq f' \;\text{ and }\; g \simeq g' \quad\Longrightarrow\quad g \circ f \simeq g' \circ f'.$$

Proof. We leave the elementary proof of the lemma to the reader. ■

Definition. We define the chain homotopy category $\mathcal{HomCplx}$

$$\begin{aligned} \mathrm{Ob}(\mathcal{HomCplx}) \;&:=\; \text{all chain complexes,} \\ \mathrm{Mor}(C_*, D_*) \;&:=\; \text{set of chain homotopy equivalence classes of chain maps } C_* \to D_* \end{aligned}$$

with the composition

$$\begin{array}{ccc} \mathrm{Mor}(C_*, D_*) \times \mathrm{Mor}(D_*, E_*) & \to & \mathrm{Mor}(C_*, D_*) \\ ([f], [g]) & \mapsto & [g \circ f]. \end{array}$$

Definition. Let C_* and D_* be two chain complexes.

(1) A chain map $f\colon C_* \to D_*$ is called a chain homotopy equivalence between C_* and D_*, if it admits a chain homotopy inverse, i.e. a chain map $g\colon D_* \to C_*$ with $g \circ f \simeq \mathrm{id}_{C_*}$ and $f \circ g \simeq \mathrm{id}_{D_*}$.

(2) If there exists a chain homotopy equivalence between C_* and D_*, then we say that the chain complexes C_* and D_* are chain homotopy equivalent and we write $C_* \simeq D_*$.

Lemma 36.3. (Chain Homotopy Equivalence-Reformulation Lemma) Let C_* and D_* be two chain complexes and let $f\colon C_* \to D_*$ be a chain map. The following statements are equivalent:

(1) The element $[f] \in \mathrm{Mor}_{\mathcal{HomCplx}}(C_*, D_*)$ is an isomorphism in the chain homotopy category $\mathcal{HomCplx}$.
(2) f is a chain homotopy equivalence.

Proof. As in the case of the proof of the Homotopy Equivalence-Reformulation Lemma 9.4 the statements follow almost immediately from the definitions. ■

Corollary 36.4. (Chain Homotopy Equivalence-H_*-Corollary) If $f\colon C_* \to D_*$ is a chain homotopy equivalence between two chain complexes, then for each $n \in \mathbb{N}_0$ the induced map $f_*\colon \mathrm{H}_n(C) \to \mathrm{H}_n(D)$ is an isomorphism.

Proof. First note that it follows from the Chain Homotopy–H_*-Lemma 36.1 that the homology functor $\mathrm{H}_n\colon \mathcal{ChCplx} \to \mathcal{AbGroup}$ from the category of chain complexes to the category of abelian groups factors through a functor $\mathcal{HomCplx} \to \mathcal{AbGroup}$ from the homotopy category of chain complexes to the category of abelian groups. It follows from this observation, the Chain Homotopy Equivalence-Reformulation Lemma 36.3 and the fact that functors send isomorphisms to isomorphisms that chain homotopies give rise to isomorphisms of abelian groups. ■

36.2. Homology and homotopic maps.

Proposition 36.5. (Homotopy-H_*-Proposition) Let X and Y be two topological spaces and let $f, g\colon X \to Y$ be two continuous maps. If f and g are homotopic, then the following hold:

(1) The induced maps f_* and g_* from $\mathrm{C}_*(X)$ to $\mathrm{C}_*(Y)$ are chain homotopic.
(2) For each $n \in \mathbb{N}_0$ we have $f_* = g_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$.

Corollary 36.6. (Homotopy Category-H_*-Functor Corollary) Each of our homology functors $\mathrm{H}_n\colon \mathcal{Top} \to \mathcal{AbGroup}$ defines a functor $\mathrm{H}_n\colon \mathcal{HomTop} \to \mathcal{AbGroup}$ from the homotopy category of topological spaces to the category of abelian groups.

Proof. This statement follows immediately from the Homotopy-H_*-Proposition 36.5 and the definitions. ■

Corollary 36.7. (Homotopy Equivalence-H_*-Corollary) Let X and Y be topological spaces and let $n \in \mathbb{N}_0$.

(1) (a) If $f\colon X \to Y$ is a homotopy equivalence, then the induced map $f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ is an isomorphism.
    (b) If X is a deformation retract of Y, then the inclusion induced map $\mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ is an isomorphism.
(2) If X and Y are homotopy equivalent, then $\mathrm{H}_n(X) \cong \mathrm{H}_n(Y)$.

Proof.

(1) (a) This statement follows immediately from the Homotopy Category-H_*-Functor Corollary 36.6 and the fact that functors send isomorphisms to isomorphisms.
    (b) This statement follows immediately from (1a) and the Deformation Retract–Homotopy Equivalence Lemma 9.3.

(2) This statement follows immediately from (1a).

■

Corollary 36.8. (Contractible-H$_*$-Corollary) If X is contractible (i.e. if X is homotopy equivalent to a singleton), then we have $\mathrm{H}_0(X) \cong \mathbb{Z}$ and we have $\mathrm{H}_n(X) = 0$ for all $n \in \mathbb{N}$.

Proof. This statement follows from the Homotopy Equivalence-H$_*$-Corollary 36.7 (2) and the Singleton-H$_*$-Lemma 35.11. ■

Example. On page 111 we saw that every convex subset of $\mathbb{R}^n$, e.g. the open disk B^n, the closed disk $\overline{B}^n$, any simplex or $\mathbb{R}^n$, is contractible. Thus it follows from the Contractible-H$_*$-Corollary 36.8, that for any $i \in \mathbb{N}_0$ we have

$$\mathrm{H}_i(\underbrace{\text{convex subset of } \mathbb{R}^n}_{\text{e.g. } \mathbb{R}^n,\ \overline{B}^n,\ B^n}) \ \cong\ \begin{cases} \mathbb{Z}, & \text{if } i = 0, \\ 0, & \text{if } i > 0. \end{cases}$$

36.3. Proof of the Homotopy-H$_*$-Proposition 36.5.

Notation. Let $n \in \mathbb{N}_0$.

(1) For $i = 0, \dots, n$ we write

$$\begin{aligned} v_i &:= (0, \dots, 1, 0, \dots, 0) \times \{0\} \ \in\ \mathbb{R}^{n+1} \times \{0\}, \\ \text{and} \qquad w_i &:= (0, \dots, 1, 0, \dots, 0) \times \{1\} \ \in\ \mathbb{R}^{n+1} \times \{1\}. \end{aligned}$$

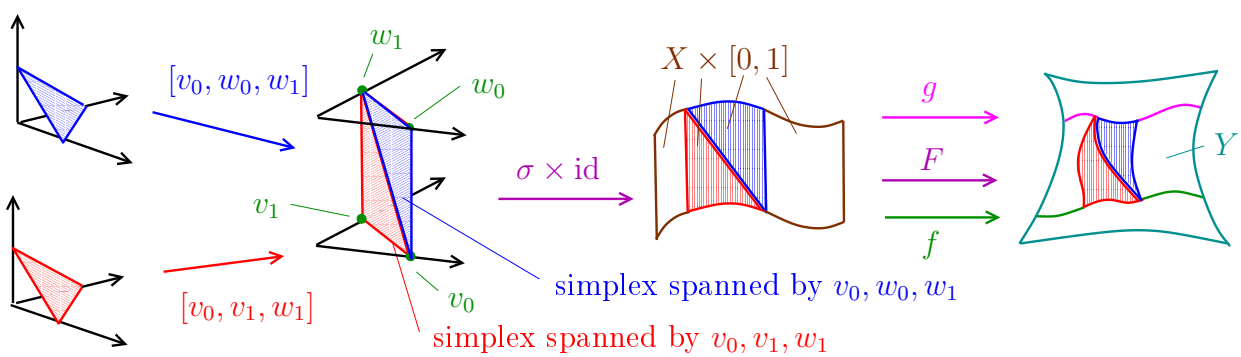
(2) For $a_0, \dots, a_k \in \Delta^n \times [0,1]$ we consider the map

$$\begin{aligned} [a_0, \dots, a_k] \colon \Delta^k &\to \Delta^n \times [0,1] \\ (\lambda_0, \dots, \lambda_k) &\mapsto \sum_{j=0}^{k} \lambda_j \cdot a_j. \end{aligned}$$

The map $[a_0, \dots, a_k]$ thus sends the standard k-simplex Δ^k to the simplex spanned by $a_0, \dots, a_k$ in $\Delta^n \times [0,1]$.

(3) We set

$$\Omega_n \ := \ \sum_{j=0}^{n} (-1)^j \cdot \underbrace{[v_0, \dots, v_j, w_j, \dots, w_n]}_{\text{map } \Delta^{n+1} \to \Delta^n \times [0,1]} \ \in\ \mathrm{C}_{n+1}(\Delta^n \times [0,1]).$$



Lemma 36.9. (Prism Boundary Lemma) Let $n \in \mathbb{N}_0$. Given $t \in \{0,1\}$ we write

$$\begin{array}{rcl} \eta_t \colon \Delta^n & \to & \Delta^n \times [0,1] \\ x & \mapsto & (x,t). \end{array}$$

With this notation we have

$$\partial_{n+1}(\Omega_n) + \sum_{j=0}^{n} (-1)^j \cdot (i_j^n \times \mathrm{id}_{[0,1]})_*\big(\underbrace{\Omega_{n-1}}_{\in \mathrm{C}_n(\Delta^{n-1}\times[0,1])} \big) \;=\; \eta_1 - \eta_0 \;\in\; \mathrm{C}_n(\Delta^n \times [0,1]).$$

Proof. The proof of the lemma is basically an elementary, albeit slightly confusing calculation. First note that

$$\begin{array}{rcl} \partial\Omega_n & = & \partial\Big(\sum\limits_{j=0}^{n} (-1)^j \cdot [v_0, \ldots, v_j, w_j, \ldots, w_n] \Big) \\ & = & \sum\limits_{k=0}^{n+1} (-1)^k \cdot \sum\limits_{j=0}^{n} (-1)^j \cdot [v_0, \ldots, v_j, w_j, \ldots, w_n] \circ i_k^{n+1}. \end{array}$$

It is straightforward to see that for any $k \in \{0, \ldots, n+1\}$ we have

$$[v_0, \ldots, v_j, w_j, \ldots, w_n] \circ i_k^{n+1} \;=\; \begin{cases} [v_0, \ldots, \widehat{v_k}, \ldots, v_j, w_j, \ldots, w_n], & \text{if } k \le j, \\ [v_0, \ldots, v_j, w_j, \ldots, \widehat{w_{k-1}}, \ldots, w_n], & \text{if } k > j. \end{cases}$$

Thus it follows that

$$\begin{array}{rcl} \partial\Omega_n & = & \sum\limits_{0 \le k \le j \le n} (-1)^k \cdot (-1)^j \cdot [v_0, \ldots, \widehat{v_k}, \ldots, v_j, w_j, \ldots, w_n] \\ & & + \sum\limits_{n+1 \ge k > j \ge 0} (-1)^k \cdot (-1)^j \cdot [v_0, \ldots, v_j, w_j, \ldots, \widehat{w_{k-1}}, \ldots, w_n]. \end{array}$$

Now we split the first sum into the cases $k = j$ and $k < j$ and we split the second sum into the cases $k = j+1$ and $k > j+1$. This leads us to the following equality:

$$\begin{array}{rcl} \partial\Omega_n & = & \sum\limits_{k=0}^{n} (-1)^{2k} \cdot [v_0, \ldots, v_{k-1}, w_k, \ldots, w_n] \\ & & + \sum\limits_{k=1}^{n+1} (-1)^{2k+1} \cdot [v_0, \ldots, v_{k-1}, w_k, \ldots, w_n] \\ & & + \sum\limits_{0 \le k < j \le n} (-1)^k \cdot (-1)^j \cdot [v_0, \ldots, \widehat{v_k}, \ldots, v_j, w_j, \ldots, w_n] \\ & & + \sum\limits_{n+1 \ge k > j+1 \ge 1} (-1)^k \cdot (-1)^j \cdot [v_0, \ldots, v_j, w_j, \ldots, \widehat{w_{k-1}}, \ldots, w_n]. \end{array}$$

Next we consider the first two sums. The summands cancel each other, except for the summands for $k = 0$ and $k = n+1$. Thus we obtain that

$$\begin{array}{rcl} \partial\Omega_n & = & [w_0, \ldots, w_n] - [v_0, \ldots, v_n] \\ & & + \sum\limits_{0 \le k < j \le n} (-1)^k \cdot (-1)^j \cdot [v_0, \ldots, \widehat{v_k}, \ldots, v_j, w_j, \ldots, w_n] \\ & & + \sum\limits_{n+1 \ge k > j+1 \ge 1} (-1)^k \cdot (-1)^j \cdot [v_0, \ldots, v_j, w_j, \ldots, \widehat{w_k}, \ldots, w_n]. \end{array} \qquad (*)$$

The first summand is just η_1 and the second is $-\eta_0$. It remains to show that the remaining summand corresponds to the remaining term in the formula. Thus we continue our proof

with the discussion of the one term in the statement of the lemma that we have not studied yet. We have

$$\sum_{j=0}^{n}(-1)^j\cdot(i_j^n\times\mathrm{id}_{[0,1]})_*(\Omega_{n-1}) \underset{\substack{\uparrow \\ \text{definition of } (i_j^n\times\mathrm{id}_{[0,1]})_* \text{ and substitution } r=j}}{=} \sum_{r=0}^{n}\sum_{s=0}^{n-1}(-1)^r\cdot(-1)^s\cdot(i_r^n\times\mathrm{id}_{[0,1]})\circ[v_0,\ldots,v_s,w_s,\ldots,w_{n-1}].$$

It follows from an elementary (but arguably confusing) argument, using only the definitions, that we have the following equalities of maps $\Delta^n\to\Delta^n\times[0,1]$:

$$(i_r^n\times\mathrm{id}_{[0,1]})\circ[v_0,\ldots,v_s,w_s,\ldots,w_{n-1}] = \begin{cases}[v_0,\ldots,\widehat{v_r},\ldots,v_{s+1},w_{s+1},\ldots,w_n], & \text{if } r\le s,\\ [v_0,\ldots,v_s,w_s,\ldots,\widehat{w_{r+1}},\ldots,w_n], & \text{if } r>s.\end{cases}$$

Thus we see that

$$\begin{aligned}\sum_{j=0}^{n}(-1)^j\cdot(i_j^n\times\mathrm{id}_{[0,1]})_*(\Omega_{n-1}) &= \sum_{0\le r\le s\le n-1}(-1)^s\cdot(-1)^r\cdot[v_0,\ldots,\widehat{v_r},\ldots,v_{s+1},w_{s+1},\ldots,w_n]\\ &\quad+\sum_{n\ge r>s\ge 0}(-1)^s\cdot(-1)^r\cdot[v_0,\ldots,v_s,w_s,\ldots,\widehat{w_{r+1}},\ldots,w_n].\end{aligned}$$

In the first term we perform the substitutions $r=k$ and $s=j-1$ and in the second term we perform the substitutions $r=k-1$, $s=j$. We obtain the following equality:

$$\begin{aligned}\sum_{j=0}^{n}(-1)^j\cdot(i_j^n\times\mathrm{id}_{[0,1]})_*(\Omega_{n-1}) &= \sum_{0\le k\le j-1\le n-1}(-1)^{j-1}\cdot(-1)^k\cdot[v_0,\ldots,\widehat{v_k},\ldots,v_j,w_j,\ldots,w_n]\\ &\quad+\sum_{n\ge k-1>j\ge 0}(-1)^j\cdot(-1)^{k-1}\cdot[v_0,\ldots,v_j,w_j,\ldots,\widehat{w_k},\ldots,w_n].\end{aligned}$$

We see immediately that these terms are just the negatives of the corresponding terms in the above equation $(*)$. ■

Proof of the Homotopy-H$_*$-Proposition 36.5. Let X be a topological space and let $F\colon X\times[0,1]\to Y$ be a homotopy from a map f to a map g.

(1) This statement follows immediately from the following claim.

Claim. Given any $n\in\mathbb{N}_0$ we define

$$P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(Y)$$

via

$$(\sigma\colon\Delta^n\to X) \mapsto (F\circ(\sigma\times\mathrm{id}_{[0,1]}))_*(\underset{\substack{\uparrow \\ \text{we defined } \Omega_n\in\mathrm{C}_{n+1}(\Delta^n\times[0,1]) \\ \text{on page 312}}}{\Omega_n}).$$

For any $n\in\mathbb{N}_0$ we have

$$\partial P_n + P_{n-1}\partial = g_* - f_*$$

as maps from $\mathrm{C}_n(X)$ to $\mathrm{C}_n(Y)$.

Proof. As always it suffices to verify the equality for each singular n-simplex. Thus let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. We calculate that

$$\begin{aligned}
(\partial P_n + P_{n-1}\partial)(\sigma) &= \partial(P_n(\sigma)) + P_{n-1}\Big(\sum_{j=0}^{n}(-1)^j\cdot\sigma\circ i_j^n\Big)\\
&= \partial(P_n(\sigma)) + \sum_{j=0}^{n}(-1)^j\cdot P_{n-1}(\sigma\circ i_j^n)\\
&= \partial((F\circ(\sigma\times \mathrm{id}_{[0,1]}))_*(\Omega_n)) + \sum_{j=0}^{n}(-1)^j\cdot(F\circ(\sigma\circ_j^n\times \mathrm{id}_{[0,1]}))_*(\Omega_{n-1})\\
&\underset{\uparrow}{=} (F\circ(\sigma\times \mathrm{id}_{[0,1]}))_*\Big(\partial(\Omega_n)) + \sum_{j=0}^{n}(-1)^j\cdot(i_j^n\times \mathrm{id}_{[0,1]})_*(\Omega_{n-1})\Big)
\end{aligned}$$

by the C_*–Functor Lemma 35.5 (1) and since we have $(\sigma\times\mathrm{id})\circ(i_j^n\times\mathrm{id}) = (\sigma\circ i_j^n)\times\mathrm{id}$ as maps $\Delta^n\times[0,1]\times Y\times[0,1]$

$$\underset{\uparrow}{=} (F\circ(\sigma\times \mathrm{id}_{[0,1]}))_*(\eta_1-\eta_0) \underset{\uparrow}{=} f\circ\sigma - g\circ\sigma = g_*(\sigma) - f_*(\sigma).$$

by the Prism Boundary Lemma 36.9 since F is a homotopy from f to g ⊞

(2) This statement follows immediately from Statement (1) together with the Chain Homotopy–H_*-Lemma 36.1. ■

Exercise 36.1. Let $\varphi\colon A\to B$ be a homomorphism of free abelian groups and let $C\subset A$ and $D\subset B$ be subgroups such that the natural map $C/D\to A/B$ is an isomorphism. Show that the chain complex $0\to A\xrightarrow{\varphi} B\to 0$ is chain homotopy equivalent to the chain complex $0\to C\xrightarrow{\varphi} D\to 0$.

37. Exact sequences and long exact sequences

37.1. Exact sequences.

Definition. A sequence of homomorphisms of abelian groups

$$\ldots \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \to \ldots$$

is called exact, if for each n we have

$$\mathrm{im}(f_{n+1}\colon A_{n+1} \to A_n) \;=\; \ker(f_n\colon A_n \to A_{n-1}).$$

Lemma 37.1. (Exactness Lemma)

(1) In any exact sequence the composition of two consecutive maps is the zero map.

(2) We have the following equivalences of statements:

$$\begin{aligned}
&\text{the sequence } 0 \to A \xrightarrow{f} B \quad \text{is exact} &&\iff f \text{ is injective}\\
&\text{the sequence } A \xrightarrow{f} B \to 0 \text{ is exact} &&\iff f \text{ is surjective}\\
&\text{the sequence } 0 \to A \xrightarrow{f} B \to 0 \text{ is exact} &&\iff f \text{ is an isomorphism.}
\end{aligned}$$

(3) Given a monomorphism $i\colon A \to B$ and an epimorphisms $p\colon B \to C$ of groups we have

$$\begin{aligned}
&0 \to A \xrightarrow{i} B \xrightarrow{f} C \text{ is exact} &&\iff i \text{ induces an isomorphism } A \xrightarrow{\cong} \ker(f)\\
&A \xrightarrow{f} B \xrightarrow{p} C \to 0 \text{ is exact} &&\iff p \text{ descends to an isomorphism } \mathrm{coker}(f) \xrightarrow{\cong} C.
\end{aligned}$$

Proof. The lemma follows easily from the definitions. We leave it to the reader to provide the details. ■

Example. Let $f\colon A \to B$ be a homomorphism of groups. It follows from the Exactness Lemma 37.1 that the following sequence is exact

$$0 \;\to\; \ker(f) \;\to\; A \xrightarrow{f} B \;\to\; \mathrm{coker}(f) \;\to\; 0.$$

37.2. Split exact sequences.

Definition. We say that a short exact sequence

$$0 \;\to\; A \xrightarrow{i} B \xrightarrow{p} C \;\to\; 0$$

of groups splits, if there exists a splitting $s\colon C \to B$, i.e. a homomorphism $s\colon C \to B$ such that $p \circ s = \mathrm{id}_C$.

Example.

(1) The short exact sequence

$$0 \;\to\; \mathbb{Z} \xrightarrow{k \mapsto (k,0)} \mathbb{Z} \oplus \mathbb{Z}_n \xrightarrow{(k,l) \mapsto l} \mathbb{Z}_n \;\to\; 0$$

splits. In fact a splitting is given by the homomorphism $b \mapsto (0, b)$. On the other hand the short exact sequence

$$0 \;\to\; \mathbb{Z} \xrightarrow{k \mapsto n \cdot k} \mathbb{Z} \;\to\; \mathbb{Z}_n \;\to\; 0.$$

does not split.

(2) Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups. It follows from the Free Abelian Group–Splitting Lemma 18.5, that if the last group C is a free abelian group, then the short exact sequence splits.

Lemma 37.2. (Splitting Lemma) Let

$$0 \;\to\; A \;\xrightarrow{i}\; B \;\xrightarrow{p}\; C \;\to\; 0$$

be a short exact sequence of abelian groups. The following three statements are equivalent:

(1) The short exact sequence splits.
(2) There exists a homomorphism $t\colon B \to A$ such that $t \circ i = \mathrm{id}_A$.
(3) There exists an isomorphism $\Phi\colon B \to A \times C$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & B \arrow[dr, "p"] \arrow[dd, "\cong\ \Phi"'] & & \\
0 \arrow[r] & A \arrow[ur, "i"] \arrow[dr, "a \mapsto (a,0)"'] & & C \arrow[r] & 0 \\
 & & A \times C \arrow[ur, "(a,c) \mapsto c"'] & &
\end{tikzcd}$$

where the lower left diagonal map is the obvious inclusion and the lower right diagonal map is the natural projection.

Proof. Let

$$0 \;\to\; A \;\xrightarrow{i}\; B \;\xrightarrow{p}\; C \;\to\; 0$$

be a short exact sequence of abelian groups.

(1)⇒(3). Suppose that $s\colon C \to B$ is a splitting, i.e. suppose that s is a homomorphism such that $p \circ s = \mathrm{id}_C$. We consider the map

$$\begin{array}{rcl} \Psi\colon A \times C & \to & B \\ (a,c) & \mapsto & i(a) + s(c). \end{array}$$

Using the fact that B is abelian one can easily verify that the map is a homomorphism.

Claim. The above homomorphism $\Psi\colon A \times C \to B$ is an isomorphism and the inverse $\Phi := \Psi^{-1}\colon B \to A \times C$ has all desired properties.

Proof. First we show that Ψ is an epimorphism. Let $b \in B$. Note that

$$p(b - s(p(b))) \;\underset{\substack{\uparrow \\ \text{since } p \text{ is a homomorphism}}}{=}\; p(b) - \underbrace{(p \circ s)}_{=\mathrm{id}}(p(b)) \;=\; p(b) - p(b) \;=\; 0.$$

It follows from this calculation and the exactness of the short exact sequence that there exists a unique $a \in A$ with $i(a) = b - s(p(b))$. Then $\Psi(a, p(b)) = i(a) + s(p(b)) = b$.

Next we show that Ψ is a monomorphism. Let $(a,c) \in A \times B$ with $\Psi(a,c) = 0$. Then

$$0 \;=\; p(0) \;=\; p(\Psi(a,c)) \;=\; p(i(a) + s(c)) \;=\; \underbrace{(p \circ i)}_{=0}(a) + \underbrace{(p \circ s)}_{=\mathrm{id}_C}(c) \;=\; c.$$

Thus we have shown that $c = 0$. It follows that $0 = \Psi(a,c) = i(a) + s(c) = i(a)$. Since i is a monomorphism we see that $a = 0$. In summary we have shown that $(a,c) = 0$.

It remains to show that the following diagram commutes:

$$\begin{tikzcd}
 & & B \arrow[dr, "p"] & & \\
0 \arrow[r] & A \arrow[ur, "i"] \arrow[dr, "a\mapsto(a,0)"'] & & C \arrow[r] & 0 \\
 & & A\times C \arrow[uu, "\Psi", "\cong"'] \arrow[ur, "(a,c)\mapsto c"'] & &
\end{tikzcd}$$

It is clear that the left hand triangle commutes. Furthermore it follows from $p \circ i = 0$ and $p \circ s = \mathrm{id}_C$ that the triangle to the right commutes.
(2)⇒(3). Let $t\colon B \to A$ be a homomorphism such that $t \circ i = \mathrm{id}_A$. We define

$$\begin{array}{rcl} \Phi\colon B & \to & A \times C \\ b & \mapsto & (t(b), p(b)). \end{array}$$

As in proof of (1)⇒(2) it follows easily from the exactness of the initial sequence that this map is in fact an isomorphism. It is straightforward to verify that Φ has all the other desired properties.
(3)⇒(1),(2). Now suppose we are given an isomorphism $\Phi\colon B \to A \times C$ such that the given diagram commutes. It follows immediately from the definitions and given properties that the maps

$$B \xrightarrow{\Phi} A \times C \xrightarrow{(a,c)\mapsto a} A \quad \text{and} \quad C \xrightarrow{c\mapsto(0,c)} A \times C \xrightarrow{\Phi^{-1}} B$$

have the required properties. ■

37.3. **Long exact sequence of homology groups.** In the previous two sections we studied (short) exact sequences of abelian groups. In this section we will study short exact sequences of chain complexes and we will see that they give rise to an exact sequence of homology groups.

Definition. A short exact sequence of chain complexes

$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$$

consists of chain complexes A_*, B_*, C_* and chain maps $i\colon A_* \to B_*$ and $p\colon B_* \to C_*$ such that for each n the maps

$$0 \to A_n \xrightarrow{i} B_n \xrightarrow{p} C_n \to 0$$

form a short exact sequence.

Definition. Let $0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$ be short exact sequence of chain complexes. Let $n \in \mathbb{N}_0$. We define a homomorphism

$$\partial_n\colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$$

as follows:
(0) Let $z \in \mathrm{H}_n(C)$.
(1) We pick an n-cycle $c \in C_n$ with $[c] = z$.
(2) Since $0 \to A_n \xrightarrow{i} B_n \xrightarrow{p} C_n \to 0$ is exact we know by the Exactness Lemma 37.1 that $p\colon B_n \to C_n$ is an epimorphism. Thus there exists a $b \in B_n$ with $p(b) = c$.
(3) We have

$$p(\partial b) \underset{\substack{\uparrow \\ \text{since } p \text{ is a chain map}}}{=} \partial p(b) = \partial c \underset{\substack{\uparrow \\ \text{since } c \text{ is a cycle}}}{=} 0.$$

(4) Since $0 \to A_{n-1} \xrightarrow{i} B_{n-1} \xrightarrow{p} C_{n-1} \to 0$ is exact it follows from (3) that there exists a unique $a \in A_{n-1}$ with $i(a) = \partial b$.

(5) We set $\partial_n(z) := [a] \in \mathrm{H}_{n-1}(A)$.

We refer to this map $\partial_n \colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$ as the connecting homomorphism.

(2) since $p \colon B_n \to C_n$ is an epimorphism
there exists a $b \in B_n$ with $p(b_n) = c$

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "i"] \arrow[dd] & B_n \arrow[r, "p"] \arrow[dd] & C_n \arrow[r] \arrow[dd] & 0 \\
 & & b \arrow[r, mapsto] \arrow[d, mapsto, "\partial_n"] & c \arrow[d, mapsto, "\partial_n"'] & \\
0 \arrow[r] & A_{n-1} \arrow[r] & B_{n-1} \arrow[r] & C_{n-1} \arrow[r] & 0 \\
 & a \arrow[uur, mapsto] & \partial_n b \arrow[r, mapsto] & 0 &
\end{tikzcd}$$

(1) we pick a cycle $c \in C_n$ with $[c] = z \in \mathrm{H}_n(C_*)$

since c is a cycle

(3) since the diagram commutes

(4) since the horizontal sequence is exact
there exists an $a \in A_{n-1}$ with $i(a) = \partial(b)$

Lemma 37.3. (Connecting Homomorphism Lemma)

(1) Let $0 \to A_* \to B_* \to C_* \to 0$ be a short exact sequence of chain complexes. For each $n \in \mathbb{N}_0$ the connecting homomorphism $\partial_n \colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$ is well-defined and it is indeed a homomorphism.

(2) The connecting homomorphism is natural, i.e. if

$$\begin{tikzcd}
0 \arrow[r] & A_* \arrow[r, "i"] \arrow[d, "a"] & B_* \arrow[r, "p"] \arrow[d, "b"] & C_* \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A'_* \arrow[r, "i"] & B'_* \arrow[r, "p"] & C'_* \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram of chain maps where the horizontal sequences are exact, then the following diagram commutes for every $n \in \mathbb{N}$:

$$\begin{tikzcd}
\mathrm{H}_n(C) \arrow[r, "\partial_n"] \arrow[d, "c_*"'] & \mathrm{H}_{n-1}(A) \arrow[d, "a_*"] \\
\mathrm{H}_n(C') \arrow[r, "\partial_n"] & \mathrm{H}_{n-1}(A')
\end{tikzcd}$$

Proof.

(1) We start out with the following claim.

Claim 1. The map
$$\partial_n \colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$$
is well-defined.

Proof. Let $z \in \mathrm{H}_n(C)$. We denote by $c \in C_n$, $b \in B_n$ and $a \in A_{n-1}$ the elements that arise in the definition of $\partial_n(z)$. We need to show that the element a is indeed a cycle and that the definition of $\partial_n(z)$ does not depend on the choices of c and b.

The proofs of these two statements are not particularly difficult and they are done via "diagram chasing". First we note that

$$i(\partial a) \underset{\substack{\uparrow \\ \text{since } i \text{ is a chain map}}}{=} \partial i(a) = \partial(\partial b) = 0.$$

Since i is injective we obtain that $\partial a = 0$, i.e. a is indeed a cycle. Now we show that the homology class $[a] \in \mathrm{H}_{n-1}(A)$ depends only on the homology class $z \in \mathrm{H}_n(C)$.

(a) If we chose a different element $\widetilde{b}$ in B_n with $p(\widetilde{b}) = p(b) = c$, then it follows from the fact that the horizontal sequences are exact that this element would be of the form $\widetilde{b} = b + i(a')$ for some $a' \in A_n$. But then we obtain the following equality:

$$\partial\widetilde{b} = \partial(b + i(a')) = \partial b + \partial(i(a')) \overset{\substack{\text{since } i \text{ is a chain map} \\ \downarrow}}{=} \underbrace{\partial b}_{=i(a)} + i(\partial a') = i(a + \partial a').$$

Since $[a] = [a + \partial a'] \in \mathrm{H}_{n-1}(A)$ this does not change the homology class.

(b) If we picked a different representative $\widetilde{c}$ for $z \in \mathrm{H}_n(C)$, then this representative would be of the form $\widetilde{c} = c + \partial c'$ for some $c' \in C_{n+1}$. Since the map $p\colon B_n \to C_n$ is an epimorphism there exists a $b' \in B_{n+1}$ with $p(b') = c'$. It follows that

$$p(b + \partial b') = p(b) + p(\partial b') \underset{\substack{\uparrow \\ \text{since } p \text{ is a chain map}}}{=} c + \partial p(b') = c + \partial c' = \widetilde{c}.$$

Since $\partial(b + \partial b') = \partial b$ we see that we have ended up with the same element in B_{n-1} as before. ⊞

Claim 2. The map

$$\partial_n\colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$$

is a homomorphism.

Proof. Let $[c_1], [c_2] \in \mathrm{H}_n(C)$. For $j = 1, 2$ we choose $b_j \in B_n$ as in the definition of the connecting homomorphism $\partial_n\colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$ and we denote by $a_j \in A_{n-1}$ the uniquely determined element with $i(a_j) = \partial b_j$. Since i and p are homomorphisms we also have $p(b_1 + b_2) = c_1 + c_2$ and $i(a_1 + a_2) = i(a_1) + i(a_2) = \partial b_1 + \partial b_2 = \partial(b_1 + b_2)$. It follows that

$$\partial_n([c_1 + c_2]) = [a_1 + a_2] = [a_1] + [a_2] = \partial_n([c_1]) + \partial_n([c_2]).$$ ⊞

(2) It follows easily from the definitions that the connecting homomorphisms are natural. ■

Proposition 37.4. (LES-Proposition)

(1) Let

$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$$

be a short exact sequence of chain complexes. The sequence

$$\ldots \to \mathrm{H}_n(A) \xrightarrow{i_*} \mathrm{H}_n(B) \xrightarrow{p_*} \mathrm{H}_n(C) \xrightarrow{\partial} \mathrm{H}_{n-1}(A) \xrightarrow{i_*} \mathrm{H}_{n-1}(B) \xrightarrow{p_*} \ldots$$

is exact. This sequence is called the long exact sequence in homology corresponding to the short exact sequence of chain complexes.

(2) If

$$\begin{tikzcd} 0 \arrow[r] & A_* \arrow[r, "i"] \arrow[d, "a"] & B_* \arrow[r, "p"] \arrow[d, "b"] & C_* \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & A'_* \arrow[r, "i"] & B'_* \arrow[r, "p"] & C'_* \arrow[r] & 0 \end{tikzcd}$$

is a commutative diagram of chain maps where the horizontal sequences are exact, then the following diagram commutes:

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{H}_n(A) \arrow[r, "i_*"] \arrow[d, "a_*"] & \mathrm{H}_n(B) \arrow[r, "p_*"] \arrow[d, "b_*"] & \mathrm{H}_n(C) \arrow[r, "\partial"] \arrow[d, "c_*"] & \mathrm{H}_{n-1}(A) \arrow[r] \arrow[d, "a_*"] & \dots \\ \dots \arrow[r] & \mathrm{H}_n(A') \arrow[r, "i_*"] & \mathrm{H}_n(B') \arrow[r, "p_*"] & \mathrm{H}_n(C') \arrow[r, "\partial"] & \mathrm{H}_{n-1}(A') \arrow[r] & \dots \end{tikzcd}$$

Proof of the LES-Proposition 37.4. First note that Statement (2) is an immediate consequence of the Chain Complex-H_*-Functor Lemma 35.1 (2) together with the Connecting Homomorphism Lemma 37.3. Thus let us turn to the actual meat, namely the proof of Statement (1).

Let

$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$$

be a short exact sequence of chain complexes. The sequence

$$\dots \to \mathrm{H}_{n+1}(C) \xrightarrow{\partial} \mathrm{H}_n(A) \xrightarrow{i_*} \mathrm{H}_n(B) \xrightarrow{p_*} \mathrm{H}_n(C) \xrightarrow{\partial} \mathrm{H}_{n-1}(A) \xrightarrow{i_*} \mathrm{H}_{n-1}(B) \xrightarrow{p_*} \dots$$

is exact. We examine the exactness at the three groups $\mathrm{H}_n(A), \mathrm{H}_n(B)$ and $\mathrm{H}_n(C)$. Put differently, at each of the three groups we want to show the following two statements:

(i) The image of the map to the left is contained in the kernel of the map to the right, or equivalently, the composition of the map to the left and the map to the right is zero,

(ii) The kernel of the map to the right is contained in the image of the map to the left.

Now we perform these steps:

(A) (i) Let $[c] \in \mathrm{H}_{n+1}(C)$. We want to show that $i_*(\partial[c]) = 0$. We choose $b \in B_{n+1}$ with $p(b) = c$ and we denote by $a \in A_n$ the element with $i(a) = \partial b$. Then we have

$$i(\partial[c]) \underset{\substack{\uparrow \\ \text{definition of } \partial\colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)}}{=} i([a]) = [\partial b] = 0 \in \mathrm{H}_n(B).$$

(ii) Let $[a] \in \ker(i_*)$. Then $i(a) = \partial b$ for some $b \in B_{n+1}$. It follows immediately from the definitions that $\partial([p(b)]) = [a]$.

(B) (i) From $p \circ i = 0$ it follows that $p_* \circ i_* = 0$, i.e. $\mathrm{im}(i_*) \subset \ker(p_*)$.

(ii) Let $b \in \ker(p_*)$. Then $p(b) = \partial c'$ for some $c' \in C_{n+1}$. Since p is surjective, there exists a $b' \in B_{n+1}$ with $p(b') = c'$. Since p is a chain map we see that $p(\partial b') = \partial p(b') = \partial c' = p(b)$. Thus it follows that $p(b - \partial b') = 0$, i.e. $b - \partial b' = i(a)$ for some $a \in A_n$. But then it follows that $[b] = [b - \partial b'] = [i(a)] \in \mathrm{im}(i_*)$.

(C) (i) Let $[b] \in \mathrm{H}_n(B)$. We have to show that $\partial(p_*([b])) = 0$. In the definition of $\partial p_*([b])$ we need to choose a $\widetilde{b} \in B_n$ with $[p(\widetilde{b})] = p_*([b])$. Of course we take $\widetilde{b} = b$. Since $\partial b = 0$ it follows immediately that $\partial p_*([b]) = 0$.

(ii) Let $[c] \in \ker(\partial\colon \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A))$. We choose $b \in B_n$ with $p(b) = c$. Recall that there exists a uniquely determined $a \in A_{n-1}$ with $i(a) = \partial b$. Since we have $[a] = \partial[c] = 0 \in \mathrm{H}_{n-1}(A)$ there exists an $a' \in A_n$ with $\partial a' = a$. It follows that

$$\partial(b - i(a')) \;=\; \partial b - \partial i(a') \;\underset{\substack{\uparrow \\ \text{since } i \text{ is a chain map}}}{=}\; \partial b - i(\partial a') \;=\; \partial b - i(a) \;=\; 0,$$

i.e. $b - i(a') \in B_n$ is a cycle. On the other hand note that we have the equalities $p(b - i(a')) = p(b) - (p \circ i)(a') = p(b)$. Thus we see that $p_*([b - i(a')]) = [c]$. We have thus shown that $\ker(\partial) \subset \mathrm{im}(p_*)$.

Thus we have proved the claim and therefore we have also completed the proof of the proposition. ■

Lemma 37.5. (Five Lemma) Let

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, two heads, "a"] & B \arrow[r, "j"] \arrow[d, "b", "\cong"'] & C \arrow[r, "k"] \arrow[d, "c"] & D \arrow[r, "l"] \arrow[d, "d", "\cong"'] & E \arrow[d, hook, "e"] \\
A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "l'"] & E'
\end{tikzcd}$$

be a diagram of groups (here for once we do not suppose that the groups are abelian) such that the horizontal sequences are exact and such that every squares commute (if all the groups are abelian, then it suffices that the squares commute up to a sign). If a is surjective, if b and d are isomorphisms, and if e is injective, then c is an isomorphism.

Proof. One can prove the lemma via a "diagram chase". We leave the pleasure of the chase as an exercise to the reader. ■

38. Relative homology

38.1. Definition of the relative homology groups.

Definition. Let (X, A) be a pair of topological spaces. We view each chain group $\mathrm{C}_n(A)$ as a subgroup of $\mathrm{C}_n(X)$. Thus we can define

$$\mathrm{C}_n(X, A) \;:=\; \mathrm{C}_n(X)/\mathrm{C}_n(A)$$

and we consider the boundary map

$$\begin{aligned} \partial_n \colon \mathrm{C}_n(X, A) &\to \mathrm{C}_{n-1}(X, A) \\ [c] &\mapsto [\partial_n c]. \end{aligned}$$

We refer to

$$\ldots \xrightarrow{\partial_{n+2}} \mathrm{C}_{n+1}(X, A) \xrightarrow{\partial_{n+1}} \mathrm{C}_n(X, A) \xrightarrow{\partial_n} \mathrm{C}_{n-1}(X, A) \xrightarrow{\partial_{n-1}} \ldots$$

as the chain complex of the pair (X, A). Given any $n \in \mathbb{N}_0$ we define the n-th relative homology group of (X, A) as follows:

$$\mathrm{H}_n(X, A) \;:=\; \mathrm{H}_n(\mathrm{C}_*(X, A)).$$

Examples.

(1) If $A = \varnothing$, then for all $n \in \mathbb{N}_0$ we have $\mathrm{C}_n(\varnothing) = 0$ and thus we have a natural isomorphism $\mathrm{C}_n(X, \varnothing) = \mathrm{C}_n(X)/\{0\} \xrightarrow{\cong} \mathrm{C}_n(X)$, i.e. we have a natural isomorphism $\mathrm{H}_n(X, \varnothing) \xrightarrow{\cong} \mathrm{H}_n(X)$. In the following we will use these natural isomorphisms to freely go back and forth between the notations $\mathrm{H}_n(X, \varnothing)$ and $\mathrm{H}_n(X)$.

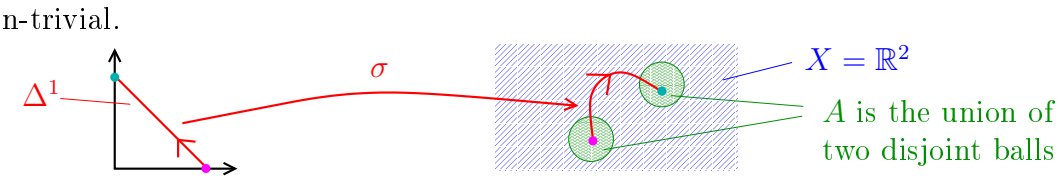
(2) Whereas it is often difficult to write down singular cycles in absolute chain complexes $\mathrm{C}_*(X)$, it can be surprisingly easy to write down cycles in a relative chain complex $\mathrm{C}_*(X, A)$: Let $\sigma \colon \Delta^n \to X$ be a continuous map with $\sigma(\partial\Delta^n) \subset A$. Note that

$$\partial_n(\sigma) \;=\; \sum_{j=0}^{n} (-1)^j \cdot \sigma \circ i_j^n \;\underset{\uparrow}{=}\; 0 \;\in\; \mathrm{C}_{n-1}(X, A) = \mathrm{C}_{n-1}(X)/\mathrm{C}_{n-1}(A).$$

since $\sigma \circ i_j^n \colon \Delta^{n-1}$ takes values in A we see that $\sigma \circ i_j^n \in \mathrm{C}_{n-1}(A)$

Thus we have shown that *any continuous map* $\sigma \colon \Delta^n \to X$ *with* $\sigma(\partial\Delta^n) \subset A$ *defines a cycle in the chain complex* $\mathrm{C}_n(X, A)$.

As an example we consider $X = \mathbb{R}^2$ with the subspace $A = B_1 \sqcup B_2$ that is given by the union of two disjoint closed disks in $\mathbb{R}^2$. Let $\sigma \colon \Delta^1 \to X$ be a singular 1-simplex with $\sigma(0, 1) \in B_1$ and $\sigma(1, 0) \in B_2$. It follows from the above discussion that σ is a cycle in $\mathrm{C}_*(X, A)$. Later, on page 326, we will see whether or not $[\sigma] \in \mathrm{H}_1(X, A)$ is non-trivial.



Definition.

(1) We refer to the category that is given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{PairTop}) &:= \text{all pairs of topological spaces,} \\ \mathrm{Mor}((X,A),(Y,B)) &:= \text{all continuous maps } f\colon X \to A \text{ with } f(A) \subset B \end{aligned}$$

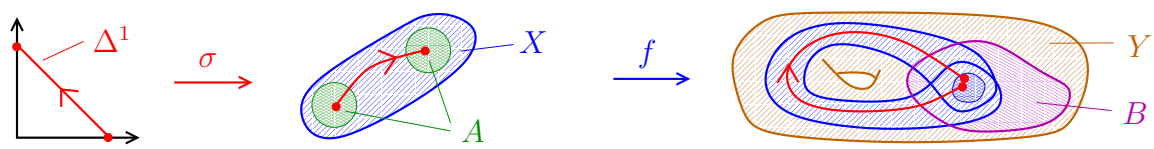
with the usual composition of maps as the category of pairs of topological spaces.

(2) Let $f\colon (X,A) \to (Y,B)$ be a continuous map of pairs of topological spaces, i.e. a continuous map $f\colon X \to Y$ with $f(A) \subset B$. If follows from $f(A) \subset B$ that the induced chain maps $f_*\colon \mathrm{C}_n(X) \to \mathrm{C}_n(Y)$ from the H_n–Functor Lemma 35.7 descends to a chain map

$$f_*\colon \mathrm{C}_n(X,A) \ \to\ \mathrm{C}_n(Y,B).$$

In particular we also get induced homomorphisms

$$f_*\colon \mathrm{H}_n(X,A) \ \to\ \mathrm{H}_n(Y,B).$$



Lemma 38.1. (Relative-H_*–Functor Lemma) The maps

$$\begin{aligned} (X,A) &\to (\mathrm{C}_*(X,A),\partial_*) \\ (f\colon (X,A) \to (Y,B)) &\mapsto (f_*\colon \mathrm{C}_*(X,A) \to \mathrm{C}_*(Y,B)) \end{aligned}$$

define a covariant functor from the category $\mathcal{PairTop}$ of pairs of topological spaces to the category $\mathcal{ChCplx}$ of chain complexes and for each $n \in \mathbb{N}_0$ the maps

$$\begin{aligned} (X,A) &\to \mathrm{H}_n(X,A) \\ (f\colon (X,A) \to (Y,B)) &\mapsto (f_*\colon \mathrm{H}_n(X,A) \to \mathrm{H}_n(Y,B)) \end{aligned}$$

define a covariant functor from the category $\mathcal{PairTop}$ of pairs of topological spaces to the category $\mathcal{AbGroup}$ of abelian groups.

Proof. The lemma follows almost immediately from the definitions. ■

38.2. The long exact sequence of a triple.

Definition.

(1) A triple of topological spaces is a triple (X,B,A) where X is a topological space and $A \subset B \subset X$ are subspaces.

(2) A map $f\colon (X,B,A) \to (X',B',A')$ between triples of topological spaces is a continuous map $f\colon X \to X'$ such that $f(B) \subset B'$ and $f(A) \subset A'$.

Proposition 38.2. (LES-of-Triple Proposition) Let X be a topological space and let $A \subset B \subset X$ be two subspaces. We denote by $i\colon (B,A) \to (X,A)$ and by $p\colon (X,A) \to (X,B)$ the obvious maps.

(1) For each $n \in \mathbb{N}_0$ the following map is a well-defined homomorphism:

$$\begin{array}{rcl} \partial_n\colon \mathrm{H}_n(X,B) & \to & \mathrm{H}_{n-1}(B,A) \\ \Big[\sum\limits_{j=1}^{k} s_j \cdot (\sigma_j\colon \Delta^n \to X)\Big] & \mapsto & \Big[\sum\limits_{j=1}^{k} s_j \cdot \underbrace{\partial\sigma_j}_{\substack{\uparrow \\ \text{singular } (n-1)\text{-chain in } B}}\Big]. \end{array}$$

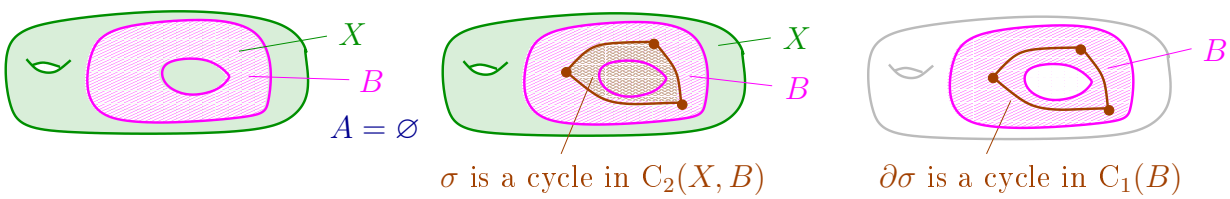
We refer to ∂_n as a connecting homomorphism of the triple (X,B,A).

(2) The connecting homomorphisms are natural.

(3) The maps

$$\ldots \to \mathrm{H}_n(B,A) \xrightarrow{i_*} \mathrm{H}_n(X,A) \xrightarrow{p_*} \mathrm{H}_n(X,B) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(B,A) \to \ldots$$

form an exact sequence. It is called the called the long exact sequence (LES) of the triple (X,B,A).



Proof.

(1) First note that it follows immediately from the definitions and the third isomorphism theorem in group theory that

$$0 \to \mathrm{C}_*(B,A) \xrightarrow{i_*} \mathrm{C}_*(X,A) \xrightarrow{p_*} \mathrm{C}_*(X,B) \to 0$$

is a short exact sequence of chain complexes.

Claim. The maps ∂_n introduced in Statement (1) of the propositions are precisely the connecting homomorphisms, as defined on page 319, of the above short exact sequence of chain complexes. In particular the maps ∂_n are well-defined homomorphisms.

Proof. The claim follows basically immediately from the definitions. ⊞

(2) It follows from the above claim together with the Connecting Homomorphism Lemma 37.3 that the connecting homomorphisms are natural.

(3) It follows from the above claim together with the LES-Proposition 37.4 that the given sequence is exact. ■

Notation. Let X be a topological space, let $A \subset X$ be a subspace and let $n \in \mathbb{N}_0$. We introduce the following homomorphisms

$$\begin{tikzcd} \mathrm{H}_n(X) \arrow[r, "="] \arrow[rr, bend right, "=:\text{я}:=\text{Я}_{X,A}"'] & \mathrm{H}_n(X,\varnothing) \arrow[r] & \mathrm{H}_n(X,A) \end{tikzcd}$$

Proposition 38.3. (LES-of-Pair Proposition)

(1) Let X be a topological space and let $B \subset X$ be a subspace. The connecting homomorphism of the LES-of-Triple Proposition 38.2 applied to $A = \varnothing$ gives rise to an exact sequence

$$\begin{array}{ccccccccccc} \dots & \to & \mathrm{H}_n(B) & \xrightarrow{i_*} & \mathrm{H}_n(X) & \xrightarrow{\text{я}} & \mathrm{H}_n(X,B) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(B) & \to & \dots \\ & & & & & & \Big[\sum\limits_{j=1}^{k} s_j \cdot \sigma_j\Big] & \mapsto & \Big[\sum\limits_{j=1}^{k} s_j \cdot \partial\sigma_j\Big]. & & \end{array}$$

(2) The connecting homomorphisms in (1) are natural.

Proof. This proposition follows immediately from the LES-of-Triple Proposition 38.2 applied to $A = \varnothing$. ■

Example. As on page 323 we consider topological space let $X = \mathbb{R}^2$ and the subspace $A = B_1 \sqcup B_2$ that is given by the disjoint union of two closed disks in $\mathbb{R}^2$. We want to determine the relative homology groups of (X, A). We denote by $i\colon A \to X$ the inclusion map. By the LES-of-Pair Proposition 38.3 we have a long exact sequence

$$\mathrm{H}_2(X) \to \mathrm{H}_2(X,A) \to \mathrm{H}_1(A) \xrightarrow{i_*} \mathrm{H}_1(X) \to \mathrm{H}_1(X,A) \to \mathrm{H}_0(A) \to \mathrm{H}_0(X) \to \mathrm{H}_0(X,A) \to 0.$$

By the Contractible-H_*-Corollary 36.8 together with the Path-Component–H_*–Lemma 35.8 we have $\mathrm{H}_i(X) = \mathrm{H}_i(\mathbb{R}^2) = 0$ and also $\mathrm{H}_i(A) = \mathrm{H}_i(B_1) \oplus \mathrm{H}_i(B_2) = 0$ for $i \geq 1$. Thus we see from the above long exact sequence that $\mathrm{H}_i(X,A) = 0$ for $i \geq 2$.

Next let $\epsilon\colon \mathrm{H}_0(\mathbb{R}^2) \to \mathbb{Z}$ be the augmentation map. Since $\mathbb{R}^2$ is path-connected we know by the H_0-Proposition 35.10 (1) that this map is an isomorphism. Similarly we denote by $\mu_1\colon \mathrm{H}_0(B_1) \to \mathbb{Z}$ and $\mu_2\colon \mathrm{H}_0(B_2) \to \mathbb{Z}$ the isomorphisms that are given by the augmentation maps. We obtain the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}_1(X,A) \arrow[r, "\partial"] \arrow[d, "\mathrm{id}"'] & \mathrm{H}_0(A) \arrow[r, "i_*"] \arrow[d, "\mu_1\oplus\mu_2"', "\cong"] & \mathrm{H}_0(X) \arrow[r] \arrow[d, "\cong"', "\epsilon"] & \mathrm{H}_0(X,A) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & \mathrm{H}_1(X,A) \arrow[r] & \mathbb{Z}^2 \arrow[r] & \mathbb{Z} \arrow[r] & \mathrm{H}_0(X,A) \arrow[r] & 0. \end{tikzcd}$$

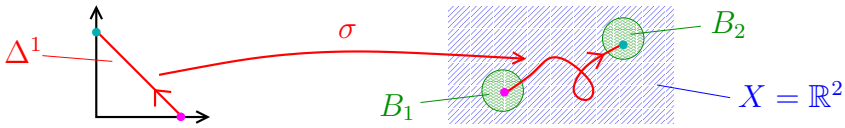
It follows easily from the Augmentation Lemma 35.9 that the homomorphism $\mathbb{Z}^2 \to \mathbb{Z}$ is given by the matrix $(1\ 1)$. It follows that

$$\mathrm{H}_1(X,A) \underset{\substack{\uparrow \\ \text{by the Exactness Lemma 37.1 (3)}}}{\cong} \ker\big(\mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z}\big) \cong \mathbb{Z} \qquad \text{and} \qquad \mathrm{H}_0(X,A) \underset{\substack{\uparrow \\ \text{by the Exactness Lemma 37.1 (3)}}}{\cong} \operatorname{coker}\big(\mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z}\big) \cong 0.$$

Finally we want to find an explicit generator of $\mathrm{H}_1(X,A) \cong \mathbb{Z}$. As on page 323 let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex with $\sigma(0,1) \in B_2$ and $\sigma(1,0) \in B_1$. Note that

$$((\mu_1 \oplus \mu_2) \circ \partial_1)[\sigma]) \underset{\substack{\uparrow \\ \text{see the description of the connecting} \\ \text{homomorphism } \partial_1\colon \mathrm{H}_1(X,A) \to \mathrm{H}_0(A) \text{ in} \\ \text{the LES-of-Triple Proposition 38.2}}}{=} ((\mu_1 \oplus \mu_2)([\partial_1\sigma]) = (\mu_1 \oplus \mu_2)([\underbrace{\sigma(0,1)}_{\in B_2)}] - [\underbrace{\sigma(1,0)}_{\in B_1)}]) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Since $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a generator of the kernel of $\mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z}$ we see that $[\sigma]$ is in fact a generator of $\mathrm{H}_1(X,A) \cong \mathbb{Z}$. In particular it is non-zero.



Corollary 38.4. (Homotopy Equivalence-Relative-H$_*$-Corollary) In the following let $f\colon (X,A) \to (Y,B)$ be a continuous map of pairs of topological spaces. If $f\colon X \to Y$ is a homotopy equivalence and if $f\colon A \to B$ is a homotopy equivalence , then for every $n \in \mathbb{N}_0$ the induced map $f_*\colon \mathrm{H}_n(X,A) \to \mathrm{H}_n(Y,B)$ is an isomorphism.

Example. It follows from the Homotopy Equivalence-Relative-H$_*$-Corollary 38.4 that for each $k \in \mathbb{N}_0$ the inclusion induced homomorphisms $\mathrm{H}_n(\overline{B}^k, S^{k-1}) \to \mathrm{H}_n(\overline{B}^k, \overline{B}^k \setminus \{0\})$ are isomorphisms.

Proof. We consider the following diagram

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}_n(A) \arrow[r, "i_*"] \arrow[d, "f_*"] & \mathrm{H}_n(X) \arrow[r, "\text{я}"] \arrow[d, "f_*"] & \mathrm{H}_n(X,A) \arrow[r, "\partial_n"] \arrow[d, "f_*"] & \mathrm{H}_{n-1}(A) \arrow[r, "i_*"] \arrow[d, "f_*"] & \mathrm{H}_{n-1}(X) \arrow[r] \arrow[d, "f_*"] & \dots \\
\dots \arrow[r] & \mathrm{H}_n(B) \arrow[r, "i_*"] & \mathrm{H}_n(Y) \arrow[r, "\text{я}"] & \mathrm{H}_n(Y,B) \arrow[r, "\partial_n"] & \mathrm{H}_{n-1}(B) \arrow[r, "i_*"] & \mathrm{H}_{n-1}(Y) \arrow[r] & \dots
\end{tikzcd}$$

By the LES-of-Pair Proposition 38.3 (2) we know that the diagram commutes. Furthermore note that it follows from our hypothesis on f and the Homotopy Equivalence-H$_*$-Corollary 36.7 that the two vertical maps to the left and the two vertical maps to the right are isomorphisms. It follows from the Five Lemma 37.5 that the middle vertical map is, as promised, an isomorphism. ■

Corollary 38.5. (Deformation Retract-Relative H$_*$-Corollary) Let X be a topological space. If $A \subset X$ is a deformation retract of X, then $\mathrm{H}_n(X,A) = 0$ for all $n \in \mathbb{N}_0$.

Proof. Let $i\colon A \to X$ be the inclusion map. By the LES-of-Pair Proposition 38.3 we have a long exact sequence

$$\dots \to \mathrm{H}_n(A) \xrightarrow[\uparrow]{i_*} \mathrm{H}_n(X) \to \mathrm{H}_n(X,A) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A) \xrightarrow[\uparrow]{i_*} \mathrm{H}_{n-1}(X) \to \dots$$

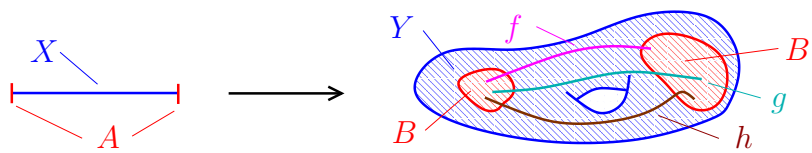
it follows from the Deformation Retract–Homotopy Equivalence Lemma 9.3 that $i\colon A \to X$ is a homotopy equivalence, thus it follows from the Homotopy Equivalence-H$_*$-Corollary 36.7 that i_* is an isomorphism

Since the maps $i_*\colon \mathrm{H}_n(A) \to \mathrm{H}_n(X)$ are isomorphisms and since the sequence is exact we see that the relative homology groups $\mathrm{H}_n(X,A)$ are all zero. ■

38.3. Relative homology groups and homotopies.

Definition. We say that two continuous maps $f, g\colon (X,A) \to (Y,B)$ between pairs of topological spaces are homotopic, if there exists a continuous map $F\colon X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$, and such that $F(a,t) \in B$ for all $a \in A$ and all $t \in [0,1]$.

Proposition 38.6. (Homotopy-H$_*$-Proposition) Let $f, g\colon (X,A) \to (Y,B)$ be two continuous maps between pairs of topological spaces. If f and g are homotopic, then the chain maps $f_*, g_*\colon \mathrm{C}_*(X,A) \to \mathrm{C}_*(Y,B)$ are chain homotopic, in particular for every



$f, g\colon (X, A) \to (Y, B)$ are homotopic but $f, h\colon (X, A) \to (Y, B)$ are not homotopic

$n \in \mathbb{N}_0$ we have
$$f_* \;=\; g_*\colon \mathrm{H}_n(X, A) \;\to\; \mathrm{H}_n(Y, B).$$

Proof. Let $f, g\colon (X, A) \to (Y, B)$ be two continuous maps between pairs of topological spaces and let $F\colon X \times [0, 1] \to Y$ be a homotopy between f and g as maps of pairs of topological spaces. Let $\Omega_n \in \mathrm{C}_{n+1}(\Delta^n \times [0, 1])$ be the singular chain that we defined on page 312. On page 314 we showed that the homomorphisms

$$\begin{aligned} P_n\colon \mathrm{C}_n(X) \;&\to\; \mathrm{C}_{n+1}(Y) \\ \sum_{i=1}^{k} s_i \cdot (\sigma\colon \Delta^n \to X) \;&\mapsto\; \sum_{i=1}^{k} s_i \cdot (F \circ (\sigma \times \mathrm{id}_{[0,1]}))_*(\Omega_n) \end{aligned}$$

define a chain homotopy between the maps $f_*, g_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$, i.e. for each $n \in \mathbb{N}_0$ we have
$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; f_* - g_*\colon \mathrm{C}_n(X) \;\to\; \mathrm{C}_n(Y).$$
Since F restricts to a map $A \times [0, 1] \to B$ we see that these homomorphisms P_n have the property that $\mathrm{C}_n(A)$ gets sent to $\mathrm{C}_{n+1}(B)$. Thus the maps $P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(Y)$ descend to homomorphisms $\overline{P}_n\colon \mathrm{C}_n(X, A) \to \mathrm{C}_{n+1}(Y, B)$. These satisfy
$$\partial_{n+1} \circ \overline{P}_n + \overline{P}_{n-1} \circ \partial_n \;=\; f_* - g_*\colon \mathrm{C}_n(X, A) \;\to\; \mathrm{C}_n(Y, B),$$
i.e. the maps $\overline{P}_n$ form a chain homotopy between the maps f_* and g_* from $\mathrm{C}_*(X, A)$ to $\mathrm{C}_*(Y, B)$. It follows from this observation together with the Chain Homotopy–H_*-Lemma 36.1 that $f_* = g_*\colon \mathrm{H}_n(X, A) \to \mathrm{H}_n(Y, B)$. ■

39. Reduced homology

39.1. Definition and basic properties of reduced homology.

Definition. Let X be a topological space. By the Augmentation Lemma 35.9 (1) we know that the following is a generalized chain complex

$$\begin{array}{ccccccccc} & & & & \text{degree } 0 & & \text{degree } -1 & & \\ & & & & \downarrow & & \downarrow & & \\ \dots & \xrightarrow{\partial_3} & \mathrm{C}_2(X) & \xrightarrow{\partial_2} & \mathrm{C}_1(X) & \xrightarrow{\partial_1} & \mathrm{C}_0(X) & \xrightarrow{\epsilon_X} & \mathbb{Z} \to 0. \\ & & & & & & \sum_{i=1}^{k} r_i \cdot \sigma_i & \mapsto & \sum_{i=1}^{k} r_i \\ & & & & & & & \uparrow & \\ & & & & & & & \text{augmentation map from page 306} & \end{array}$$

We introduce the following two definitions.

(1) We refer to this chain complex as the augmented chain complex $\widetilde{\mathrm{C}}_*(X)$ of X.

(2) Given $n \in \mathbb{Z}_{\geq -1}$ we refer to $\widetilde{\mathrm{H}}_n(X) := \mathrm{H}_n(\widetilde{\mathrm{C}}_*(X))$ as the n-th reduced homology group of X.

Lemma 39.1. ($\widetilde{\mathrm{H}}_n$–Functor Lemma)

(1) If $f\colon X \to Y$ is a continuous map between two topological spaces, then we denote by $f_*\colon \widetilde{\mathrm{C}}_*(X) \to \widetilde{\mathrm{C}}_*(Y)$ the map that is the usual map on $\widetilde{\mathrm{C}}_n(X) = \mathrm{C}_n(X)$ for $n \geq 0$ and that is the identity on the final $\mathbb{Z}$-term. This map is a chain map and thus induces homomorphisms $f_*\colon \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(Y)$ on the reduced homology groups.

(2) For any $n \in \mathbb{Z}_{\geq -1}$ the maps

$$\begin{array}{rcl} X & \mapsto & \widetilde{\mathrm{H}}_n(X) \\ (f\colon X \to Y) & \mapsto & (f_*\colon \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(Y)) \end{array}$$

define a covariant functor from the category of topological spaces to the category of abelian groups.

(3) For any $n \in \mathbb{Z}_{\geq -1}$ the maps

$$\begin{array}{rcl} \text{ж}\colon \widetilde{\mathrm{H}}_n(X) & \to & \mathrm{H}_n(X) \\ [\sigma] & \mapsto & [\sigma] \end{array}$$

are well-defined homomorphisms and they define a natural transformation from the functor $X \mapsto \widetilde{\mathrm{H}}_n(X)$ to the functor $X \mapsto \mathrm{H}_n(X)$.

Proof.

(1) Let $f\colon X \to Y$ be a continuous map between two topological spaces. As in the Augmentation Lemma 35.9 (3) it is a straightforward consequence of the definition of the augmentation maps that the following diagram commutes:

$$\begin{tikzcd} \mathrm{C}_0(X) \arrow[r, "f_*"] \arrow[d, "\epsilon_X"'] & \mathrm{C}_0(Y) \arrow[d, "\epsilon_Y"] \\ \mathbb{Z} \arrow[r, "\mathrm{id}"] & \mathbb{Z}. \end{tikzcd}$$

It follows from this observation together with the C_*–Functor Lemma 35.5 that the induced maps $f_*\colon \widetilde{\mathrm{C}}_*(X) \to \widetilde{\mathrm{C}}_*(Y)$ are chain maps. Furthermore we obtain from the Chain Complex-H_*-Functor Lemma 35.1 (1) that we get induced maps on reduced homology.

(2) This statement is basically obvious.
(3) This statement follows easily from the definitions. ■

Lemma 39.2. (Reduced Homology Lemma) For every topological space X the following statements hold:

(0) We have
$$\widetilde{\mathrm{H}}_{-1}(X) \;=\; \begin{cases} 0, & \text{if } X \neq \varnothing, \\ \mathbb{Z}, & \text{if } X = \varnothing. \end{cases}$$

(1) (a) If X is non-empty, then the sequence
$$0 \;\to\; \widetilde{\mathrm{H}}_0(X) \;\xrightarrow{\text{ж}}\; \mathrm{H}_0(X) \;\xrightarrow[\text{Augmentation Lemma 35.9}]{\text{augmentation map } \epsilon_X \text{ from the}}\; \mathbb{Z} \;\to\; 0$$
is exact.

(b) Given any point $P \in X$ the map
$$\begin{array}{rcl} \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z} & \to & \mathrm{H}_0(X) \\ [\sigma] \oplus n & \mapsto & [\sigma] + [n \cdot P] \end{array}$$
is an isomorphism.

(c) If X has $n \in \mathbb{N}$ path-components, then $\widetilde{\mathrm{H}}_0(X) \cong \mathbb{Z}^{n-1}$. In particular X is path-connected if and only if $\widetilde{\mathrm{H}}_0(X) = 0$.

(2) For every $n \geq 1$ the natural map $\text{ж}\colon \widetilde{\mathrm{H}}_n(X) \to \mathrm{H}_n(X)$ is the identity.

Proof. Let X be a topological space.

(0) First note that clearly we have $\widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$. Now we assume that X is non-empty. It follows immediately from the definitions that the augmentation map ϵ_X is an epimorphism which entails that $\widetilde{\mathrm{H}}_{-1}(X) = 0$.

(1) (a) We consider the following short exact sequence of generalized chain complexes:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathrm{C}_1(X) \arrow[r, "\mathrm{id}"] \arrow[d, "\partial_1"] & \mathrm{C}_1(X) \arrow[r] \arrow[d, "\partial_1"] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathrm{C}_0(X) \arrow[r, "\mathrm{id}"] \arrow[d, "\epsilon_X"] & \mathrm{C}_0(X) \arrow[r] \arrow[d, "\partial_0"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\mathrm{id}"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

The desired statement follows easily from the corresponding long exact sequence of homology groups that is given by the LES-Proposition 37.4.

(b) Let $P \in X$ be a point. It is straightforward to show that the maps
$$\begin{array}{rcl} \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z} & \to & \mathrm{H}_0(X) \\ [\sigma] \oplus n & \mapsto & [\sigma] + [n \cdot P] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{H}_0(X) & \to & \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z} \\ [\sigma] & \mapsto & [\sigma - \epsilon_X(\sigma) \cdot P] + \epsilon_X(\sigma) \end{array}$$
are inverses of one another. In particular both maps are isomorphisms.

(c) This statement follows almost immediately from (1a), the H_0-Proposition 35.10 (1) together with the Direct Sum-H_*-Lemma 35.2.

(2) This statement is clear. ■

Proposition 39.3. (Homotopy-$\widetilde{\mathrm{H}}_*$-Proposition) Let X and Y be two topological spaces and let $f, g\colon X \to Y$ be two continuous maps. If f and g are homotopic, then the following hold:

(1) The induced maps f_* and g_* from $\widetilde{\mathrm{C}}_*(X)$ to $\widetilde{\mathrm{C}}_*(Y)$ are chain homotopic.
(2) For each $n \in \mathbb{N}_0$ we have $f_* = g_*\colon \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(Y)$.

Sketch of proof.

(1) This statement follows easily from the Homotopy-$\widetilde{\mathrm{H}}_*$-Proposition 39.3 (2).
(2) This statement follows immediately from (1). ■

Corollary 39.4. (Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary) Let X and Y be topological spaces and let $n \in \mathbb{N}_0$.

(1) (a) If $f\colon X \to Y$ is a homotopy equivalence, then the induced map $f_*\colon \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(Y)$ is an isomorphism.
    (b) If X is a deformation retract of Y, then the inclusion induced map $\widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(Y)$ is an isomorphism.
(2) If X and Y are homotopy equivalent, then $\widetilde{\mathrm{H}}_n(X) \cong \widetilde{\mathrm{H}}_n(Y)$.

Sketch of proof. This corollary can be obtained easily from the Homotopy-$\widetilde{\mathrm{H}}_*$-Proposition 39.3. ■

Corollary 39.5. (Contractible-$\widetilde{\mathrm{H}}_*$-Corollary) If X is contractible (i.e. if X is homotopy equivalent to a singleton), then we have $\widetilde{\mathrm{H}}_n(X) = 0$ for all $n \in \mathbb{Z}_{\geq -1}$.

Example. On page 111 we saw that every convex subset of $\mathbb{R}^n$ is contractible. Thus it follows from the Contractible-H_*-Corollary 36.8, that for any $i \in \mathbb{Z}_{\geq -1}$ we have

$$\widetilde{\mathrm{H}}_i(\underbrace{\text{convex subset of } \mathbb{R}^n}_{\text{e.g. } \mathbb{R}^n,\ \overline{B}^n,\ B^n \text{ or simplex}}) \ = \ 0.$$

Proof. This corollary follows immediately from the Contractible-H_*-Corollary 36.8 together with the Reduced Homology Lemma 39.2. ■

39.2. The long exact sequence of a pair in reduced homology.

Notation. Let X be a topological space, let $A \subset X$ and let $n \in \mathbb{N}_0$. We introduce the following map:

$$\begin{tikzcd} \widetilde{\mathrm{H}}_n(X) \arrow[r, "ж"] \arrow[rr, bend right, "=:ж_{X,A}"'] & \mathrm{H}_n(X) \arrow[r, "я"] & \mathrm{H}_n(X, A), \end{tikzcd}$$

where $ж\colon \widetilde{\mathrm{H}}_n(X) \to \mathrm{H}_n(X)$ is the natural homomorphism introduced in the $\widetilde{\mathrm{H}}_n$–Functor Lemma 39.1 and $я\colon \mathrm{H}_n(X) \to \mathrm{H}_n(X, A)$ is the natural homomorphism that we introduced on page 325.

Proposition 39.6. (Reduced LES-of-Pair Proposition)

(1) Let X be a topological space and let $A \subset X$ be a subspace. The following sequence is exact

$$\begin{array}{ccccccccc} \dots & \to & \widetilde{\mathrm{H}}_n(A) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_n(X) & \xrightarrow{\text{ж}_{X,A}} & \mathrm{H}_n(X,A) & \xrightarrow{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(A) & \to & \dots \\ & & & & & & \left[\sum\limits_{j=1}^{k} s_j \cdot \sigma_j\right] & \mapsto & \left[\sum\limits_{j=1}^{k} s_j \cdot \partial\sigma_j\right] & & \end{array}$$

(2) The connecting homomorphisms in (1) are natural.

Proof.

(1) We consider the following short exact sequence of chain complexes:

$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\ 0 \arrow[r] & \mathrm{C}_1(A) \arrow[r] \arrow[d, "\partial_1"] & \mathrm{C}_1(X) \arrow[r] \arrow[d, "\partial_1"] & \mathrm{C}_1(X,A) \arrow[r] \arrow[d, "\partial_1"] & 0 \\ 0 \arrow[r] & \mathrm{C}_0(A) \arrow[r] \arrow[d, "\epsilon_A"] & \mathrm{C}_0(X) \arrow[r] \arrow[d, "\epsilon_X"] & \mathrm{C}_0(X,A) \arrow[r] \arrow[d, "\partial_0"] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "\mathrm{id}"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0. & \end{tikzcd}$$

From the LES-Proposition 37.4 we obtain a long exact sequence of homology groups, which in this case involve the reduced homology groups of A and X, and the relative homology groups of (X, A). Furthermore, as in the proof of the LES-of-Triple Proposition 38.2 we see that the connecting homomorphism is given by the map

$$\begin{array}{ccc} \mathrm{H}_n(X,A) & \to & \widetilde{\mathrm{H}}_{n-1}(A) \\ \left[\sum\limits_{j=1}^{k} s_j \cdot \sigma_j\right] & \mapsto & \left[\sum\limits_{j=1}^{k} s_j \cdot \partial\sigma_j\right] \end{array}$$

Finally note that one can easily easily verify that the maps $\widetilde{\mathrm{H}}_k(X) \to \mathrm{H}_k(X,A)$ are precisely the above maps $\text{ж}_{X,A}$.

(2) This statement follows easily from the naturality of the connecting homomorphism proved in the LES-Proposition 37.4 ■

Lemma 39.7. (Reduced-to-Relative-H$_*$-Lemma) Let X be a topological space and let $n \in \mathbb{N}_0$. For each $x_0 \in X$ the map

$$\text{ж}_{X,\{x_0\}} \colon \widetilde{\mathrm{H}}_n(X) \to \mathrm{H}_n(X, \{x_0\})$$

is a natural isomorphism.

Proof. Let X be a topological space and let $x_0 \in X$. We apply the Reduced LES-of-Pair Proposition 39.6 to the pair $(X, \{x_0\})$ and we obtain a long exact sequence

$$\dots \to \underbrace{\widetilde{\mathrm{H}}_n(\{x_0\})}_{=0} \xrightarrow{i_*} \widetilde{\mathrm{H}}_n(X) \xrightarrow{\text{ж}_{X,\{x_0\}}} \mathrm{H}_n(X, \{x_0\}) \xrightarrow{\partial_n} \underbrace{\widetilde{\mathrm{H}}_{n-1}(\{x_0\})}_{=0} \to \dots$$

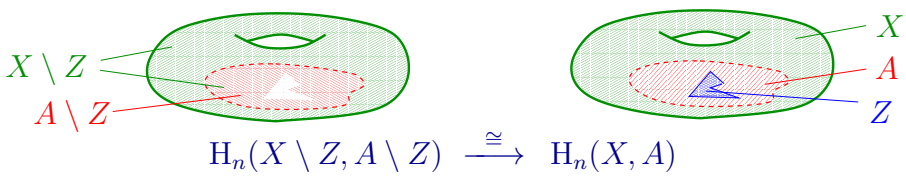
The lemma follows from the exactness of this sequence and the observation that it follows from the Singleton-H$_*$-Lemma 35.11 and the Reduced Homology Lemma 39.2 that all reduced homology groups of $\{x_0\}$ vanish. ■

40. The Excision Theorem

40.1. The Excision Theorem.

Theorem 40.1. (Excision Theorem) Let X be a topological space and let $Z \subset A \subset X$ be subsets. If the closure of Z is contained in the interior of A, then the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

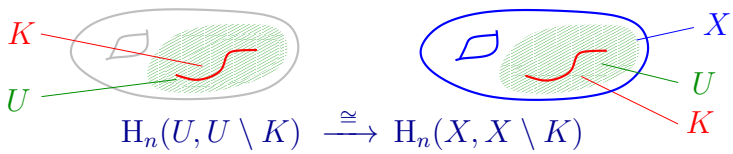
$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} \mathrm{H}_n(X, A).$$



40.2. Variations on the Excision Theorem 40.1.

Theorem 40.2. (Excision Theorem) Let X be a topological space, let K be a subset of X and let U be a neighborhood of the closure of K. The inclusion $(U, U \setminus K) \to (X, X \setminus K)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(U, U \setminus K) \xrightarrow{\cong} \mathrm{H}_n(X, X \setminus K).$$



Proof of Theorem 40.2 assuming Theorem 40.1. We have

$$\mathrm{H}_n(U, U \setminus K) \underset{\substack{\uparrow \\ \text{set } A := X \setminus K \text{ and } Z := X \setminus U}}{=} \mathrm{H}_n(X \setminus Z, A \setminus Z) \underset{\substack{\uparrow \\ \text{since } U \text{ is a neighborhood of the closure of } K \text{ we see} \\ \text{that the closure of } Z \text{ is contained in the interior of } A, \\ \text{thus we can apply the Excision Theorem 40.1}}}{\xrightarrow{\cong}} \mathrm{H}_n(X, A) = \mathrm{H}_n(X, X \setminus K).$$

■

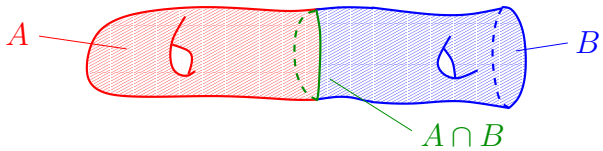
Theorem 40.3. (Manifold–Excision Theorem) Let X be an m-dimensional topological manifold and let $A, B \subset X$ be two codimension-zero submanifolds such that the following statements hold:

(a) $X = A \cup B$.
(b) $A \cap B$ is a union of components of ∂A.
(c) A and B are closed subsets of X.

For every $n \in \mathbb{N}_0$ the inclusion induced map

$$\mathrm{H}_n(B, A \cap B) \to \mathrm{H}_n(X, A)$$

is an isomorphism.

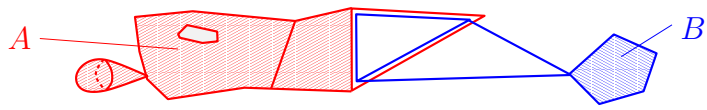


the inclusion induced map $\mathrm{H}_n(B, A \cap B) \to \mathrm{H}_n(X, A)$ is an isomorphism

Theorem 40.4. (CW-Excision Theorem) Let X be a CW-complex and let A and B be two subcomplexes of X with $X = A \cup B$. For every $n \in \mathbb{N}_0$ the inclusion induced map

$$\mathrm{H}_n(B, A \cap B) \xrightarrow{\cong} \mathrm{H}_n(X, A)$$

is an isomorphism.



the inclusion induced map $\mathrm{H}_n(B, A \cap B) \to \mathrm{H}_n(X, A)$ is an isomorphism

Theorem 40.5. (Excision-via-Deformation Theorem) Let X be a topological space and let $Z \subset A \subset X$ be two subsets. We suppose that there exists a subset $U \subset Z$ such that the closure of U is contained in the interior of A and such that the following statements hold:

(1) $A \setminus Z$ is a deformation retract of $A \setminus U$.
(2) $X \setminus Z$ is a deformation retract of $X \setminus U$

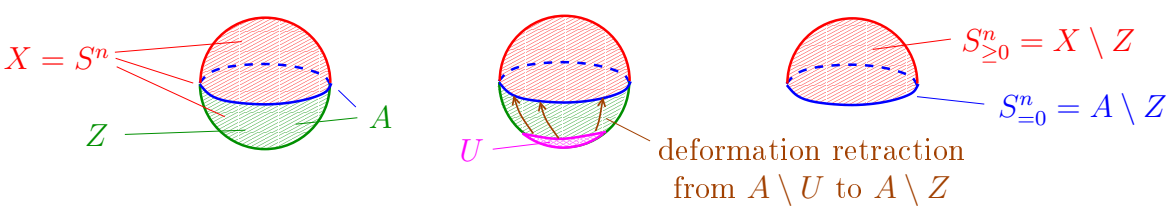
Then the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} \mathrm{H}_n(X, A).$$

Example. Let $n \in \mathbb{N}$. As usual we denote by $S^n_{\geq 0}$ and $S^n_{\leq 0}$ the upper and lower hemisphere of S^n and we write $S^n_{=0} = S^n_{\geq 0} \cap S^n_{\leq 0}$. It follows from the Excision-via-Deformation Theorem 40.5 and the choices of X, A, Z and U illustrated in the figure below that for each $k \in \mathbb{N}_0$ the inclusion induced map

$$\mathrm{H}_k(S^n_{\geq 0}, S^n_{n=0}) \to \mathrm{H}_k(S^n, S^n_{\leq 0})$$

is an isomorphism.



Proof of Theorem 40.5 assuming the Excision Theorem 40.1. We consider the following commutative diagram of inclusion induced maps

$$\begin{tikzcd}
\mathrm{H}_n(X \setminus Z, A \setminus Z) \arrow[r] \arrow[d] & \mathrm{H}_n(X, A) \\
\mathrm{H}_n(X \setminus U, A \setminus U). \arrow[ur, "\cong"'] &
\end{tikzcd}$$

Since by hypothesis we know that $\overline{U} \subset \mathring{A}$ we obtain from the Excision Theorem 40.1 that the diagonal map is an isomorphism. Thus it suffices to show that the vertical map is an isomorphism.

To do so we consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}_n(A\setminus Z) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{H}_n(X\setminus Z) \arrow[r] \arrow[d] & \mathrm{H}_n(X\setminus Z, A\setminus Z) \arrow[r, "\partial_n"] \arrow[d] & \mathrm{H}_{n-1}(A\setminus Z) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{H}_{n-1}(X\setminus Z) \arrow[d] \\
\mathrm{H}_n(A\setminus Z) \arrow[r] \arrow[d] & \mathrm{H}_n(X\setminus U) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{H}_n(X\setminus U, A\setminus Z) \arrow[r, "\partial_n"] \arrow[d] & \mathrm{H}_{n-1}(A\setminus Z) \arrow[r] \arrow[d] & \mathrm{H}_{n-1}(X\setminus U) \arrow[d, "\mathrm{id}"] \\
\mathrm{H}_n(A\setminus U) \arrow[r] & \mathrm{H}_n(X\setminus U) \arrow[r] & \mathrm{H}_n(X\setminus U, A\setminus U) \arrow[r, "\partial_n"] & \mathrm{H}_{n-1}(A\setminus U) \arrow[r] & \mathrm{H}_{n-1}(X\setminus U).
\end{tikzcd}$$

Here the horizontal maps are pieces of the long exact sequences of the various pairs of topological spaces provided by the LES-of-Pair Proposition 38.3 (1). Note that by the LES-of-Pair Proposition 38.3 (2) we know that the diagram commutes.

Furthermore the vertical maps are induced by the various inclusion maps. On the upper level the first and the fourth vertical maps are the identity. Furthermore it follows from the hypothesis that $X\setminus Z$ is a deformation retract and Homotopy Equivalence-H_*-Corollary 36.7 that the second and the fifth vertical map are also isomorphisms. It follows from the Five Lemma 37.5 that the upper middle vertical map is also an isomorphism.

Using the hypothesis that $A\setminus Z$ is a deformation retract of $A\setminus U$ one can show exactly the same way that the lower middle vertical map is also an isomorphism. Combining the two middle vertical maps we obtain the desired isomorphism. ■

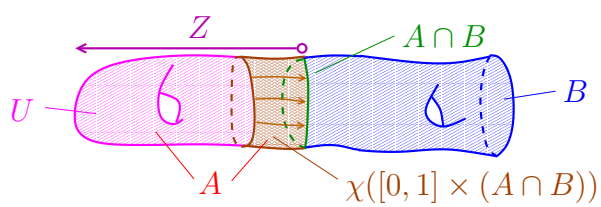
Proof of Theorem 40.3 assuming Theorem 40.5. Let X be an m-dimensional topological manifold and let $A, B \subset X$ be two codimension-zero submanifolds such that the following statements hold:

(a) $X = A \cup B$.
(b) $A \cap B$ is a union of components of ∂A.
(c) A and B are closed subsets of X.

It follows from (b) together with the Collar Neighborhood Theorem 23.2 that there exists a continuous map $\chi\colon [0,1] \times (A\cap B) \to A$ with the following three properties:

(d) The map χ is an embedding.
(e) The image of $[0,1) \times (A\cap B)$ is an open subset of A.
(f) For every $P \in A\cap B$ we have $\chi(0,P) = P$.

We set $Z := X\setminus B$ and we set $U := X\setminus(B\cup\chi([0,1)\times(A\cap B))) = A\setminus\chi([0,1)\times(A\cap B))$.



Claim.

(0) The closure of U is contained in the interior of A.
(1) $A\setminus Z = A\cap B$ is a deformation retract of $A\setminus U = \chi([0,1]\times(A\cap B))$.
(2) $X\setminus Z = B$ is a deformation retract of $X\setminus U = (\chi([0,1]\times(A\cap B))\cup B$.

Proof.

(0) It follows from (b) that $A \setminus (A \cap B) = X \setminus B$ is an open subset of X. Furthermore it follows from (c), (e), (f) and the Cover-Open Subset Lemma 1.5 (2b) that U is a closed subset of X. It follows from this discussion with the fact that $U \subset X \setminus B$ that the closure of U is contained in the interior of A.

(1) We have an obvious deformation retraction from $[0,1] \times (A \cap B)$ to $\{0\} \times (A \cap B)$. It follows from (d) and (f) that this deformation retraction gives rise to a a deformation retraction from $\chi([0,1] \times (A \cap B))$ to $A \cap B$.

(2) It follows from (c) and the Homotopy Combination Lemma 8.2 that we can extend the deformation retraction from (2) in an obvious way to the desired deformation retraction from $X \setminus Z$ to $X \setminus U$. ⊞

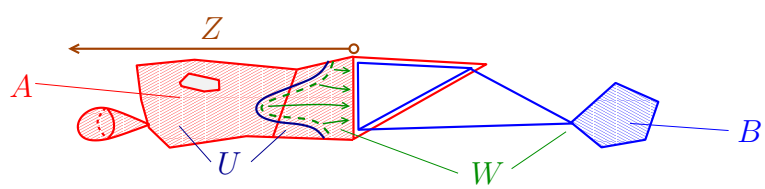
It follows from the claim that we can apply the Excision-via-Deformation Theorem 40.5 which tells us that the inclusion induced map

$$\mathrm{H}_n(\underbrace{X \setminus Z}_{=B}, \underbrace{A \setminus Z}_{=A \cap B}) \xrightarrow{\cong} \mathrm{H}_n(X, A)$$

is an isomorphism. But that is exactly what we needed to show. ■

Proof of the CW-Excision Theorem 40.4 assuming Theorem 40.5. Let X be a CW-complex and let A and B be two subcomplexes of X with $X = A \cup B$. Note that by the Subcomplex-Neighborhood Proposition 31.8, applied to the subcomplex B, there exists an open neighborhood W of B and a deformation retraction $F\colon W \times [0,1] \to W$ from W to B with the extra property that the map F restricts to a deformation retraction $F\colon (A \cap W) \times [0,1] \to A \cap W$ from $A \cap W$ to $A \cap B$.

We set $Z := X \setminus B$ and we set $U := X \setminus W$. Since W is an open neighborhood of B we see that the closure of U is contained in the interior of A.



Recall that F is a deformation retraction from $W = X \setminus U$ to $B = X \setminus Z$ and that F restricts to a deformation retraction from $A \cap W = A \setminus U$ to $A \cap B = A \setminus Z$. Therefore it follows from the Excision-via-Deformation Theorem 40.5 that the inclusion induced map

$$\mathrm{H}_n(\underbrace{X \setminus Z}_{=B}, \underbrace{A \setminus Z}_{=A \cap B}) \xrightarrow{\cong} \mathrm{H}_n(X, A)$$

is an isomorphism. But that is exactly what we needed to show. ■

40.3. The proof of the Excision Theorem 40.1.

Definition. Let again X be a topological space.

(1) On page 32 we defined a cover of X to be a family $\{U_i\}_{i \in I}$ of subsets of X such that $X = \bigcup_{i \in I} U_i$.

(2) We say a cover $\{U_i\}_{i \in I}$ is comfortable if $X = \bigcup_{i \in I} \mathring{U}_i$, i.e. if the interiors of the U_i also form a cover of X.

Definition. Let again X be a topological space. Given a comfortable cover $\mathcal{U} = \{U_i\}_{i\in I}$ we consider the following subgroup of $\mathrm{C}_n(X)$:

$$\mathrm{C}_n^{\mathcal{U}}(X) := \left\{ \sum_{j=1}^{k} a_j \cdot \sigma_j \,\middle|\, \text{for each } j \text{ there exists a } U_i \in \mathcal{U} \text{ such that the image of } \sigma_j \text{ lies in } U_i \right\}.$$

It follows quite easily from the definitions that the restriction of the usual boundary map $\partial_n \colon \mathrm{C}_n(X) \to \mathrm{C}_{n-1}(X)$ to $\mathrm{C}_n^{\mathcal{U}}(X)$ restricts to a map $\partial_n \colon \mathrm{C}_n^{\mathcal{U}}(X) \xrightarrow{\partial} \mathrm{C}_{n-1}^{\mathcal{U}}(X)$. Since this map is the restriction of the usual boundary map it follows in particular that the composition $\partial_n \circ \partial_{n+1} \colon \mathrm{C}_{n+1}^{\mathcal{U}}(X) \xrightarrow{\partial} \mathrm{C}_{n-1}^{\mathcal{U}}(X)$ is the zero map. Thus we can define

$$\begin{aligned} \mathrm{H}_n^{\mathcal{U}}(X) &:= \mathrm{H}_n\big(C_*^{\mathcal{U}}(X), \partial_*\big) \\ &= \ker\big(\partial_n \colon \mathrm{C}_n^{\mathcal{U}}(X) \to \mathrm{C}_{n-1}^{\mathcal{U}}(X)\big) \,/\, \operatorname{im}\big(\partial_{n+1} \colon \mathrm{C}_{n+1}^{\mathcal{U}}(X) \to \mathrm{C}_n^{\mathcal{U}}(X)\big). \end{aligned}$$

Proposition 40.6. (Comfortable Covering Proposition) Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be a comfortable cover of X. The inclusion map

$$\mathrm{C}_*^{\mathcal{U}}(X) \;\to\; \mathrm{C}_*(X)$$

induces for every $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n^{\mathcal{U}}(X) \;\xrightarrow{\cong}\; \mathrm{H}_n(X).$$

We have to introduce a few more definitions.

(1) Given $n \in \mathbb{N}_0$ we denote by $B_n := (\frac{1}{n+1}, \dots, \frac{1}{n+1}) \in \Delta^n$ the *barycenter* of the standard n-simplex Δ^n.

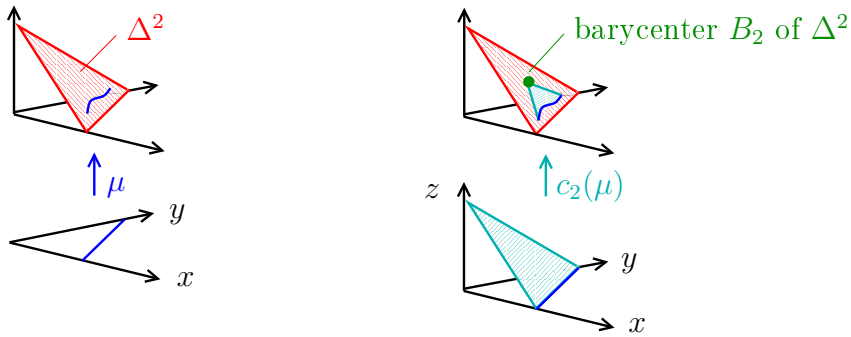
(2) Given a singular l-simplex $\mu \colon \Delta^l \to \Delta^n$ we consider the singular $(l+1)$-simplex $c_{l+1}(\mu) \colon \Delta^{l+1} \to \Delta^n$ which is defined as the composition of the (inverses) of the following two maps:

$$\begin{array}{rcl} & \text{homeomorphism by the Compact-Hausdorff Proposition 2.10 (3)} & \text{continuous by the Topological-Quotient Proposition 6.1 (1b)} \\ & \downarrow & \downarrow \\ c_{l+1}(\mu) \colon \Delta^{l+1} & \xleftarrow{\cong} & (\Delta^l \times [0,1])/\Delta^l \times \{1\} \;\to\; \Delta^n \\ ((t_0,\dots,t_l)\cdot(1-s), s) & \longmapsfrom & [((t_0,\dots,t_l), s)] \\ & & [((t_0,\dots,t_l), s)] \;\mapsto\; \underbrace{\mu(t_0,\dots,t_l)\cdot(1-s) + B\cdot s.}_{\text{contained in } \Delta^n \text{ since } \Delta^n \text{ is convex}} \end{array}$$

Pictorially speaking the singular $(l+1)$-simplex $c_{l+1}(\mu)$ thus corresponds to the "cone" which is given by the singular l-simplex μ and the barycenter B_n. Note that by the Free Abelian Group–Existence Lemma 18.4 this map $\mu \mapsto c_{l+1}(\mu)$ defines a linear map $\mathrm{C}_l(\Delta^n) \to \mathrm{C}_{l+1}(\Delta^n)$ that we also denote by c_{l+1} and which we call the $(l+1)$-st cone map.

(3) For $n = 0, 1, 2, 3, \dots$ we inductively do the following:
   (a) we define an element $\chi_n \in \mathrm{C}_n(\Delta^n)$, and
   (b) for each topological space X we define a homomorphism $u_n \colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$.

   We do this inductive procedure as follows:
   (i) We set $\chi_0 = \mathrm{id} \colon \Delta^0 \to \Delta^0$ and for each topological space X we define u_0 to be the identity on $\mathrm{C}_0(X)$.



(ii) Now we suppose that $\chi_{n-1} \in \mathrm{C}_{n-1}(\Delta^{n-1})$ is already defined and we suppose that the homomorphisms $u_{n-1}\colon \mathrm{C}_{n-1}(X) \to \mathrm{C}_{n-1}(X)$ are already defined. Then we set

$$\chi_n \;:=\; (-1)^n \cdot c_n(u_{n-1}(\underbrace{\partial\,\mathrm{id}_{\Delta^n}}_{\in \mathrm{C}_{n-1}(\Delta^n)})) \;\in\; \mathrm{C}_n(\Delta^n).$$

Furthermore, for each topological space X we define

$$\begin{aligned} u_n\colon \mathrm{C}_n(X) &\to \mathrm{C}_n(X) \\ \sum_{l=1}^{r} a_l \cdot (\sigma_l\colon \Delta^n \to X) &\mapsto \sum_{l=1}^{r} a_l \cdot \underset{\substack{\uparrow \\ \text{map } \sigma_l\colon \Delta^l \to X \text{ induces a map } (\sigma_l)_*\colon \mathrm{C}_n(\Delta^l) \to \mathrm{C}_n(X)}}{(\sigma_l)_*}(\underbrace{\chi_n}_{\in \mathrm{C}_n(\Delta^n)}). \end{aligned}$$

Note that with this notation we have $u_n(\mathrm{id}_{\Delta^n}) = \chi_n$.

Lemma 40.7. (Subdivision–Natural Lemma) The maps $u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$ are natural, in particular for any continuous map $f\colon X \to Y$ between topological spaces and any $n \in \mathbb{N}_0$ we have

$$f_* \circ u_n \;=\; u_n \circ f_*\colon \mathrm{C}_n(X) \to \mathrm{C}_n(Y).$$

Proof. This lemma follows immediately from the definitions. Indeed, let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. Then

$$f_*(u_n(\sigma)) \underset{\substack{\uparrow \\ \text{by definition of } u_n(\sigma)}}{=} f_*(\sigma_*(\chi_n)) \underset{\substack{\uparrow \\ \text{since } (f\circ\sigma)_* = f_*\circ\sigma_* \\ \text{as maps } \mathrm{C}_n(\Delta^n)\to\mathrm{C}_n(Y)}}{=} (f\circ\sigma)_*(\chi_n) \underset{\substack{\uparrow \\ \text{by definition of } u_n(f_*(\sigma))}}{=} u_n(f_*(\sigma)).$$

Since the desired equality holds for a generating set of $\mathrm{C}_n(X)$ it also holds for all of $\mathrm{C}_n(X)$. ■

Lemma 40.8. (Chain Subdivision Lemma) Let X be a topological space. The maps $u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$, $n \in \mathbb{N}_0$, form a chain map from the chain complex $(\mathrm{C}_*(X), \partial)$ to itself.

Proof of the Chain Subdivision Lemma 40.8. We adopt the following notation and we make the following remarks:

(1) We denote the barycenter of Δ^n by b.

(2) Given points $x_0, \ldots, x_k \in \Delta^n$ we denote by $[x_0, \ldots, x_k]$ the singular k-chain $\Delta^k \to \Delta^n$ that is given by

$$\begin{array}{rcl} \Delta^k & \to & \Delta^n \\ (t_0,\dots,t_k) & \mapsto & \sum\limits_{i=0}^{k} t_i \cdot x_i, \end{array}$$

we refer to such a singular k-simplex as a *linear* singular simplex.

(3) We denote by $\mathrm{C}_k^{\mathrm{lin}}(\Delta^n)$ the subgroup of $\mathrm{C}_k(\Delta^n)$ generated by all the linear singular k-simplices. Note that the boundary map $\partial\colon \mathrm{C}_k(\Delta^n) \to \mathrm{C}_{k-1}(\Delta^n)$ restricts to a boundary map $\partial\colon \mathrm{C}_k^{\mathrm{lin}}(\Delta^n) \to \mathrm{C}_{k-1}^{\mathrm{lin}}(\Delta^n)$.

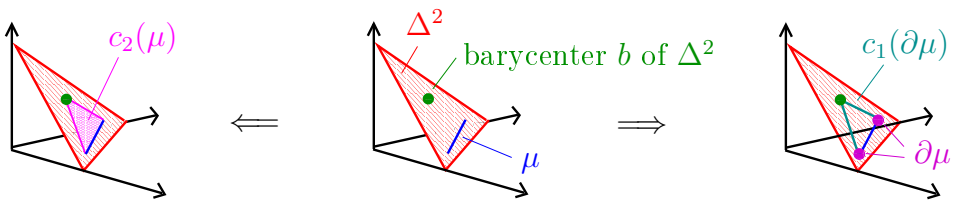
(4) For $x_0,\dots,x_k \in \Delta^n$ we write

$$[x_0,\dots,\widehat{x_i},\dots,x_k] \;:=\; [x_0,\dots,x_{i-1},x_{i+1},\dots,x_k].$$

The proof of the lemma will be broken into three claims.

Claim 1. For any $\mu \in \mathrm{C}_k^{\mathrm{lin}}(\Delta^n)$ we have

$$\partial(c_{k+1}(\mu)) \;=\; c_k(\partial\mu) + (-1)^{k+1}\cdot\mu \;\in\; \mathrm{C}_k(\Delta^n).$$



Proof. To prove the claim we first note that for any choice of points $x_0,\dots,x_{l-1} \in \Delta^n$ we have by definition of c_l that

$$c_l([x_0,\dots,x_{l-1}]) \;=\; [x_0,\dots,x_{l-1},b].$$

It follows that for any $w_0,\dots,w_k \in \Delta^n$ we have the following equality in $\mathrm{C}_k(\Delta^n)$:

$$\begin{aligned} \partial(c_{k+1}([w_0,\dots,w_k])) &\overset{\text{by the above equality}}{\overset{\downarrow}{=}} \partial([w_0,\dots,w_k,b]) \\ &\underset{\text{definition of } \partial \text{ and by the above equality}}{\underset{\uparrow}{=}} \sum_{j=0}^{k}(-1)^j\cdot[w_0,\dots,\widehat{w_j},\dots,w_k,b] + (-1)^{k+1}\cdot[w_0,\dots,w_k] \\ &\overset{\downarrow}{=} c_k(\partial([w_0,\dots,w_k])) + (-1)^{k+1}\cdot[w_0,\dots,w_k]. \end{aligned}$$

The claim immediately follows from this equality. ⊞

Claim 2. For any $n \in \mathbb{N}_0$ and any $\nu \in \mathrm{C}_n^{\mathrm{lin}}(\Delta^n)$ we have

$$\partial u_n(\nu) \;=\; u_{n-1}(\partial\nu).$$

Proof. We will prove this claim by induction on n. The case $n = 0$ is trivial. So now suppose we have already shown the case $n-1$. It suffices to show the claim for all linear

singular n-simplices. So let $\nu\colon \Delta^n \to \Delta^n$ be a linear singular n-simplex. We then have

$$\begin{aligned}
\partial u_n(\nu) &\overset{\substack{\text{definition of } u_n \text{ and } \chi_n\text{, see page 338}\\ \downarrow}}{=} \partial(\nu_*(\chi_n)) \overset{\downarrow}{=} (-1)^n \cdot \partial(\nu_*(c_n(u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})))) \overset{\substack{\text{since } \partial\circ\nu_* = \nu_*\circ\partial\\ \downarrow}}{=} (-1)^n \cdot \nu_*\big(\partial(c_n(\underbrace{u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})}_{=:\mu}))\big)\\
&\underset{\substack{\uparrow\\ \text{we can apply the previous claim since } \mu := u_{n-1}(\partial\,\mathrm{id}_{\Delta^n}) \in \mathrm{C}^{\mathrm{lin}}_{n-1}(\Delta^n)}}{=} (-1)^n \cdot \nu_*\big(c_{n-1}(\partial(u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})) + (-1)^n \cdot u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})\big)\\
&\underset{\substack{\uparrow\\ \text{by induction hypothesis we have } \partial u_{n-1}(\rho) = u_{n-2}(\partial\rho) \text{ for } \rho \in \mathrm{C}^{\mathrm{lin}}_{n-1}(\Delta^n)\text{, applied here to } \rho = \partial\,\mathrm{id}_{\Delta^n}}}{=} (-1)^n \cdot \nu_*\big(c_{n-1}(u_{n-2}(\underbrace{\partial\partial\,\mathrm{id}_{\Delta^n}}_{=0}))\big) \;+\; \nu_*(u_{n-1}(\partial\,\mathrm{id}_{\Delta^n}))\\
&\underset{\substack{\uparrow\\ \text{Subdivision–Natural}\\ \text{Lemma 40.7 applied}\\ \text{to } f=\nu}}{=} u_{n-1}(\nu_*(\partial\,\mathrm{id}_{\Delta^n})) \underset{\substack{\uparrow\\ \text{by definition}}}{=} u_{n-1}(\partial\nu).
\end{aligned}$$

⊞

The following claim is just a reformulation of the statement of the lemma.

Claim 3. For any singular n-simplex σ in X we have $u_{n-1}(\partial\sigma) = \partial(u_n(\sigma))$.

Proof. So let $\sigma\colon \Delta^n \to X$ be a singular n-simplex in X. Then we have

$$\begin{aligned}
\partial(u_n(\sigma)) &= \partial(u_n(\sigma_*(\mathrm{id}_{\Delta^n}))) \overset{\substack{\text{by the Subdivision–Natural Lemma 40.7 applied to } f=\sigma\\ \downarrow}}{=} \partial(\sigma_*(u_n(\mathrm{id}_{\Delta^n}))) \overset{\substack{\text{since } \sigma_* \text{ is a chain map}\\ \downarrow}}{=} \sigma_*(\partial(u_n(\mathrm{id}_{\Delta^n})))\\
&\underset{\substack{\uparrow\\ \text{by the previous claim}}}{=} \sigma_*(u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})) \underset{\substack{\uparrow\\ \text{by the Subdivision–Natural}\\ \text{Lemma 40.7 applied to } f=\sigma}}{=} u_{n-1}(\sigma_*(\partial\,\mathrm{id}_{\Delta^n})) \underset{\substack{\uparrow\\ \text{since } \sigma_* \text{ is a chain map}}}{=} u_{n-1}(\partial\sigma_*(\mathrm{id}_{\Delta^n})) = u_{n-1}(\partial\sigma).
\end{aligned}$$

■

Lemma 40.9. (Subdivision–Chain Homotopy-Lemma) Let X be a topological space. The chain maps

$$u_n\colon \mathrm{C}_n(X) \;\to\; \mathrm{C}_n(X)$$

are chain homotopic to the identity.

Proof. Showing that the chain maps $(u_n)_{n\in\mathbb{N}}$ are chain homotopic to the identity requires us to find maps $P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(X), n \in \mathbb{N}_0$ such that for each $n \in \mathbb{N}_0$ the following equality holds:

$$\partial_{n+1} \circ P_n \;=\; -P_{n-1} \circ \partial_n + u_n - \mathrm{id}\,.$$

First we consider the case that X is a convex subset of some $\mathbb{R}^m$. Recall that we defined $\mathrm{C}_{-1}(X) = 0$, accordingly we define $P_{-1}\colon \mathrm{C}_{-1}(X) \to \mathrm{C}_0(X)$ to be the zero map. Let us suppose that we have already found maps $P_{-1}, \ldots, P_{n-1}$ with the desired properties.

We want to define the next map $P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(X)$. This map has to have the property that for each singular n-simplex σ we have

$$\partial_{n+1}(P_n(\sigma)) \;=\; -(P_{n-1} \circ \partial_n)(\sigma) \;+\; (u_n - \mathrm{id})(\sigma).$$

Put differently, to define $P_n(\sigma)$ we have to find a singular $(n+1)$-chain $E_\sigma \in \mathrm{C}_{n+1}(X)$ with

$$\partial E_\sigma \;=\; -(P_{n-1} \circ \partial)(\sigma) + (u_n - \mathrm{id})(\sigma),$$

so that we can then define $P_n(\sigma) := E_\sigma$. Thus let $\sigma\colon \Delta^n \to X$ be a singular n-simplex.

(1) For $n = 0$ we have

$$\big(\underbrace{P_{-1}}_{=0} \circ \partial_0 + \underbrace{u_0}_{=\mathrm{id}} - \mathrm{id}\big)(\sigma) \;=\; 0 \in \mathrm{C}_0(X),$$

and we put $E_\sigma := 0 \in \mathrm{C}_1(X)$.

(2) Now suppose that $n > 0$. Since X is convex we know from the discussion on page 312 that $\mathrm{H}_n(X) = 0$. Therefore to show that an element $c \in \mathrm{C}_n(X)$ is the boundary of some element in $\mathrm{C}_{n+1}(X)$ it suffices to show that $\partial c = 0$. In our specific situation we calculate

$$\begin{aligned}\partial\big(\,(-P_{n-1}\partial_n + u_n - \mathrm{id})\,(\sigma)\big) &\overset{\substack{\text{by the Chain Subdivision Lemma 40.8 the subdivision maps}\\ \text{form a chain map, i.e. } \partial\circ u_n = u_{n-1}\circ\partial}}{=} -\partial P_{n-1}\partial\sigma + u_{n-1}\partial\sigma - \partial\sigma \\ &= \underbrace{(-\partial P_{n-1} + u_{n-1} - \mathrm{id})}_{=P_{n-2}\partial}(\partial\sigma) \;=\; P_{n-2}(\underbrace{\partial\partial\sigma}_{=0}) \;=\; 0.\end{aligned}$$

As we had just remarked, this implies that there exists an $E_\sigma \in \mathrm{C}_{n+1}(X)$ which satisfies $\partial(E_\sigma) = (P_{n-1}\partial_n - u_{n-1} - \mathrm{id})(\sigma)$.

Now we denote by $P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(X)$ the map which assigns to each singular n-simplex σ the element $E_\sigma \in \mathrm{C}_{n+1}(X)$. It follows from the definition of E_σ that

$$\partial_{n+1}\circ P_n + P_{n-1}\circ\partial_n \;=\; u_n - \mathrm{id}$$

as maps $\mathrm{C}_n(X) \to \mathrm{C}_n(X)$. This concludes the proof of the Subdivision–Chain Homotopy-Lemma 40.9 if X is a convex subset of some $\mathbb{R}^m$.

Now we deal with the general case. So let X be a topological space. We consider

$$\begin{aligned} P_n\colon \mathrm{C}_n(X) &\to \mathrm{C}_{n+1}(X) \\ \sum_{l=1}^r a_l\cdot(\sigma_l\colon \Delta^n \to X) &\mapsto \sum_{l=1}^r a_l\cdot(\sigma_l)_*(\underbrace{P_n(\mathrm{id}_{\Delta^n})}_{\substack{\uparrow\\ \text{defined since } \Delta^n \text{ is convex}}}).\end{aligned}$$

It is straightforward to see that this is a chain homotopy from the subdivision map to the identity map. ■

Definition. Given a topological space X and $m \in \mathbb{N}_0$ we denote by u_n^m the m-fold composition of $u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$.

Lemma 40.10. (Iterated Subdivision Lemma) For each topological space X and every $m \in \mathbb{N}$ the maps $\{u_n^m\}_{n\geq 0}$ form a chain map $\mathrm{C}_*(X) \to \mathrm{C}_*(X)$ that is chain homotopic to the identity.

Proof. Recall that in the Chain Subdivision Lemma 40.8 we already saw that the maps $\{u_n\}_{n\geq 0}$ form a chain map $\mathrm{C}_*(X) \to \mathrm{C}_*(X)$. Note that the composition of chain maps is again a chain map. Thus for any $m \in \mathbb{N}_0$ the maps $\{u_n^m\}_{n\geq 0}$ are also chain maps.

By the Subdivision–Chain Homotopy-Lemma 40.9 we have $u_* \simeq \mathrm{id}$. By the Chain Homotopy-Lemma 36.2 (2) this implies $u_*^2 \simeq u_*$, and by the Chain Homotopy-Lemma 36.2 (1) this implies $u_*^2 \simeq \mathrm{id}$. An induction argument now shows that $u_*^m \simeq \mathrm{id}$ for all m. ■

Definition. Given a singular chain c we can write it uniquely as $c = a_1\cdot\sigma_1 + \cdots + a_s\cdot\sigma_s$ with $a_1,\ldots,a_s \in \mathbb{Z}\setminus\{0\}$ and where the σ_i are pairwise different. We call $\sigma_1,\ldots,\sigma_s$ the pieces of c.

Definition.

(1) Given a bounded subset A of $\mathbb{R}^{n+1}$ the diameter is defined as

$$\operatorname{diam}(A) \;:=\; \sup\big\{\|a-b\| \,\big|\, a,b\in A\big\} \in \mathbb{R}_{\geq 0}.$$

(2) Given a singular simple $\sigma\colon \Delta^n \to X$ we denote by $|\sigma|$ the image of σ.

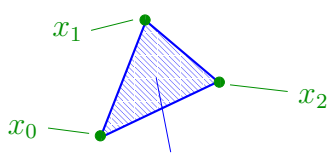
Definition. For points $z_0,\dots,z_k \in \mathbb{R}^n$ we consider the map

$$\begin{aligned} [z_0,\dots,z_k]\colon \Delta^k &\to \mathbb{R}^n \\ (t_0,\dots,t_k) &\mapsto \sum_{j=0}^{k} t_j\cdot z_j. \end{aligned}$$

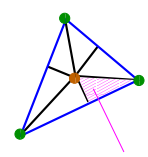
The map $[z_0,\dots,z_k]$ thus sends the standard k-simplex Δ^k to the simplex spanned by $z_0,\dots,z_k$ in $\mathbb{R}^n$.

Lemma 40.11. (Subdivision–Diameter Lemma) For any $x_0,\dots,x_k \in \mathbb{R}^n$ and any piece τ of $u_k([x_0,\dots,x_k])$ we have

$$\operatorname{diam}(|\tau|) \;\leq\; \tfrac{n}{n+1}\cdot \operatorname{diam}(|[x_0,\dots,x_k]|).$$



the image of $[x_0,x_1,x_2]$



the image of a piece τ of $u_2([x_0,x_1,x_2])$

Lemma 40.12. (Simplex–Diameter Lemma) Let $y_0,\dots,y_k \in \mathbb{R}^n$.

(1) We have the equality

$$\operatorname{diam}(|[y_0,\dots,y_k]|) \;=\; \max\big\{\|y_i-y_j\| \,\big|\, i,j=1,\dots,k\big\}.$$

(2) We denote by $b=\frac{1}{k}\cdot\sum_{i=0}^{k} y_i$ the barycenter of $y_0,\dots,y_k$. For any $i=0,\dots,k$ we have

$$\|y_i-b\| \;\leq\; \operatorname{diam}(|[y_0,\dots,y_k]|).$$

Proof of the Simplex–Diameter Lemma 40.12.

(1) The inequality "$\geq$" follows immediately from the definitions and the observation that each y_i lies in

$$|[y_0,\dots,y_k]| \;=\; \Big\{\sum_{i=0}^{k} t_i\cdot y_i \,\Big|\, t_i\in[0,1] \text{ and } \sum_{i=0}^{k} t_i = 1\Big\}.$$

We turn to the proof of the reverse inequality "$\leq$". So we pick $a,b\in|[y_0,\dots,y_k]|$. By definition we can write $a=\sum_{i=0}^{k} s_i\cdot y_i$ and $b=\sum_{i=0}^{k} t_i\cdot y_i$ with $s_i,t_i\in[0,1]$ and such that

$\sum_{i=0}^{k} s_i = \sum_{i=0}^{k} t_i = 1$. We then see that

$$\begin{aligned}
\|a-b\| &= \Big\|a - \sum_{i=0}^{k} t_i \cdot y_i\Big\| \overset{\text{since } \sum_{i=0}^{k} t_i = 1}{=} \Big\|\sum_{i=0}^{k} t_i \cdot (a-y_i)\Big\| \le \sum_{i=0}^{k} \|t_i \cdot (a-y_i)\| \\
&\le \sum_{i=0}^{k} t_i \cdot \max_{i=0,\dots,k}\{\|a-y_i\|\} \underset{\text{since } \sum_{i=0}^{k} t_i = 1}{=} \max_{i=0,\dots,k}\{\|a-y_i\|\} \underset{\text{same argument applied to } a = \sum_{j=0}^{k} s_j \cdot y_j}{\le} \max_{i,j=0,\dots,k}\{\|y_j - y_i\|\}
\end{aligned}$$

(2) We denote by b_i the barycenter of $y_0, \dots, \widehat{y_i}, \dots, y_k$, i.e.

$$b_i = \tfrac{1}{k} \cdot \sum_{j \ne i} y_j.$$

We then see that

$$b = \frac{1}{k+1} \cdot y_i + \frac{k}{k+1} \cdot b_i.$$

It follows that

$$\begin{aligned}
\|b - y_i\| &= \Big\|\frac{1}{k+1} \cdot y_i + \frac{k}{k+1} \cdot b_i - y_i\Big\| = \Big\|\frac{k}{k+1} \cdot b_i - \frac{k}{k+1} \cdot y_i\Big\| \\
&= \frac{k}{k+1} \cdot \|b_i - y_i\| \le \frac{k}{k+1} \cdot \operatorname{diam}(|[y_0, \dots, y_k]|).
\end{aligned}$$

Thus we have obtained the desired inequality. ■

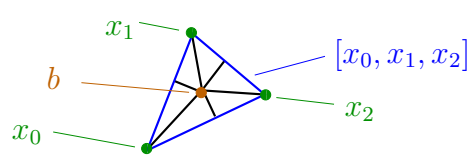
Proof of the Subdivision–Diameter Lemma 40.11. Recall that we need to show that for any $x_0, \dots, x_k \in \mathbb{R}^n$ and any piece τ of $u_k([x_0, \dots, x_k])$ we have

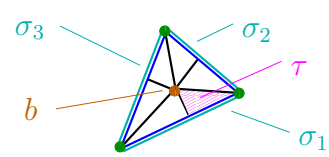
$$\operatorname{diam}(|\tau|) \le \frac{n}{n+1} \cdot \operatorname{diam}(|[x_0, \dots, x_k]|).$$

We prove the claim by induction on k. For $k = 0$ all the diameters are zero, so the inequality holds trivially. Now suppose that the claim holds for $k-1$. Let $x_0, \dots, x_k \in \Delta^n$. We denote by $b = \frac{1}{k} \cdot \sum_{i=0}^{k} x_i$ the barycenter of $x_0, \dots, x_k$. We denote by $\sigma_1, \dots, \sigma_r$ the pieces of $\partial[x_0, \dots, x_k]$.

Now let τ be a piece of $u_k([x_0, \dots, x_k])$. By the definition of the subdivision map given on page 338 we see that $|\tau| = |[w_0, \dots, w_{k-1}, b]|$ where $[w_0, \dots, w_{k-1}]$ is a piece of $u_{k-1}(\sigma_s)$ for some $s \in \{1, \dots, r\}$. By the Simplex–Diameter Lemma 40.12 (1) it now suffices to prove the following claim.

Claim.
(1) For any $i, j = 0, \dots, k-1$ we have $\|w_i - w_j\| \le \frac{k}{k+1} \cdot \operatorname{diam}(|[x_0, \dots, x_k]|)$.
(2) For any $i = 0, \dots, k-1$ we have $\|w_i - b\| \le \frac{k}{k+1} \cdot \operatorname{diam}(|[x_0, \dots, x_k]|)$.





Proof. First let $i, j \in \{0, \dots, k-1\}$. Then

$$\|w_i - w_j\| \;\le\; \operatorname{diam}([w_0, \dots, w_{k-1}]) \;\underset{\substack{\uparrow \\ \text{induction hypothesis} \\ \text{applied to } \sigma_s}}{\le}\; \frac{k-1}{k} \cdot \operatorname{diam}(\sigma_s) \;\underset{\substack{\uparrow \\ \text{since } |\sigma_s| \subset |[x_0, \dots, x_k]| \text{ and } \frac{k-1}{k} \le \frac{k}{k+1}}}{\le}\; \frac{k}{k+1} \cdot \operatorname{diam}([x_0, \dots, x_k]).$$

Now let $i \in \{0, \dots, k-1\}$. We then see that

$$\|w_i - b\| \;\underset{\substack{\uparrow \\ \text{by the Simplex–Diameter Lemma 40.12 (2)}}}{\le}\; \frac{k}{k+1} \cdot \operatorname{diam}(|[w_0, \dots, w_{k-1}]|) \;\underset{\substack{\uparrow \\ \text{since } |[w_0, \dots, w_{k-1}]| \subset |[x_0, \dots, x_k]|}}{\le}\; \frac{k}{k+1} \cdot \operatorname{diam}(|[x_0, \dots, x_k]|)$$

■

Lemma 40.13. (Simplex–Reduction Lemma) Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. For every singular n-simplex $\sigma \colon \Delta^n \to X$ there exists an $m \in \mathbb{N}_0$ such that $u_n^m(\sigma)$ lies in $\mathrm{C}_n^{\mathcal{U}}(X)$.

Proof. Let X be a topological space, let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X and let $\sigma \colon \Delta^n \to X$ be a singular n-simplex. By the Lebesgue Lemma 11.9 (2) there exists a $\delta > 0$ such that for any subset A of Δ^n with diameter $\operatorname{diam}(A) < \delta$ there exists an $i \in I$ such that $A \subset \sigma^{-1}(\mathring{U}_i)$.

We pick an $m \in \mathbb{N}$ such that $\left(\frac{n}{n+1}\right)^m < \delta$. We calculate that

$$\begin{aligned} u_n^m(\sigma) \;=\; u_n^m(\sigma_*(\mathrm{id}_{\Delta^n})) \;\overset{\substack{\text{by the Subdivision–Natural Lemma 40.7} \\ \downarrow}}{=}\; \sigma_*(u_n^m(\mathrm{id}_{\Delta^n})) \;\overset{\substack{\text{by the Subdivision–Diameter Lemma 40.11 and } \left(\frac{n}{n+1}\right)^m < \delta \\ \downarrow}}{=}\; & \sigma_*\left(\begin{matrix}\text{simplicial chain such that the image} \\ \text{has diameter less than } \delta\end{matrix}\right) \\ \underset{\substack{\uparrow \\ \text{by definition of } \delta}}{=}\; & \sigma_*\left(\begin{matrix}\text{simplicial chain such that the image} \\ \text{of each piece lies in some } \sigma^{-1}(\mathring{U}_i)\end{matrix}\right). \end{aligned}$$

But this means precisely that $u_n^m(\sigma) \in \mathrm{C}_n^{\mathcal{U}}(X)$. ■

Proof of the Comfortable Covering Proposition 40.6. Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. We want to show that the inclusion map $\iota \colon \mathrm{C}_*^{\mathcal{U}}(X) \to \mathrm{C}_*(X)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

$$\iota_* \colon \mathrm{H}_n^{\mathcal{U}}(X) \;\xrightarrow{\cong}\; \mathrm{H}_n(X).$$

We first show that the map $\iota_* \colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ is surjective. Let $z \in \mathrm{H}_n(X)$ be given. We choose a cycle $c = \sum_{j=1}^{r} a_j \cdot \sigma_j \in \mathrm{C}_n(X)$ that represents $z \in \mathrm{H}_n(X)$. It follows from the Simplex–Reduction Lemma 40.13, applied to the singular n-simplices $\sigma_1, \dots, \sigma_r$, that there exists an $m \in \mathbb{N}_0$ such that $u_n^m(c) \in \mathrm{C}_n^{\mathcal{U}}(X)$. By the Iterated Subdivision Lemma 40.10 the maps $\{u_n^m\}_{n \in \mathbb{N}_0}$ form a chain map that is chain homotopic to the identity. By definition this means that there exists a chain homotopy $P_k \colon \mathrm{C}_k(X) \to \mathrm{C}_{k+1}(X)$, $k \in \mathbb{N}_0$. Thus we obtain that

$$z \;=\; [c] \;\underset{\substack{\uparrow \\ \text{since } \partial_{n+1}P_n + P_{n-1}\partial_n \;=\; u_n^m - \mathrm{id}}}{=}\; \big[u_n^m(c) - P_{n-1}\overbrace{\partial_n(c)}^{=0} - \partial_{n+1}P_n(c)\big] \;=\; \big[u_n^m(c) - \partial_{n+1}P_n(c)\big] \;\underset{\substack{\uparrow \\ \text{equality in } \mathrm{H}_n(X)}}{=}\; \overbrace{[u_n^m(c)]}^{\in \iota_*(\mathrm{H}_n^{\mathcal{U}}(X))}.$$

Thus we have shown that $\iota_* \colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ is surjective.

The argument that the map $\iota_*\colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ is injective is very similar. Indeed, let $[c] \in \mathrm{H}_n^{\mathcal{U}}(X)$ be an element such that $[c] = 0 \in \mathrm{H}_n(X)$. This means that there exists an $e \in \mathrm{C}_{n+1}(X)$ with $\partial e = c$. From the Simplex–Reduction Lemma 40.13 it follows, as above, that there exists an $m \in \mathbb{N}_0$ with $u_{n+1}^m(e) \in \mathrm{C}_{n+1}^{\mathcal{U}}(X)$. Again we choose a chain homotopy $\{P_k\}_{k\in\mathbb{N}_0}$ between the chain maps $\{u_k^m\}_{k\in\mathbb{N}_0}$ and id. Then we have

$$c \;=\; \partial e \;\underset{\substack{\uparrow \\ \text{since } \partial P_{n+1} + P_n\partial = u_{n+1}^m - \mathrm{id}}}{=}\; \partial\big(u_{n+1}^m(e) - \partial P_{n+1}(e) - P_n\partial e\big) \;=\; \partial \underbrace{u_{n+1}^m(e)}_{\in \mathrm{C}_{n+1}^{\mathcal{U}}(X)} \;-\; \partial \underbrace{P_n(c)}_{\in \mathrm{C}_{n+1}^{\mathcal{U}}(X)}.$$

Thus have we have shown that c is indeed the boundary of an $(n+1)$-dimensional singular chain in $C_{n+1}^{\mathcal{U}}(X)$ i.e. $[c] = 0 \in \mathrm{H}_n^{\mathcal{U}}(X)$. Put differently, we have now proved that $\iota_*\colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ is injective. ■

Proof of the Excision Theorem 40.1. Let X be a topological space. Furthermore let $Z \subset A \subset X$ be subsets such that the closure of Z is contained in the interior of A. We have to show that the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \;\xrightarrow{\cong}\; \mathrm{H}_n(X, A).$$

We put $B := X \setminus Z$. Note that $A \cap B = A \setminus Z$. It follows from our hypothesis on Z and A that X is the union of the interiors of A and B. Put differently, the two sets $\{A, B\}$ form a comfortable cover of X.

For each $n \in \mathbb{N}_0$ we consider the following commutative diagram of abelian groups:

$$\begin{tikzcd}
 & & & \mathrm{C}_n(B, A\cap B) \arrow[d, "\cong"', "{[\sigma]\mapsto[\sigma]}"] & \\
0 \arrow[r] & \mathrm{C}_n(A) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{C}_n^{\{A,B\}}(X) \arrow[r] \arrow[d, hook] & \mathrm{C}_n^{\{A,B\}}(X)/\mathrm{C}_n(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{C}_n(A) \arrow[r] & \mathrm{C}_n(X) \arrow[r] & \mathrm{C}_n(X, A) \arrow[r] & 0.
\end{tikzcd}$$

We make the following two observations:

(1) By definition the horizontal sequences are short exact sequences of chain complexes.
(2) It follows immediately from the definition of the group $\mathrm{C}_n^{\{A,B\}}(X)$ that the inclusion $\mathrm{C}_n(B) \to \mathrm{C}_n^{\{A,B\}}(X)$ induces a chain map $\mathrm{C}_n(B, A\cap B) \to \mathrm{C}_n^{\{A,B\}}(X)/\mathrm{C}_n(A)$ which is in fact an isomorphism.

Now we consider the long exact sequence of homology groups arising from the above short exact sequences of chain complexes. It follows from the LES-Proposition 37.4, together with the above vertical isomorphism, that we obtain the following commutative diagram

where the horizontal sequences are exact:

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}_n(A) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{H}_n^{\{A,B\}}(X) \arrow[r] \arrow[d] & \overbrace{\mathrm{H}_n(B, A\cap B)}^{=\mathrm{H}_n(X\setminus Z, A\setminus Z)} \arrow[r, "\partial_n"] \arrow[d] & \mathrm{H}_{n-1}(A) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{H}_{n-1}^{\{A,B\}}(X) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & \mathrm{H}_n(A) \arrow[r] & \mathrm{H}_n(X) \arrow[r] & \mathrm{H}_n(X, A) \arrow[r, "\partial_n"] & \mathrm{H}_{n-1}(A) \arrow[r] & \mathrm{H}_{n-1}(X) \arrow[r] & \dots
\end{tikzcd}$$

From the Comfortable Covering Proposition 40.6 it follows that the second and fifth vertical map are isomorphisms. Since the first and fourth map are evidently isomorphisms it now follows from the Five Lemma 37.5 that the middle map is also an isomorphism. ■

41. Applications of the Excision Theorem

41.1. Suspensions and homology groups.

Definition. Let X be a non-empty topological space.

(1) We define the suspension of X to be the topological space

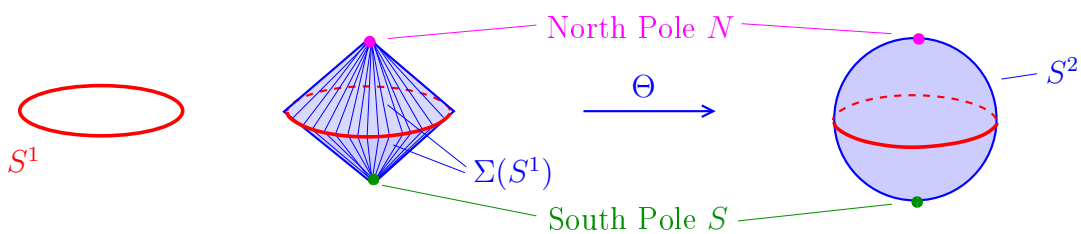
$$\Sigma(X) \;:=\; ((X \times [-1,1])/(X \times \{-1\}))/(X \times \{1\}).$$

(2) We call $S := [X \times \{-1\}]$ the South Pole and we call $N := [X \times \{1\}]$ the North Pole of $\Sigma(X)$.

Example. One can easily verify that the map

$$\begin{array}{rcl} \Theta\colon \Sigma(S^n) & \xrightarrow{\cong} & S^{n+1} \subset \mathbb{R}^{n+1} \times \mathbb{R} \\ [(P,t)] & \mapsto & \big(\underbrace{P \cdot \cos\left(\frac{\pi}{2}t\right)}_{\in \mathbb{R}^{n+1}}, \underbrace{\sin\left(\frac{\pi}{2}t\right)}_{\in \mathbb{R}}\big) \end{array}$$

is a homeomorphism.



Definition.

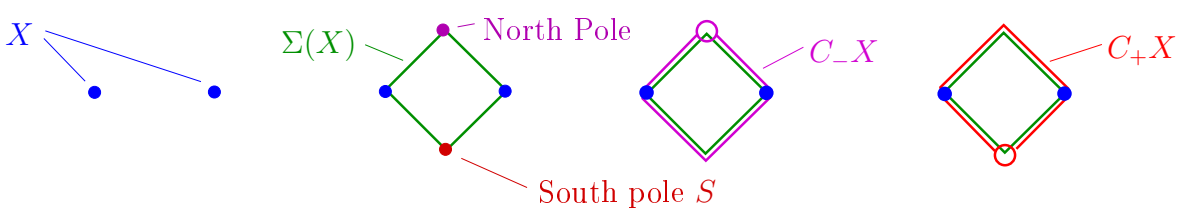
(1) We set

$$C_-X \;:=\; \Sigma(X) \setminus \{N\} \qquad \text{and} \qquad C_+X \;:=\; \setminus\{S\}.$$

(2) Given a continuous map $f\colon X \to Y$ between topological spaces we refer to

$$\begin{array}{rcl} \Sigma(f)\colon \Sigma(X) & \to & \Sigma(Y) \\ [(x,t)] & \mapsto & [(f(x),t)] \end{array}$$

as the suspension of f.



Proposition 41.1. (Suspension-H$_*$-Proposition) Let X be a topological space and let $n \in \mathbb{Z}$.

(1) We write $I := (-1, 1)$. The following maps are isomorphisms:

$$\begin{tikzcd}[column sep=large]
\widetilde{\mathrm{H}}_{n+1}(\Sigma(X)) \arrow[r, "\cong", "\text{the natural map defined on page 331}"'] & \mathrm{H}_{n+1}(\Sigma(X), C_-X) & \mathrm{H}_{n+1}(C_+X, X\times I) \arrow[l, "\text{induced by the inclusion}"'] \arrow[r, "\partial_{n+1}", "\text{connecting homomorphism of the pair } (C_+X, X\times I) \text{ from the Reduced LES-of-Pair Proposition 39.6}"'] & \widetilde{\mathrm{H}}_n(X\times I) \arrow[r, phantom, "\leftarrow"] & \widetilde{\mathrm{H}}_n(X).
\end{tikzcd}$$

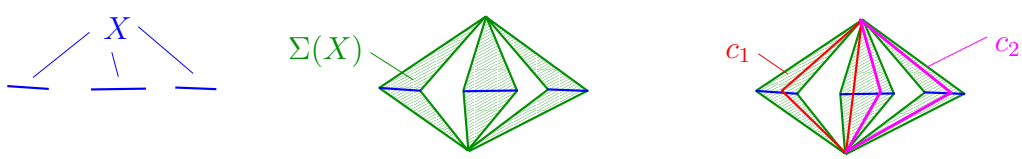
induced by the embedding $x \mapsto (x, 0)$

(2) The resulting isomorphism

$$\Sigma_X \colon \widetilde{\mathrm{H}}_{n+1}(\Sigma(X)) \xrightarrow{\cong} \widetilde{\mathrm{H}}_n(X)$$

is natural.

Example. Let $n \in \mathbb{N}$ and let X be the disjoint union of $n \in \mathbb{N}$ intervals. It follows from the Suspension-H$_*$-Proposition 41.1 together with the Reduced Homology Lemma 39.2 that we have isomorphisms $\mathrm{H}_1(\Sigma(X)) \cong \widetilde{\mathrm{H}}_0(X) \cong \mathbb{Z}^{n-1}$. In the figure below we show a generating set for $\mathrm{H}_1(\Sigma(X))$ for $n = 3$. We leave it to the reader to deduce from the proof of the Suspension-H$_*$-Proposition 41.1 that the given cycles do indeed form a generating set for $\mathrm{H}_1(\Sigma(X))$.



$\{c_1, c_2\}$ is a basis for $\mathrm{H}_1(\Sigma(X))$

Proof.

(1) Let X be a topological space. For each $m \in \mathbb{N}_0$ we consider the following homomorphisms:

$$\begin{tikzcd}[column sep=small]
\longrightarrow \widetilde{\mathrm{H}}_{n+1}(C_-X) \arrow[r] & \widetilde{\mathrm{H}}_{n+1}(\Sigma(X)) \arrow[r, "\cong"] & \mathrm{H}_{n+1}(\Sigma(X), C_-X) \arrow[r, "\partial_{n+1}"] & \widetilde{\mathrm{H}}_n(C_-X) \arrow[r] & \dots \\
\dots \arrow[r] & \widetilde{\mathrm{H}}_{n+1}(X\times I) \arrow[r] & \widetilde{\mathrm{H}}_{n+1}(C_+X) \arrow[r] & \mathrm{H}_{n+1}(C_+X, X\times I) \arrow[u, "\cong"] \arrow[r, "\partial_{n+1}", "\cong"'] & \widetilde{\mathrm{H}}_n(X\times I) \arrow[r] & \widetilde{\mathrm{H}}_n(C_+X) \arrow[r] & {} \\
& & & & \widetilde{\mathrm{H}}_n(X) \arrow[u, "i_*", "\cong"'] & &
\end{tikzcd}$$

We make the following clarifications and explanations:

(a) The top horizontal sequence is the long exact sequence in reduced homology of the pair $(\Sigma(X), C_-X)$ that is given by the Reduced LES-of-Pair Proposition 39.6.

(b) The bottom horizontal sequence is the long exact sequence in reduced homology of the pair $(C_+X, X \times I)$ that is given by the Reduced LES-of-Pair Proposition 39.6.

(c) As is explained in [Fri23], both C_-X and C_+X admit a deformation retraction to a point. It follows from the Deformation Retract–Homotopy Equivalence Lemma 9.3 together with the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5 that for all $k \in \mathbb{N}_0$ we have $\widetilde{\mathrm{H}}_k(C_-X) = 0$ and that we have $\widetilde{\mathrm{H}}_k(C_+X) = 0$.

(d) It follows from (c) and the exactness of the horizontal sequences that the horizontal maps shown in red are isomorphisms.

(e) As before let $S = [X \times \{-1\}]$ be the South Pole of the suspension. Note that $\Sigma(X) \setminus \{S\} = C_+X$ and $C_-X \setminus \{S\} = C_-X \cap C_+X = X \times I$. Using this observation we obtain immediately from the Excision Theorem 40.1 that the vertical map $\mathrm{H}_{n+1}(C_+X, X \times I) \to \mathrm{H}_{n+1}(\Sigma(X), C_-X)$ is an isomorphism.

(f) We denote by $i\colon X \to X \times I$ the natural embedding given by $x \mapsto (x, 0)$. Since I is an interval one can easily show, using the Deformation Retract–Homotopy Equivalence Lemma 9.3, that i is a homotopy equivalence.

(g) It follows from (f) together with the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 39.4 that the induced map $i_*\colon \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(X \times I)$ is an isomorphism.

We have now shown that all red homomorphisms are isomorphisms. But that is exactly what he had to do.

(2) Let $f\colon X \to Y$ be a continuous map between two topological spaces. We consider the following diagram:

$$\begin{tikzcd}
\widetilde{\mathrm{H}}_{n+1}(\Sigma(X)) \arrow[r, "\rightsquigarrow"] \arrow[rrr, bend left, "\Sigma_X"] \arrow[d, "\Sigma(f)_*"] & \mathrm{H}_{n+1}(\Sigma(X), C_-X) \arrow[d, "\Sigma(f)_*"] & \mathrm{H}_{n+1}(C_+X, X \times I) \arrow[l] \arrow[r, "\partial_{n+1}"] \arrow[d, "(f\times\mathrm{id})_*"] & \widetilde{\mathrm{H}}_n(X \times I) \arrow[d, "(f\times\mathrm{id})_*"] & \widetilde{\mathrm{H}}_n(X) \arrow[l, "i_*"'] \arrow[d, "f_*"] \\
\widetilde{\mathrm{H}}_{n+1}(\Sigma(Y)) \arrow[r, "\rightsquigarrow"] \arrow[rrr, bend right, "\Sigma_Y"'] & \mathrm{H}_{n+1}(\Sigma(Y), C_-Y) & \mathrm{H}_{n+1}(C_+Y, Y \times I) \arrow[l] \arrow[r, "\partial_{n+1}"] & \widetilde{\mathrm{H}}_n(Y \times I) & \widetilde{\mathrm{H}}_n(Y). \arrow[l, "i_*"']
\end{tikzcd}$$

The first, second and fourth square commutes by functoriality of (reduced) homology. The third square commutes since in the LES-of-Triple Proposition 38.2 (2) we showed that the connecting homomorphism is natural. We have thus shown that the diagram commutes, which implies that Σ_X itself is natural. ■

41.2. The homology groups of spheres.

Proposition 41.2. (Sphere-H_n-Proposition) For any $n \in \mathbb{Z}_{\geq -1}$ we have

$$\widetilde{\mathrm{H}}_k(S^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = n, \text{ and} \\ 0, & \text{if } k \neq n. \end{cases}$$

Proof. For each $m \in \mathbb{Z}_{\geq -1}$ we have

$$\begin{tikzcd}[column sep=small]
\widetilde{\mathrm{H}}_m(S^n) \arrow[d, phantom, "\cong"] & \widetilde{\mathrm{H}}_m(\Sigma(S^{n-1})) \arrow[r, phantom, "\overset{\text{by the Suspension-H}_*\text{-Proposition 41.1}}{\cong}"] & \widetilde{\mathrm{H}}_{m-1}(S^{n-1}) \arrow[r, phantom, "\overset{\text{iterating the argument}}{\cong}"] & \dots \arrow[r, phantom, "\cong"] & \widetilde{\mathrm{H}}_{m-n}(S^0) \arrow[d, phantom, "\cong"] & \begin{cases} \mathbb{Z}, & \text{if } m-n=0, \\ 0, & \text{otherwise.} \end{cases} \\
\text{since } S^n \text{ is homeomorphic to the suspension } \Sigma(S^{n-1}) & & & & \text{by the Singleton-H}_*\text{-Lemma 35.11, the Path-Component–H}_*\text{–Lemma 35.8 and the Reduced Homology Lemma 39.2} &
\end{tikzcd}$$

■

Proposition 41.3. ($\mathbb{R}^k$-$\mathbb{R}^l$-Non Homeomorphism–Proposition) For any $k \neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic.

Proof. Let $k, l \in \mathbb{N}_0$ and suppose there exists a homeomorphism $f\colon \mathbb{R}^k \to \mathbb{R}^l$. Then we obtain that

$$\mathbb{Z} \underset{\substack{\uparrow \\ \text{by the Sphere-H}_n\text{-Proposition 41.2}}}{\cong} \widetilde{\mathrm{H}}_{k-1}(S^{k-1}) \overset{\substack{\text{this follows from the Homotopy Equivalence-}\widetilde{\mathrm{H}}_*\text{-Corollary 39.4 (1a), since for} \\ \text{any } P \in \mathbb{R}^m \text{ the topological space } \mathbb{R}^m \setminus \{P\} \text{ is homotopy equivalent to } S^{m-1} \\ \downarrow}}{\cong} \widetilde{\mathrm{H}}_{k-1}(\mathbb{R}^k \setminus \{0\}) \underset{\substack{\uparrow \\ \text{isomorphism since } f \text{ is a homeomorphism}}}{\xrightarrow{f_*}} \widetilde{\mathrm{H}}_{k-1}(\mathbb{R}^l \setminus \{f(0)\}) \overset{\substack{\downarrow \\ \cong}}{} \widetilde{\mathrm{H}}_{k-1}(S^{l-1}).$$

So we have $\widetilde{\mathrm{H}}_{k-1}(S^{l-1}) \cong \mathbb{Z}$. But by the Sphere-H$_n$-Proposition 41.2 that is only possible if $k = l$. ■

Lemma 41.4. (Ball-rel-Sphere-H$_*$-Lemma) Given any $n \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(\overline{B}^n, S^{n-1}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We consider the long exact sequence in reduced homology of the pair $(\overline{B}^n, S^{n-1})$ which is provided for us by the Reduced LES-of-Pair Proposition 39.6:

$$\ldots \to \underbrace{\widetilde{\mathrm{H}}_k(\overline{B}^n)}_{= 0\text{, see page 331}} \to \mathrm{H}_k(\overline{B}^n, S^{n-1}) \xrightarrow{\partial} \underbrace{\widetilde{\mathrm{H}}_{k-1}(S^{n-1})}_{\substack{\text{calculated in the} \\ \text{Sphere-H}_n\text{-Proposition 41.2}}} \to \underbrace{\widetilde{\mathrm{H}}_{k-1}(\overline{B}^n)}_{= 0\text{, see page 331}} \to \ldots$$

The statement of the lemma is now an immediate consequence of the Sphere-H$_n$-Proposition 41.2. ■

41.3. The Brouwer Fixed Point Theorem.

Proposition 41.5. (Sphere-Non Retract Proposition) For every $n \in \mathbb{N}$ the sphere $S^{n-1} = \partial\overline{B}^n$ is not a retract of the closed ball $\overline{B}^n$.

Proof. Suppose there exists a retraction $r\colon \overline{B}^n \to S^{n-1}$. We denote by $i\colon S^{n-1} \to \overline{B}^n$ the inclusion map. We consider the following diagram

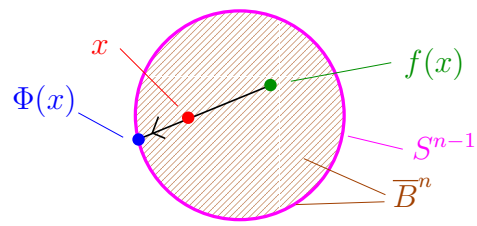
$$\begin{tikzcd} & \widetilde{\mathrm{H}}_{n-1}(\overline{B}^n) \arrow[dr, "r_*"] & \\ \widetilde{\mathrm{H}}_{n-1}(S^{n-1}) \arrow[ur, "i_*"] \arrow[rr, "(r \circ i)_*"] & & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}). \end{tikzcd}$$

This diagram commutes since it follows from the functoriality of the reduced homology groups that $r_* \circ i_* = (r \circ i)_*$. Since r is a retraction it follows that $r \circ i = \mathrm{id}_{S^{n-1}}$, i.e. $(r \circ i)_* = (\mathrm{id}_{S^{n-1}})_*$ is the identity map on $\widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ which, by the Sphere-H$_n$-Proposition 41.2, is isomorphic to $\mathbb{Z}$. The lower map is thus an isomorphism. But the upper map factorizes through the group $\widetilde{\mathrm{H}}_{n-1}(\overline{B}^n)$, which is the trivial group by the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5. This implies that the composition of i_* and r_* cannot be an isomorphism of $\mathbb{Z}$. Thus we have obtained a contradiction. ■

Theorem 41.6. (Brouwer Fixed Point Theorem) Every continuous map $f\colon \overline{B}^n \to \overline{B}^n$ admits a fixed point, i.e. for every continuous map $f\colon \overline{B}^n \to \overline{B}^n$ there exists a point $x \in \overline{B}^n$ such that $f(x) = x$.

Proof. Suppose there exists a continuous map $f\colon \overline{B}^n \to \overline{B}^n$ without a fixed point. It follows from the hypothesis that f admits no fixed points that for each $x \in \overline{B}^n$ the open ray $\{f(x) + t \cdot (x - f(x)) \,|\, t > 0\}$ which starts at $f(x)$ and which goes through x contains precisely one point in S^{n-1}. Thus we can consider the following map:

$$\begin{array}{rcl} \Phi\colon \overline{B}^n & \to & \overline{B}^n \\ x & \mapsto & \begin{array}{c}\text{the intersection point of } S^{n-1} \text{ with the} \\ \text{uniquely determined open ray from } f(x) \text{ to } x.\end{array} \end{array}$$



It is elementary to show that Φ is continuous. It is evident that $\Phi(x) = x$ for all $x \in S^{n-1}$ and that $\Phi(x) \in S^{n-1}$ for all $x \in S^{n-1}$. The map Φ is thus a retraction from $\overline{B}^n$ to S^{n-1}. But according to the Sphere-Non Retract Proposition 41.5 this not possible. Thus we have obtained a contradiction. ■

Theorem 41.7. (Perron-Frobenius Theorem) Let $A = (a_{ij})$ be an $n \times n$-real matrix. If all entries are positive, then A has a positive eigenvalue λ with an eigenvector v such that all entries of v are non-negative.

Proof. We need to introduce two players:

(1) Given $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ we write $|v| := \sum\limits_{i=1}^{n} |v_i|$.

(2) We consider the standard $(n-1)$-simplex

$$\Delta^{n-1} \;:=\; \{v \in [0,1]^n \,\big|\, |v| = 1\}.$$

We saw in the Δ^n-Homeomorphism Lemma 35.3 (2) that Δ^{n-1} is homeomorphic to $\overline{B}^{n-1}$. It follows immediately from this observation and from the Brouwer Fixed Point Theorem 41.6 that any continuous self-map $f\colon \Delta^{n-1} \to \Delta^{n-1}$ has a fixed point.

Now let $A = (a_{ij})$ be an $n \times n$-real matrix such that all entries are positive. We consider the map

$$\begin{array}{rcl} f\colon \Delta^{n-1} & \to & \Delta^{n-1} \\ v = (v_1, \ldots, v_n) & \mapsto & \frac{A \cdot v}{|A \cdot v|}. \end{array}$$

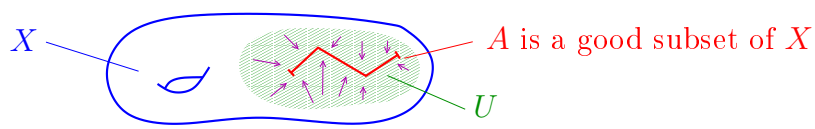
Note that this map is defined since our hypothesis that all a_{ij} are positive, the fact that all entries of v are non-negative and the fact that at least one entry of v is positive imply that $A \cdot v \neq 0$. It is clear that f is continuous. As we had just discussed above, the map $f\colon \Delta^{n-1} \to \Delta^{n-1}$ has a fixed point $v \in \Delta^{n-1}$. But this means that $A \cdot v = |A \cdot v| \cdot v$. Put differently, we have just shown that v is an eigenvector to the positive eigenvalue $|A \cdot v|$. ■

41.4. Good subsets and relative homology.

Definition. Let X be a topological space and let $A \subset X$ be a subset. We denote by $p\colon (X,A) \to (X/A, A/A)$ the natural projection. For each $n \in \mathbb{N}_0$ we define

$$\begin{tikzcd} \mathrm{H}_n(X,A) \arrow[r, "p_*"] \arrow[rr, bend right, "=:\ \text{ю}"'] & \mathrm{H}_n(X/A, A/A) & \widetilde{\mathrm{H}}_n(X/A)\ . \arrow[l, "\substack{\text{isomorphism ж from the} \\ \text{Reduced-to-Relative-H}_*\text{-Lemma 39.7}}"', "\cong"] \end{tikzcd}$$

Definition. Let X be a topological space and let $A \subset X$ be a subset. We say A is good if A is closed and if there exists an open neighborhood U of A in X such that A is a deformation retract of U.



Examples.

(1) Let $x \in X$ be a point. Basically by definition the subset $\{x\} \subset X$ is good in the above sense if and only the point is good in the sense of the definition on page 185.

(2) The sphere S^{n-1} is a good subset of the closed ball $\overline{B}^n$. Indeed, it is clear that S^{n-1} is a closed subset of $\overline{B}^n$. Furthermore, if we consider the open neighborhood $U = \overline{B}^n \setminus \{0\}$, then a deformation retraction from U to S^{n-1} is given by the map

$$\begin{aligned} (\overline{B}^n \setminus \{0\}) \times [0,1] &\to \overline{B}^n \setminus \{0\} \\ (x,t) &\mapsto \left(1 - t \cdot \tfrac{1-\|x\|}{\|x\|}\right) \cdot x. \end{aligned}$$

(3) It follows from the Subcomplex–Closed Lemma 30.6 and the Subcomplex–Neighborhood Proposition 31.8 (1) that every subcomplex A of a CW-complex X is good.

Proposition 41.8. (Relative-to-Quotient-H$_*$-Proposition) Let X be a topological space, let $A \subset X$ be a subset and let $n \in \mathbb{N}_0$. If A is a good subset, then the natural homomorphism ю$\colon \mathrm{H}_n(X,A) \to \widetilde{\mathrm{H}}_n(X/A)$ is an isomorphism.

Example. Let X be a topological space, let $A \subset X$ be a subset and let $n \in \mathbb{N}_0$. For better or worse the natural homomorphism $\mathrm{H}_n(X,A) \to \widetilde{\mathrm{H}}_n(X/A)$ is in general not an isomorphism. In fact in general the two groups $\mathrm{H}_n(X,A)$ and $\widetilde{\mathrm{H}}_n(X/A)$ are not even abstractly isomorphic.

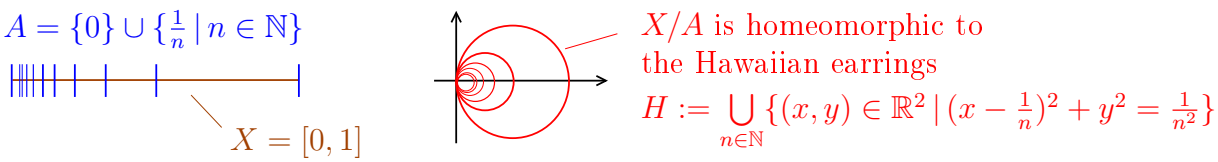
For example let us consider the topological space $X = [0,1]$ together with the subspace $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$. We leave the proof of the following two statements as an exercise to the reader.

(1) The group $\mathrm{H}_1(X,A)$ is countable.

(2) The group $\mathrm{H}_1(X/A)$ is uncountable.

Proof of the Relative-to-Quotient-H$_*$-Proposition 41.8. Let X be a topological space, let $A \subset X$ be a good subset and let $n \in \mathbb{N}_0$. We make the following simple preparations:

(1) Since A is good we can pick an open neighborhood U of A in X such that A is a deformation retract of U.



(2) We denote by $p\colon X \to X/A$ the projection.
(3) We denote by $q\colon X \setminus A \to (X/A) \setminus (A/A)$ the restriction of p to $X \setminus A$.

Now we consider the following diagram

$$\begin{tikzcd}
 & \mathrm{H}_n(X,A) \arrow[r, "(3)"] \arrow[d, "p_*"] \arrow[dd, bend right=60, "ю"'] & \mathrm{H}_n(X,U) \arrow[d, "p_*"] & \mathrm{H}_n(X\setminus A, U\setminus A) \arrow[l, "(1)"'] \arrow[d, "q_*"] \\
 & \mathrm{H}_n(X/A, A/A) \arrow[r, "(4)"'] & \mathrm{H}_n(X/A, U/A) & \mathrm{H}_n((X/A)\setminus(A/A), (U/A)\setminus(A/A)) \arrow[l, "(2)"] \\
 & \widetilde{\mathrm{H}}_n(X/A), \arrow[u, "\cong"', "ж"] & &
\end{tikzcd}$$

where all the horizontal maps are induced by inclusions of pairs of topological spaces. Recall that we need to show that ю$\colon \mathrm{H}_n(X,A) \to \widetilde{\mathrm{H}}_n(X/A)$ is an isomorphism. By the Reduced-to-Relative-H_*-Lemma 39.7 we know that ж$\colon \widetilde{\mathrm{H}}_n(X/A) \to \mathrm{H}_n(X/A, A/A)$ is an isomorphism. Thus it suffices to show that the top left vertical map $p_*\colon \mathrm{H}_n(X,A) \to \mathrm{H}_n(X/A, A/A)$ is an isomorphism.

First note that it follows easily from the functoriality of relative homology groups that the two squares of the diagram commute. Next note that it follows from our hypothesis that the subset A is closedand an elementary argument in general topology that the map $q\colon (X \setminus A, U \setminus A) \to ((X/A) \setminus (A/A), (U/A) \setminus (A/A))$ is a homeomorphism of pairs of topological spaces. We obtain from this observation that the vertical map on the right is an isomorphism. At this stage it remains to prove the following claim:

Claim. All horizontal maps are isomorphisms.

Proof.

(1) Since $A \subset X$ is closed and since $U \subset X$ is open it follows from the Excision Theorem 40.1 that both the top right horizontal map is an isomorphism.
(2) The hypothesis that $A \subset X$ is closed and that $U \subset X$ is an open neighborhood of $A \subset X$ implies, by definition of the quotient topology, that A/A is a closed subset of X/A and that U/A is an open neighborhood of $A/A \subset X/A$. Thus we can also apply the Excision Theorem 40.1 to the horizontal map on the bottom right, and we see that this map is also an isomorphism.
(3) Since A is deformation retract of U we obtain from the Homotopy Equivalence-Relative-H_*-Corollary 38.4 that the top left horizontal map is an isomorphism.
(4) Recall that A is a deformation retract of U. It follows from the Deformation Retraction-Quotient Theorem 9.2 (2) that A/A is a deformation retract of U/A. As in (3) we now see that the bottom-left horizontal map is an isomorphism. ■

41.5. **The homology groups of the wedge of topological spaces.**

Proposition 41.10. (Wedge-H_*-Proposition) Let $\{(X_k, x_k)\}_{k\in K}$ be a family of pointed topological spaces. We denote the wedge point of $\bigvee_{k\in K}(X_k, x_k)$ by $*$ and given $k \in K$ we denote by $i_k\colon X_k \to \bigvee_{k\in K}(X_k, x_k)$ the natural inclusion that we introduced on page 155. Let $n \in \mathbb{N}_0$. If each $x_k \in X_k$ is good , then the map

$$\bigoplus_{k\in K} i_{k*}\colon \bigoplus_{k\in K} \mathrm{H}_n(X_k, \{x_k\}) \to \mathrm{H}_n\Big(\bigvee_{k\in K}(X_k, x_k), \{*\}\Big)$$

is an isomorphism. The same statement also holds for reduced homology instead of relative homology.

Proof. We consider the following diagram:

$$\begin{tikzcd}
\bigoplus_{k\in K} \widetilde{\mathrm{H}}_n(X_k) \arrow[rrr, "\bigoplus_{k\in K} i_{k*}"] \arrow[dd, "\cong", "\text{induced by the maps } \varkappa_{X_k,\{x_k\}}"'] & & & \widetilde{\mathrm{H}}_n\Big(\bigvee_{k\in K}(X_k, x_k)\Big) \arrow[dd, "\cong"', "\varkappa_{\bigvee(X_k,x_k),\{*\}}"] \\
& \mathrm{H}_n\Big(\bigsqcup_{k\in K} X_k, \bigsqcup_{k\in K}\{x_k\}\Big) \arrow[r, "\cong"] & \widetilde{\mathrm{H}}_n\Big(\bigsqcup_{k\in K} X_k \Big/ \bigsqcup_{k\in K}\{x_k\}\Big) \arrow[ur, "\mathrm{id}"] & \\
\bigoplus_{k\in K} \mathrm{H}_n(X_k, \{x_k\}) \arrow[ur, "\cong", "\bigoplus_{k\in K} i_{k*}"'] \arrow[rrr, "\bigoplus_{k\in K} i_{k*}"] & & & \mathrm{H}_n\Big(\bigvee_{k\in K}(X_k, x_k), *\Big).
\end{tikzcd}$$

The proposition follows from the following clarifications and observations:

(a) It follows from the Reduced-to-Relative-H_*-Lemma 39.7 that the vertical maps to the left and right are isomorphisms.

(b) It follows from the obvious generalization of the Path-Component–H_*–Lemma 35.8 that the bottom left diagonal map is an isomorphism.

(c) By hypothesis each $x_k \in X_k$ is good. It follows easily from the definitions that $\bigsqcup_{k\in K}\{x_k\}$ is a good subset of $\bigsqcup_{k\in K} X_k$. Thus we can apply the Relative-to-Quotient-H_*-Proposition 41.8 to see that the middle horizontal map is an isomorphism.

(d) The top right diagonal map is indeed the identity, we just wrote the wedge of the (X_k, x_k) in two different ways.

(e) One can easily verify that the lower triangle commutes. This implies the desired statement for relative homology groups.

(f) One can also easily verify that the upper triangle commutes. Thus we also get the desired statement on reduced homology groups. ■

41.6. The long exact sequence for quotients.

Theorem 41.11. (Quotient–LES Theorem) Let X be a topological space. Furthermore let $A \subset X$ be a good subset. We denote by $i\colon A \to X$ the inclusion map and we

denote by $p\colon X \to X/A$ the projection. For each $n \in \mathbb{N}_0$ there exists a natural homomorphism $\partial_n\colon \widetilde{\mathrm{H}}_n(X/A) \to \widetilde{\mathrm{H}}_{n-1}(A)$ such that the sequence

$$\begin{array}{ccccccc} & & & \dots & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_{n+1}(X/A) & \xrightarrow{\partial_{n+1}} \\ \to & \widetilde{\mathrm{H}}_n(A) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_n(X) & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_n(X/A) & \xrightarrow{\partial_n} \\ \to & \widetilde{\mathrm{H}}_{n-1}(A) & \xrightarrow{i_*} & \dots & & & \end{array}$$

is exact.

Proof. Let X be a topological space and let $A \subset X$ be a good subset. We denote by $i\colon A \to X$ the inclusion and we denote by $p\colon X \to X/A$ the projection. We consider the following diagram:

$$\begin{tikzcd} \dots \arrow[r] & \widetilde{\mathrm{H}}_n(A) \arrow[r, "i_*"] & \widetilde{\mathrm{H}}_n(X) \arrow[r, "ж_{X,A}"] \arrow[dr, "p_*"'] & \mathrm{H}_n(X,A) \arrow[r, "\partial_n"] \arrow[d, "ю"] & \widetilde{\mathrm{H}}_{n-1}(A) \arrow[r, "i_*"] & \widetilde{\mathrm{H}}_{n-1}(X) \arrow[r] & \dots \\ & & & \widetilde{\mathrm{H}}_n(X/A). & & & \end{tikzcd}$$

We make the following comments and observations:

(1) The horizontal sequence is the long exact sequence coming from the Reduced LES-of-Pair Proposition 39.6.
(2) The vertical map is the natural homomorphism that we defined on page 352. It follows easily from the definitions that the triangle commutes.
(3) Since A is good we obtain from the Relative-to-Quotient-H_*-Proposition 41.8 that the vertical map ю is an isomorphism.

Now we consider the homomorphisms $\widetilde{\partial}_n := ю^{-1} \circ \partial_n\colon \widetilde{\mathrm{H}}_n(X/A) \to \widetilde{\mathrm{H}}_{n-1}(A)$. Since ю and ∂ are natural we see that the maps $\widetilde{\partial}_n = ю^{-1} \circ \partial$ are also natural. It follows from the above discussion that the sequence

$$\dots \longrightarrow \widetilde{\mathrm{H}}_n(A) \xrightarrow{i_*} \widetilde{\mathrm{H}}_n(X) \xrightarrow{p_*} \widetilde{\mathrm{H}}_n(X/A) \xrightarrow{\widetilde{\partial}_n} \widetilde{\mathrm{H}}_{n-1}(A) \xrightarrow{i_*} \widetilde{\mathrm{H}}_{n-1}(X) \longrightarrow \dots$$

is exact. ■

Proposition 41.2. (Sphere-H_n-Proposition) For each $n \in \mathbb{Z}_{\geq -1}$ we have

$$\widetilde{\mathrm{H}}_k(S^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = n, \text{ and} \\ 0, & \text{if } k \neq n. \end{cases}$$

Proof. We prove the proposition by induction on n. First note that for $n = -1$ we have $S^n = S^{-1} = \varnothing$, thus the statement follows immediately from the Reduced Homology Lemma 39.2 (0).

Now we suppose that the statement of the proposition holds for $n-1$. We want to determine the reduced homology of S^n. The key idea is to use that by the Ball Quotient-Sphere Lemma 6.5 we know that S^n is homeomorphic to $\overline{B}^n/S^{n-1}$. On page 352 we saw that S^{n-1} is a good subset of $\overline{B}^n$. Therefore we can apply the Quotient–LES Theorem 41.11

to $X = \overline{B}^n$ and the closed subset $A = S^{n-1}$ and we obtain the following exact sequence

$$\begin{array}{rcccccc} & \dots & \longrightarrow & \widetilde{\mathrm{H}}_{k+1}(\overline{B}^n) & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_{k+1}(\overline{B}^n/S^{n-1}) & \xrightarrow{\partial_{k+1}} \\ \rightarrow & \widetilde{\mathrm{H}}_k(S^{n-1}) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_k(\overline{B}^n) & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_k(\overline{B}^n/S^{n-1}) & \xrightarrow{\partial_k} \\ \rightarrow & \widetilde{\mathrm{H}}_{k-1}(S^{n-1}) & \xrightarrow{i_*} & \dots & & & \end{array}$$

By the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5 we know that $\widetilde{\mathrm{H}}_i(\overline{B}^n) = 0$ for all i. Since the sequence is exact we see that all the connecting homomorphisms are isomorphisms. Thus we see that

$$\widetilde{\mathrm{H}}_k(S^n) \underset{\substack{\uparrow \\ \text{in the Ball Quotient-Sphere} \\ \text{Lemma 6.5 we showed} \\ \text{that } S^n \cong \overline{B}^n/S^{n-1}}}{\cong} \widetilde{\mathrm{H}}_k(\overline{B}^n/S^{n-1}) \underset{\substack{\uparrow \\ \text{by the above exact} \\ \text{sequence}}}{\cong} \widetilde{\mathrm{H}}_{k-1}(S^{n-1}) \underset{\substack{\uparrow \\ \text{by induction hypothesis}}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } k-1 = n-1 \\ 0, & \text{if } k \neq n. \end{cases}$$

■

42. Local homology and explicit generators of homology groups

42.1. An explicit generator of $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$.

Lemma 42.1. (Identity-is-Generator Lemma) For every $n \in \mathbb{N}_0$ the following two statements hold:
(1) $\mathrm{H}_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$.
(2) The identity map $\mathrm{id}\colon \Delta^n \to \Delta^n$ represents a generator of $\mathrm{H}_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$.

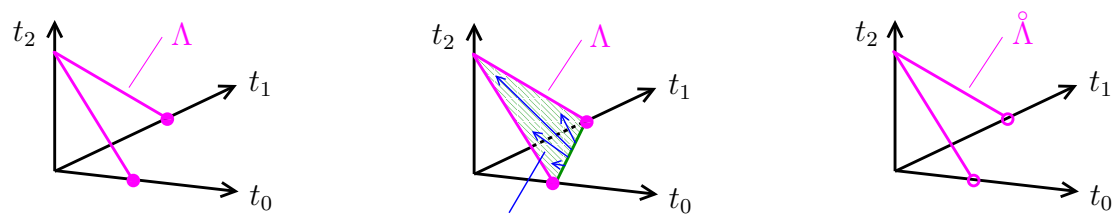
Proof. We prove the lemma by induction on $n \in \mathbb{N}_0$. It follows easily from the H_0-Proposition 35.10 (1) that the statement holds for $n = 0$. Now we suppose that the desired statement holds for $n - 1$ with $n \in \mathbb{N}$.

Our next goal is to find a connection between the two homology groups $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$ and $\mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$. To do so we introduce the following notation.

(a) We consider
$$\Lambda \;:=\; \bigcup_{i=0}^{n-1} \{(t_0, \ldots, t_n) \in \Delta^n \,|\, t_i = 0\},$$
i.e. Λ is the union of all $(n-1)$-dimensional faces of Δ^n except for the face given by $t_n = 0$. Note that Λ is a deformation retract of Δ^n.

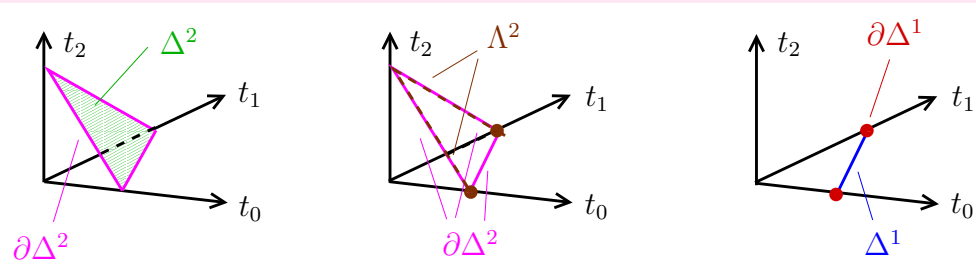
(b) We write
$$\mathring{\Lambda} \;:=\; \{(t_0, \ldots, t_n) \in \Lambda \,|\, t_n > 0\}.$$

In the following we view $\mathbb{R}^n$ as the subspace of $\mathbb{R}^{n+1}$ given by setting the last coordinate equal to zero. This point of view allows us to view Δ^{n-1} as a subspace of Δ^n. Note that with this convention we have $\partial\Delta^{n-1} \subset \Lambda$.



deformation retraction from Δ^2 to Λ

Claim. We consider the maps
$$\mathrm{H}_n(\Delta^n, \partial\Delta^n) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda) \xleftarrow{\iota_*} \mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}).$$
Here the map ∂_n on the left is the connecting homomorphism in the long exact sequence of the triple $(\Delta^n, \partial\Delta^n, \Lambda)$ and the right-hand map is induced by the inclusion. Both maps are isomorphisms and the following equality holds:
$$\partial([\mathrm{id}_{\Delta^n}]) \;=\; (-1)^n \cdot \iota_*([\mathrm{id}_{\Delta^{n-1}}]).$$



It is clear that a proof of the claim together with the induction hypothesis implies that $\mathrm{H}_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$ and that $[\mathrm{id}_{\Delta^n}]$ is a generator of $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$. In particular a proof of the claim implies the Identity-is-Generator Lemma 42.1.

Proof. We consider the two maps separately.

(a) Of the two maps in the claim we first consider the map ∂_n on the left. It follows from the LES-of-Triple Proposition 38.2 applied to the triple $(\Delta^n, \partial\Delta^n, \Lambda)$ of topological spaces that the following sequence is exact:

$$\mathrm{H}_n(\Delta^n, \Lambda) \xrightarrow{p_*} \mathrm{H}_n(\Delta^n, \partial\Delta^n) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda) \xrightarrow{\iota_*} \mathrm{H}_{n-1}(\Delta^n, \Lambda)$$

where $p\colon (\Delta^n, \Lambda) \to (\Delta^n, \partial\Delta^n)$ is the obvious inclusion. As we remarked above, one can easily show that Λ is a deformation retract of Δ^n. It follows from the Deformation Retract-Relative H_*-Corollary 38.5 that $\mathrm{H}_n(\Delta^n, \Lambda) = 0$, which implies that the connecting homomorphism $\partial_n\colon \mathrm{H}_n(\Delta^n, \partial\Delta^n) \to \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda)$ is an isomorphism. Furthermore note that we have the following equalities in $\mathrm{H}_n(\partial\Delta^n, \Lambda)$:

$$\partial_n([\mathrm{id}_{\Delta^n}]) \underset{\substack{\uparrow \\ \text{by definition of the connecting} \\ \text{homomorphism in the} \\ \text{LES-of-Triple Proposition 38.2}}}{=} [\partial_n(\mathrm{id}_{\Delta^n})] \overset{\substack{\text{see the definition of } \partial_n\colon \mathrm{C}_n(\Delta^n) \to \mathrm{C}_{n-1}(\Delta^n) \text{ on page 302} \\ \downarrow}}{=} \Bigg[\underbrace{\sum_{j=0}^{n-1} (-1)^j \cdot \mathrm{id} \circ i_j^n}_{\substack{\text{lies in } \mathrm{C}_{n-1}(\Lambda), \\ \text{thus } = 0 \text{ in } \mathrm{C}_n(\partial\Delta^n, \Lambda)}} + (-1)^n \cdot \underbrace{\mathrm{id} \circ i_n^n}_{=\mathrm{id}_{\Delta^{n-1}}} \Bigg] \underset{\substack{\uparrow \\ \text{since we work} \\ \text{in } \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda)}}{=} (-1)^n \cdot [\mathrm{id}_{\Delta^{n-1}}].$$

(b) We turn our attention to the right-hand map $\iota_*\colon \mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) \to \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda)$ in the statement of the claim. It follows from the Excision-via-Deformation Theorem 40.5, applied to $X = \partial\Delta^n$, $A = \Lambda$, $Z = \mathring{\Lambda}$ and $U = (0, \ldots, 0, 1)$ that the inclusion induced map

$$\mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) = \mathrm{H}_{n-1}(\underbrace{\partial\Delta^n \setminus \mathring{\Lambda}}_{=\Delta^{n-1}}, \underbrace{\Lambda \setminus \mathring{\Lambda}}_{=\partial\Delta^{n-1}}) \xrightarrow{\iota_*} \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda)$$

is an isomorphism. Finally note that evidently we have $\iota_*([\mathrm{id}_{\Delta^{n-1}}]) = [\mathrm{id}_{\Delta^{n-1}}]$.

Thus we have proved that both maps in the claim are isomorphisms and that

$$\partial([\mathrm{id}_{\Delta^n}]) = (-1)^n \cdot [\mathrm{id}_{\Delta^{n-1}}] = (-1)^n \cdot \iota_*([\mathrm{id}_{\Delta^{n-1}}]).$$ ■

42.2. The local homology groups of a topological space.

Definition. Given a topological space X and a point $x \in X$ the relative homology group $\mathrm{H}_n(X, X \setminus \{x\})$ is called the n-th local homology group of X at the point x.

Proposition 42.2. (Local-H_*-Proposition) Let X be a topological space and let $x \in X$.

(1) If $\{x\} \subset X$ is a closed subset , then for any neighborhood U of $x \in X$ the inclusion induced maps $\mathrm{H}_n(U, U \setminus \{x\}) \to \mathrm{H}_n(X, X \setminus \{x\})$ are isomorphisms.

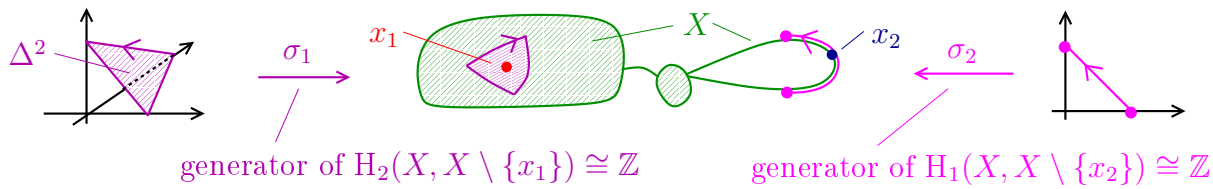
(2) For any homeomorphism $f\colon X \to Y$ the maps $f_*\colon \mathrm{H}_n(X, X \setminus \{x\}) \to \mathrm{H}_n(Y, Y \setminus \{f(x)\})$ are isomorphisms.

(3) (a) If X is an open subset of $\mathbb{R}^n$, then for any $x \in X$ and any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(X, X \setminus \{x\}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

(b) If X is an open subset of $\mathbb{H}^n = \{(t_1, \dots, t_n) \in \mathbb{R}^n \mid t_n \geq 0\}$ and if $x \in \mathbb{R}^{n-1} \times \{0\}$, then $\mathrm{H}_k(X, X \setminus \{x\}) = 0$ for all $k \in \mathbb{N}_0$.

(4) If $\sigma\colon \Delta^n \to X$ is a continuous map such that $\sigma\colon \mathring{\Delta}^n \to X$ is an open embedding, then for any $x \in \sigma(\mathring{\Delta}^n)$ we have $\mathrm{H}_n(X, X \setminus \{x\}) \cong \mathbb{Z}$ and the singular n-simplex $\sigma\colon \Delta^n \to X$ represents a generator of $\mathrm{H}_n(X, X \setminus \{x\}) \cong \mathbb{Z}$.



Proof. Let X be a topological space and let $x \in X$.

(1) Let U be a neighborhood of $x \in X$. It follows immediately from the Excision Theorem 40.2 that the inclusion induced maps $\mathrm{H}_n(U, U \setminus \{x\}) \to \mathrm{H}_n(X, X \setminus \{x\})$ are isomorphisms.

(2) This statement follows from the functoriality of relative homology groups.

(3) (a) It follows easily from (1) that it suffices to consider the case that $X = \mathbb{R}^n$. For any $k \in \mathbb{N}_0$ we see that

$$\mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \xrightarrow[\substack{\text{this is the connecting homomorphism of the long exact sequence of the pair } (\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \\ \text{in reduced homology, it is an isomorphism since on page 331 we showed that } \widetilde{\mathrm{H}}_*(\mathbb{R}^n) = 0}]{\partial_k} \widetilde{\mathrm{H}}_{k-1}(\mathbb{R}^n \setminus \{x\}) \xleftarrow[\substack{\text{isomorphism since the inclusion } S^{n-1}(x) \to \mathbb{R}^n \setminus \{x\} \\ \text{is a homotopy equivalence}}]{\cong} \widetilde{\mathrm{H}}_{k-1}(S^{n-1}(x)) \underset{\substack{\text{Sphere-H}_n\text{-Proposition 41.2}}}{\cong} \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

(b) It follows easily from (1) that it suffices to consider the case that $X = \mathbb{H}^n$ and that $x \in \mathbb{R}^{n-1} \times \{0\}$. By the Reduced LES-of-Pair Proposition 39.6 we have a long exact sequence of the form

$$\dots \to \widetilde{\mathrm{H}}_k(\mathbb{H}^n) \to \mathrm{H}_k(\mathbb{H}^n, \mathbb{H}^n \setminus \{x\}) \xrightarrow{\cong} \widetilde{\mathrm{H}}_k(\mathbb{H}^n \setminus \{x\}) \to \dots$$

It follows easily from the fact that $x \in \mathbb{R}^{n-1} \times \{0\}$ that both $\mathbb{H}^n$ and $\mathbb{H}^n \setminus \{x\}$ are contractible. It follows from the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5 that the reduced homology groups of $\mathbb{H}^n$ and $\mathbb{H}^n \setminus \{x\}$ vanish. But this implies, by exactness, that also the "middle" groups $\mathrm{H}_k(\mathbb{H}^n, \mathbb{H}^n \setminus \{x\})$ have to vanish.

(4) Let X be a topological space and let $\sigma\colon \Delta^n \to X$ be a continuous map such that $\sigma\colon \mathring{\Delta}^n \to X$ is an open embedding. Furthermore let $x \in \sigma(\mathring{\Delta}^n)$. Since $x \in \sigma(\mathring{\Delta}^n)$ and since $\sigma\colon \mathring{\Delta}^n \to X$ is an embedding there exists a unique $a \in \mathring{\Delta}^n$ with $\sigma(a) = x$.

Claim. There exists a continuous map $\tau\colon \Delta^n \to X$ with the following three properties:
(a) $\tau\colon \Delta^n \to X$ is an embedding.
(b) $\tau(\Delta^n)$ is a neighborhood of $x \in X$.
(c) We have $[\sigma] = [\tau] \in \mathrm{H}_n(X, X \setminus \{x\})$.

Proof. We introduce the following notation: We denote by $b := (\frac{1}{n+1}, \dots, \frac{1}{n+1}) \in \Delta^n$ the barycenter of Δ^n. Furthermore given $\epsilon > 0$ we write

$$\Delta^n_\epsilon \;=\; \{(x-b)\cdot(1-\epsilon)+b \,|\, x \in \Delta^n\}.$$

Note that $\Delta^n_\epsilon \subset \Delta^n$ and note that $\Delta^n_0 = \Delta^n$. It follows from an elementary argument and the hypothesis that $a \in \Delta^n \setminus \partial\Delta^n$ that there exists an $\epsilon > 0$ such that $a \in \Delta^n_\epsilon$. Next we consider the homotopy

$$\begin{array}{rcl} F\colon \Delta^n \times [0,1] & \to & \Delta^n \\ (x,t) & \mapsto & (x-b)\cdot(1-t+(1-\epsilon)\cdot t)+b. \end{array}$$

Note that $F_0 = \mathrm{id}_{\Delta^n}$ and note that F_1 defines a homeomorphism $\Delta^n \to \Delta^n_\epsilon$. We set $\tau := \sigma \circ F_1\colon \Delta^n \to X$. We have to verify that τ has the desired properties:

(a) Since $F_1\colon \Delta^n \to \Delta^n$ is an embedding with image Δ^n_ϵ, since $\Delta^n_\epsilon \subset \mathring{\Delta}^n$ and since $\sigma\colon \mathring{\Delta}^n \to X$ is an embedding we see that $\tau = \sigma \circ F_1\colon \Delta^n \to X$ is an embedding.

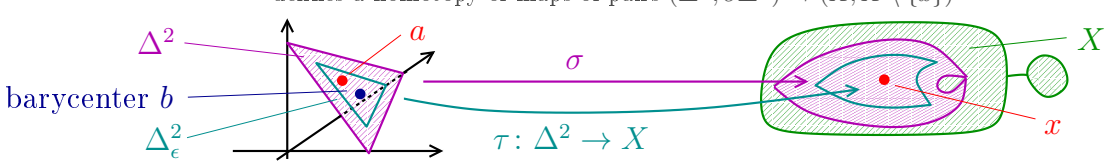
(b) It is straightforward to verify that Δ^n_ϵ is a neighborhood of $a \in \Delta^n$. By hypothesis $\sigma(\mathring{\Delta}^n)$ is an open subset of X. It follows from the Open-Closed Inclusion Lemma 2.7 that $\tau(\Delta^n) = \sigma(\Delta^n_\epsilon)$ is a neighborhood of $x = \sigma(a) \in X$.

(c) We have the following equalities in $\mathrm{H}_n(X, X\setminus\{x\})$:

$$[\sigma] \;=\; (\sigma\circ F_0)_*([\mathrm{id}_{\Delta^n}]) \;\underset{\uparrow}{=}\; (\sigma\circ F_1)_*([\mathrm{id}_{\Delta^n}]) \;=\; [\sigma\circ F_1\circ \mathrm{id}_{\Delta^n}] \;=\; [\tau].$$

follows from the Homotopy-H_*-Proposition 38.6 since $\sigma \circ F$ defines a homotopy of maps of pairs $(\Delta^n, \partial\Delta^n) \to (X, X\setminus\{x\})$

⊞



Finally we consider the following sequence of homomorphisms:

since $\tau(\Delta^n)$ is a neighborhood of x it follows from the Excision Theorem 40.2 that this inclusion induced map is an isomorphism

isomorphism since τ is an embedding

$\partial\Delta^n$ is a deformation retract of $\Delta^n \setminus \{a\}$, thus it follows from the Homotopy Equivalence–Relative-H_*-Corollary 38.4 that the map is an isomorphism

$$\begin{array}{ccccccc} \mathrm{H}_n(X, X\setminus\{x\}) & \leftarrow & \mathrm{H}_n(\tau(\Delta^n), \tau(\Delta^n)\setminus\{x\}) & \xleftarrow{\tau_*} & \mathrm{H}_n(\Delta^n, \Delta^n\setminus\{a\}) & \leftarrow & \mathrm{H}_n(\Delta^n, \partial\Delta^n). \\ [\sigma] = [\tau] & \leftarrow\!\shortmid & [\tau] & \leftarrow\!\shortmid & [\mathrm{id}_{\Delta^n}] & \leftarrow\!\shortmid & [\mathrm{id}_{\Delta^n}]. \end{array}$$

By the Identity-is-Generator Lemma 42.1 we know that id_{Δ^n} represents a generator of $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$. It follows from the above explicit isomorphisms that σ represents a generator of $\mathrm{H}_k(X, X\setminus\{x\})$. ■

Definition. Let (X, A) be a pair of topological spaces. In the following we say a singular n-chain $c = \sum_{i=1}^{m} t_i \cdot \sigma_i \in \mathrm{C}_n(X)$ is non-trivial with respect to $x \in X \setminus A$ if the following statements hold:

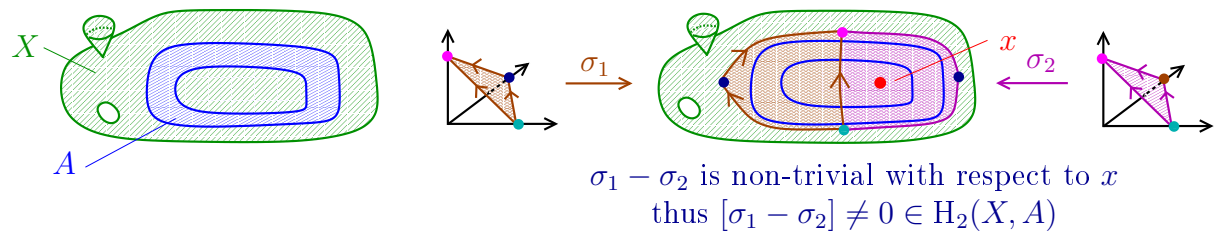
(a) We have $\partial_n(c) \in \mathrm{C}_{n-1}(A)$, in other words, c represents an n-cycle in $\mathrm{C}_n(X, A)$.

(b) There exists a $j \in \{1, \ldots, m\}$ with $t_j = \{\pm 1\}$ such that the map $\sigma_j \colon \mathring{\Delta}^n \to X$ is an open embedding, such that $x \in \sigma_j(\mathring{\Delta}^n)$ and such that for all $i \neq j$ we have $x \notin \sigma_i(\Delta^n)$.

Proposition 42.3. (Detect-Cycle Proposition) Let (X, A) be a pair of topological spaces and let $c = \sum_{i=1}^{m} t_i \cdot \sigma_i \in \mathrm{C}_n(X)$ be a singular n-chain that is non-trivial with respect to some $x \in X \setminus A$. The following statements hold:

(1) We have $\mathrm{H}_n(X, X \setminus \{x\}) \cong \mathbb{Z}$ and under the map $\mathrm{H}_n(X, A) \to \mathrm{H}_n(X, X \setminus \{x\})$ the singular n-cycle $c \in \mathrm{C}_n(X, A)$ gets sent to a generator of $\mathrm{H}_n(X, X \setminus \{x\})$.

(2) If $\mathrm{H}_n(X, A) \cong \mathbb{Z}$, then c represents a generator of $\mathrm{H}_n(X, A)$.



Proof. Let (X, A) be a pair of topological spaces and let $c = \sum_{i=1}^{m} t_i \cdot \sigma_i \in \mathrm{C}_n(X)$ be a singular n-chain with the following two properties:

(a) c represents an n-cycle in $\mathrm{C}_n(X, A)$.

(b) There exists an $x \in X \setminus A$ and a $j \in \{1, \ldots, m\}$ with $t_j = \{\pm 1\}$ such that the map $\sigma_j \colon \mathring{\Delta}^n \to X$ is an open embedding, such that $x \in \sigma_j(\mathring{\Delta}^n)$ and such that for all $i \neq j$ we have $x \notin \sigma_i(\Delta^n)$.

We turn to the actual proofs of the statements:

(1) First note that it follows from the hypothesis that $x \in \varphi_j(\mathring{\Delta}^n)$ and the hypothesis that $\sigma_j \colon \mathring{\Delta}^n \to X$ is an open embedding, together with the the Local-H_*-Proposition 42.2 that $\mathrm{H}_n(X, X \setminus \{x\}) \cong \mathbb{Z}$ and that $[\sigma_j] \in \mathrm{H}_n(X, X \setminus \{x\})$ is a generator. Next we consider the induced homomorphism $\varphi \colon \mathrm{H}_n(X, A) \to \mathrm{H}_n(X, X \setminus \{x\})$. We see that

$$\varphi([c]) \;=\; \varphi\Big(\Big[\sum_{i=1}^{m} t_i \cdot \sigma_i\Big]\Big) \;=\; \underbrace{t_j}_{=\pm 1} \cdot [\sigma_j] + \varphi\Big(\Big[\sum_{i \neq j} \underbrace{t_j \cdot \sigma_j}_{\substack{= 0 \in \mathrm{C}_n(X, X \setminus \{x\}),\\ \text{since } \sigma_j(\Delta^n) \subset X \setminus \{x\}}}\Big]\Big) \;=\; \pm \cdot \underbrace{[\sigma_j]}_{\substack{\uparrow\\ \text{generator of}\\ \mathrm{H}_n(X, X \setminus \{x\})}}.$$

(2) It follows from the hypothesis that $\mathrm{H}_n(X, A) \cong \mathbb{Z}$ and elementary algebra that any epimorphism $\mathrm{H}_n(X, A) \to \mathbb{Z}$ is actually an isomorphism. Statement (2) is an immediate consequence of this observation together with Statement (1). ■

42.3. The standard generator of $\mathrm{H}_n(S^n)$.

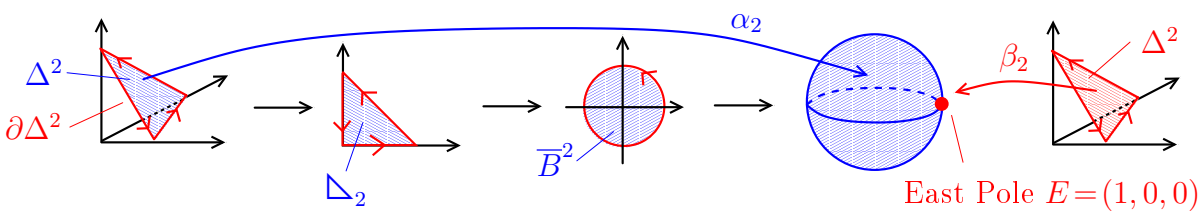
Notation. Let $n \in \mathbb{N}_0$.

(1) We consider the singular n-simplex $\alpha_n \colon \Delta^n \to S^n$ that is given by the concatenation of the following two continuous maps

$$\begin{array}{ccccccc} \Delta^n & \xrightarrow[\Delta^n\text{-Homeomorphism Lemma 35.3 (2)}]{\text{homeomorphism given by the}} & \overline{B}^n & \to & S^n & \subset & \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \\ & & r \cdot v & \mapsto & \underbrace{\begin{pmatrix} -\cos(\pi \cdot r) \\ \sin(\pi \cdot r) \cdot v \end{pmatrix}}_{\in \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}}. & & \end{array}$$

Similar to the Ball Quotient-Sphere Lemma 6.5 we see that this map $\Delta^n \to S^n$ sends $\partial\Delta^n$ to the "East Pole" $E := (1, 0, \ldots, 0)$ and that α_n descends to a homeomorphism $\Delta^n/\partial\Delta^n \to S^n$.

(2) We denote by $\beta_n \colon \Delta^n \to S^n$ the constant map that sends all points to the "East Pole" $(1, 0, \ldots, 0)$.



Proposition 42.4. (Standard Generator-$\mathrm{H}_n(S^n)$-Proposition) Let $n \in \mathbb{N}_0$.

(1) The singular n-chain $\alpha_n - \beta_n \in \widetilde{\mathrm{C}}_n(S^n)$ is a cycle and it represents a generator of $\widetilde{\mathrm{H}}_n(S^n)$.

(2) If n is odd, then $\alpha_n \in \mathrm{C}_n(S^n)$ is also a cycle with $[\alpha_n] = [\alpha_n - \beta_n] \in \widetilde{\mathrm{H}}_n(S^n)$.

Definition. Let $n \in \mathbb{N}_0$. We refer to

$$[S^n] \;:=\; (-1)^{n-1} \cdot [\alpha_n - \beta_n] \quad \in \; \widetilde{\mathrm{H}}_n(S^n)$$

as the standard generator of $\widetilde{\mathrm{H}}_n(S^n)$.

Example.

(1) For $n = 0$ we see that $[S^0] = -[\alpha_0 - \beta_0] = -[(-1) - (1)] = [(1) - (-1)] \in \widetilde{\mathrm{H}}_0(S^0)$.

(2) Next we consider the case $n = 1$. If one goes carefully through the definition of the map $\alpha_1 \colon \Delta^1 \to S^1$ one sees that

$$\alpha_1(1-t, t) \;=\; (-\cos(\pi(1-2t)), \sin(\pi(1-2t))) \;=\; (\cos(2t\pi), \sin(2t\pi)) \;=\; \exp(2\pi\,\mathrm{i}t).$$

Thus we see that the singular 1-simplex $(1-t, t) \mapsto \exp(2\pi\,\mathrm{i}t)$ represents the standard generator of $\mathrm{H}_1(S^1)$.

Proof.

(1) We start out with the following claim.

Claim. The singular n-chain $\alpha_n - \beta_n \in \widetilde{\mathrm{C}}_n(S^n)$ is a cycle.

Proof. Let $\gamma\colon \Delta^{n-1} \to S^n$ be the singular $(n-1)$-simplex that sends all points to the "East Pole" $E = (1, 0, \dots, 0)$. For $n \neq 0$ we calculate that

$$\partial_n(\alpha_n - \beta_n) \;=\; \partial_n(\alpha_n) - \partial_n(\beta_n) \;\underset{\substack{\uparrow \\ \text{see the definition} \\ \text{of } \partial_n \text{ on page 302}}}{=}\; \sum_{j=0}^{n} \cdot (-1)^j \cdot \underbrace{\alpha_n \circ i_j^n}_{\substack{= \gamma, \text{ since} \\ \alpha_n(\partial\Delta^n) = \{E\}}} \;-\; \sum_{j=0}^{n} \cdot (-1)^j \cdot \underbrace{\beta_n \circ i_j^n}_{\substack{= \gamma, \text{ since} \\ \beta_n(\partial\Delta^n) = \{E\}}} \;=\; 0.$$

For $n = 0$ the same argument also shows that $\epsilon_{S^0}(\alpha_0 - \beta_0) = 0$. ⊞

We set $x := (-1, 0, \dots, 0)$. We consider the long exact sequence in reduced homology of the pair $(S^n, S^n \setminus \{x\})$:

$$\widetilde{\mathrm{H}}_n(S^n \setminus \{x\}) \;\to\; \widetilde{\mathrm{H}}_n(S^n, S^n \setminus \{x\}) \;\xrightarrow{\text{я}}\; \widetilde{\mathrm{H}}_n(S^n) \;\xrightarrow{\partial_n}\; \widetilde{\mathrm{H}}_{n-1}(S^n \setminus \{x\}).$$

It follows from the Stereographic Projection Lemma 2.11 that $S^n \setminus \{x\}$ is homeomorphic to $\mathbb{R}^n$. This implies by the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5 that $\widetilde{\mathrm{H}}_n(S^n \setminus \{x\}) = \widetilde{\mathrm{H}}_{n-1}(S^n \setminus \{x\}) = 0$. It follows that $\text{я}\colon \widetilde{\mathrm{H}}_n(S^n) \to \widetilde{\mathrm{H}}_n(S^n \setminus \{x\})$ is an isomorphism.

Next note that it follows easily from the definitions that $\alpha_n - \beta_n$ is non-trivial with respect to the given $x \in S^n$. It follows from the Detect-Cycle Proposition 42.3 (1) that $\alpha_n - \beta_n$ represents a generator of $\widetilde{\mathrm{H}}_n(S^n, S^n \setminus \{x\})$. Thus by the above it also represents a generator of $\widetilde{\mathrm{H}}_n(S^n)$.

(2) If n is odd, then we know by the simple calculation in the proof of the Singleton-H_*-Lemma 35.11 that β_n is a cycle which is the boundary of the singular $(n+1)$-simplex that is given by the constant map $\gamma\colon \Delta^{n+1} \to (1, 0, \dots, 0)$. Thus we see that $[\alpha_n - \beta_n] = [\alpha_n - \partial\gamma] = [\alpha_n] \in \widetilde{\mathrm{H}}_n(S^n)$. ■

43. Homology and self-maps of spheres

43.1. The degree of a self-map of a sphere.

Definition. Let $(G,+)$ be a group that is isomorphic to $\mathbb{Z}$ and let $\Phi\colon G \to G$ be a homomorphism. Note that Φ is given by multiplication by a unique integer which we refer to as the degree $\deg(\Phi)$.

Definition. Let $n \in \mathbb{N}_0$ and let $f\colon S^n \to S^n$ be a continuous map. By the Sphere-H_n-Proposition 41.2 we know that $\widetilde{\mathrm{H}}_n(S^n)$ is isomorphic to $\mathbb{Z}$. Thus we can define the degree of f:

$$\deg(f) \;:=\; \deg\big(f_*\colon \widetilde{\mathrm{H}}_n(S^n) \to \widetilde{\mathrm{H}}_n(S^n)\big).$$

Lemma 43.1. (Sphere–Degree Lemma) Let $f, g\colon S^n \to S^n$ be two continuous maps.
(1) $\deg(\mathrm{id}_{S^n}) = 1$.
(2) If f is not surjective, then $\deg(f) = 0$.
(3) If f is homotopic to g, then $\deg(f) = \deg(g)$.
(4) $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.
(5) If f is a self-homeomorphism of S^n, then $\deg(f^{-1}) = \deg(f)$.

Proof.
(1) Since $(\mathrm{id}_{S^n})_* = \mathrm{id}_{\widetilde{\mathrm{H}}_n(S^n)}$ we know that $\deg(\mathrm{id}_{S^n}) = 1$.
(2) Suppose that the map $f\colon S^n \to S^n$ misses the point $P \in S^n$. We consider the commutative diagram

$$\begin{tikzcd} & S^n\setminus\{P\} \arrow[dr, hook, "i"] & \\ S^n \arrow[ur, "f"] \arrow[rr, "f"'] & & S^n \end{tikzcd}$$

which gives rise to the commutative diagram

$$\begin{tikzcd} & \widetilde{\mathrm{H}}_n(S^n\setminus\{P\}) \arrow[dr, "i_*"] & \\ \widetilde{\mathrm{H}}_n(S^n) \arrow[ur, "f_*"] \arrow[rr, "f_*"'] & & \widetilde{\mathrm{H}}_n(S^n). \end{tikzcd}$$

As we already mentioned on page 363, $S^n \setminus \{P\}$ is homeomorphic to $\mathbb{R}^n$, which implies by the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5 that $\widetilde{\mathrm{H}}_n(S^n \setminus \{P\}) = 0$. But this means that $f_*\colon \widetilde{\mathrm{H}}_n(S^n) \to \widetilde{\mathrm{H}}_n(S^n)$ factors through the zero group, i.e. f_* is the zero map.
(3) If f is homotopic to g, then it follows from the Homotopy-$\widetilde{\mathrm{H}}_*$-Proposition 39.3 that $\deg(f) = \deg(g)$.
(4) We consider the following diagram

$$\begin{tikzcd} \widetilde{\mathrm{H}}_n(S^n) \arrow[r, "f_*", "\cdot\deg(f)"'] \arrow[rr, bend right, "(f\circ g)_*"'] & \widetilde{\mathrm{H}}_n(S^n) \arrow[r, "g_*", "\cdot\deg(g)"'] & \widetilde{\mathrm{H}}_n(S^n). \end{tikzcd}$$

This diagram commutes since $(f \circ g)_* = f_* \circ g_*$. It follows that the curved map is given by multiplication by $\deg(f) \cdot \deg(g)$, i.e. $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.
(5) This statement follows immediately from (1) and (4). ■

Lemma 43.2. (Degree-of-Power Lemma) Given $k \in \mathbb{Z}$ we have

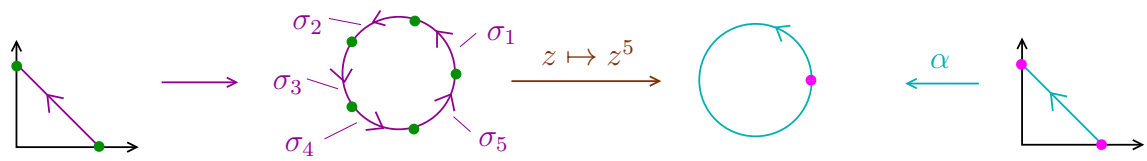
$$\deg\begin{pmatrix} S^1 & \to & S^1 \\ z & \mapsto & z^k \end{pmatrix} \;=\; k.$$

Proof. Let $k \in \mathbb{Z}$. We denote by $f\colon S^1 \to S^1$ the map that is given by $z \mapsto z^k$. We need to show that $\deg(f) = k$. Let $\alpha\colon \Delta^1 \to S^1$ be the singular 1-simplex that is given by $\alpha(1-t,t) = \exp(2\pi \mathrm{i} t)$. By the discussion on page 362 we know that $\alpha \in \mathrm{C}_1(S^1)$ is a cycle and $[\alpha] \in \mathrm{H}_1(S^1)$ is a generator.

First we first consider the case $k \geq 1$. For $j = 0, \ldots, k-1$ we denote by $\sigma_j\colon \Delta^1 \to S^1$ the singular 1-simplex that is given by $(1-t,t) \mapsto \exp(2\pi \mathrm{i}(\frac{t+j}{k}))$. Then

$$f_*([\alpha]) \underset{\substack{\uparrow \\ \text{by the Singular 1-Simplex Lemma 35.12 (3)}}}{=} f_*([\sigma_0 + \cdots + \sigma_{k-1}]) = [\underbrace{f \circ \sigma_0}_{=\alpha} + \cdots + \underbrace{f \circ \sigma_{k-1}}_{=\alpha}] = k \cdot [\alpha].$$

It follows from this calculation that $\deg(f) = k$.



The case $k < 0$ is proved is proved in a similar fashion using the Singular 1-Simplex Lemma 35.12 (2). Finally, if $k = 0$, then $f\colon S^1 \to S^1$ is not surjective, thus it follows from the Sphere–Degree Lemma 43.1 (2) that $\deg(f) = 0$. ■

Lemma 43.3. (Degree-Suspension Lemma) Let $n \in \mathbb{N}_0$.

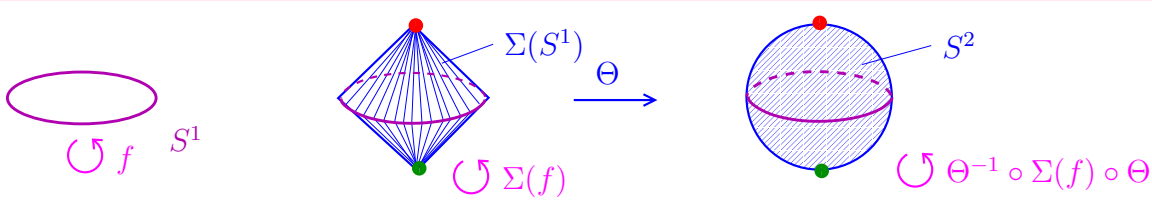
(1) The map

$$\begin{aligned} \Theta\colon \Sigma(S^n) &\to S^{n+1} \subset \mathbb{R}^{n+1} \times \mathbb{R} \\ [(P,t)] &\mapsto \big(\underbrace{P \cdot \cos\left(\tfrac{\pi}{2}t\right)}_{\in \mathbb{R}^{n+1}}, \underbrace{\sin\left(\tfrac{\pi}{2}t\right)}_{\in \mathbb{R}}\big) \end{aligned}$$

is a homeomorphism.

(2) Let $f\colon S^n \to S^n$ be a continuous map. Let $\Sigma(f)\colon \Sigma(S^n) \to \Sigma(S^n)$ be the suspension of f that we defined on page 347. Furthermore let $\Theta\colon \Sigma(S^n) \to S^{n+1}$ be any homeomorphism. Then

$$\deg(\Theta \circ \Sigma(f) \circ \Theta^{-1}\colon S^{n+1} \to S^{n+1}) = \deg(f\colon S^n \to S^n).$$



Proof. Let $n \in \mathbb{N}_0$.

(1) One can easily verify that the given map $\Theta\colon \Sigma(S^n) \to S^{n+1}$ is a homeomorphism.

(2) Let $f\colon S^n \to S^n$ be a continuous map and let $\Theta\colon \Sigma(S^n) \to S^{n+1}$ be a homeomorphism. We consider the following diagram

$$\begin{tikzcd} \widetilde{\mathrm{H}}_n(S^n) \arrow[r, "\Sigma_{S^{n+1},*}", "\cong"'] \arrow[d, "\cdot \deg(f)"', "f_*"] & \widetilde{\mathrm{H}}_n(\Sigma(S^n)) \arrow[r, "\Theta_*", "\cong"'] \arrow[d, "\Sigma(f)_*"] & \widetilde{\mathrm{H}}_{n+1}(S^{n+1}) \arrow[d, "(\Theta \circ \Sigma(f) \circ \Theta^{-1})_*"] \\ \widetilde{\mathrm{H}}_n(S^n) \arrow[r, "\Sigma_{S^{n+1},*}", "\cong"'] & \widetilde{\mathrm{H}}_{n+1}(\Sigma(S^n)) \arrow[r, "\Theta_*", "\cong"'] & \widetilde{\mathrm{H}}_{n+1}(S^{n+1}). \end{tikzcd}$$

First note that it follows from the Suspension-H_*-Proposition 41.1 (2) that the square to the left commutes and that the horizontal maps to the left are isomorphisms. It is clear that the square to the right commutes. Furthermore, since Θ is a homeomorphism we see that the horizontal maps to right are isomorphism. It follows immediately from the above discussion together with the basic observation that $\deg(\Theta \circ \Sigma(f) \circ \Theta^{-1}) = \deg(f)$. ■

Lemma 43.4. (Sphere–Degree–Matrix Lemma) Given any $A \in \mathrm{O}(n+1)$ we have

$$\deg\begin{pmatrix} S^n & \to & S^n \\ x & \mapsto & A \cdot x \end{pmatrix} = \underbrace{\det(A)}_{\in\{\pm 1\}}.$$

Example. If $f\colon S^n \to S^n$ is the reflection in the origin i.e. if $f(x) = -x$ for all $x \in S^n$, then it follows from the Sphere–Degree–Matrix Lemma 43.4 that $\deg(f) = \det(-\operatorname{id}_{n+1}) = (-1)^{n+1}$.

Proof. We start out with the following claim.

Claim. Given $n \in \mathbb{N}_0$ we denote by $\rho_n\colon S^n \to S^n$ the reflection in the $(x_1{=}0)$-hyperplane. For any $n \in \mathbb{N}_0$ we have $\deg(\rho_n) = -1$.

Proof. We start out with the case $n = 0$. We denote by $\sigma_+\colon \Delta^0 \to S^0$ and $\sigma_-\colon \Delta^0 \to S^0$ the singular 0-simplices that are given by $\{1\} \mapsto \{1\}$ and $\{1\} \mapsto \{-1\}$. With this notation let us consider the reduced chain complex of S^0:

$$\ldots \to \mathrm{C}_1(S^1) \xrightarrow[\substack{=0,\text{ see}\\ \text{page 302}}]{\partial_1} \underbrace{\mathrm{C}_0(S^0)}_{=\mathbb{Z}\cdot\sigma_+\oplus\mathbb{Z}\cdot\sigma_-} \xrightarrow[=\epsilon_{S^0}]{\substack{\sigma_+ \mapsto 1\\ \sigma_- \mapsto 1}} \mathbb{Z} \to 0.$$

We see that $\widetilde{\mathrm{H}}_0(S^0) = \ker(\epsilon_{S^0}\colon \mathbb{Z}\cdot\sigma_+ \oplus \mathbb{Z}\cdot\sigma_- \to \mathbb{Z}) = \mathbb{Z}\cdot(\sigma_+ - \sigma_-)$. Thus we see that

$$\rho_{0*}(\sigma_+ - \sigma_-) = \underbrace{\rho_0 \circ \sigma_+}_{=\sigma_-} - \underbrace{\rho_0 \circ \sigma_-}_{=\sigma_+} = \sigma_- - \sigma_+ = (-1)\cdot(\sigma_+ - \sigma_-).$$

It follows that $\deg(\rho_0) = -1$. We have thus proved the desired statement for $n = 0$.

Now we turn to the proof of the induction step. Thus we assume that we know that the statement holds for some $n \in \mathbb{N}_0$. Let

$$\begin{array}{rcl} \Theta\colon \Sigma(S^n) & \xrightarrow{\cong} & S^{n+1} \subset \mathbb{R}^{n+1} \times \mathbb{R} \\ [(P,t)] & \mapsto & (P \cdot \cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t)) \end{array}$$

be the homeomorphism from page 347. It follows immediately from the explicit definitions of the maps that for each $k \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{tikzcd} \Sigma(S^n) \arrow[r, "\Theta"] \arrow[d, "\Sigma(\rho_n)"'] & S^{n+1} \arrow[d, "\rho_{n+1}"] \\ \Sigma(S^n) \arrow[r, "\Theta"] & S^{n+1}. \end{tikzcd}$$

Thus it follows from the Degree-Suspension Lemma 43.3 and the induction hypothesis that $\deg(\rho_n) = \deg(\rho_{n-1}) = -1$. ⊞

Before we get to the denouement of the proof, it is helpful to introduce the following notation: given $B \in \mathrm{O}(n+1)$ we denote by $\mu(B)\colon S^n \to S^n$ the map that is given by $x \mapsto B \cdot x$.

Now let $A \in \mathrm{O}(n+1)$. First we consider the case $\det(A) = 1$, i.e. we assume that $A \in \mathrm{SO}(n+1)$. By the Matrix Group-π_0-Proposition 2.23 we know that $\mathrm{SO}(n+1)$ is path-connected, so there exists a continuous map $h\colon [0,1] \to \mathrm{SO}(n+1)$ with $h(0) = \mathrm{id}$ and $h(1) = A$. But then $F(x,t) = h(t) \cdot x$ defines a homotopy from $\mathrm{id}_{S^n} = \mu(\mathrm{id})$ to $\mu(A)$. It follows from the Sphere–Degree Lemma 43.1 (1) and (3) that $\deg(\mu(A)) = \deg(\mathrm{id}_{S^n}) = 1$.

Finally we consider the case that $\det(A) \neq 1$. Let $R \in \mathrm{O}(n+1)$ be the matrix such that $\mu(R) = \rho_n$. Now we see that

$$\deg(\mu(A)) \;=\; \deg(\mu(A \cdot R \cdot R^{-1})) \underset{\substack{\uparrow \\ \text{by the Sphere–Degree} \\ \text{Lemma 43.1 (4)}}}{=} \deg(\mu(A \cdot R)) \cdot \deg(\mu(R)) \underset{\substack{\uparrow \\ \text{since } \det(A\cdot R)=1 \text{ we have } \deg(\mu(A\cdot R))=1 \\ \text{and by the claim we have } \deg(\mu(R))=-1}}{=} 1 \cdot (-1) \;=\; -1.$$

■

43.2. Application: The Hairy Ball Theorem.

Theorem 43.5. (Sphere-Degree Fixed Point Theorem) Let $f\colon S^n \to S^n$ be a continuous map. If $\deg(f) \neq (-1)^{n+1}$, then f has a fixed point.

Proof. Let $f\colon S^n \to S^n$ be a continuous map that does not admit a fixed point. This means that for all $x \in S^n$ we have $f(x) \neq x$. We need to show that $\deg(f) = (-1)^{n+1}$. We will do this by showing that f is homotopic to $-\mathrm{id}$.

Our hypothesis implies that $(1-t)\cdot f(x) \neq t \cdot x$ for all $t \in [0,1]$ and all $x \in S^n$. Therefore we can consider the map

$$\begin{aligned} F\colon S^n \times [0,1] &\to S^n \\ (x,t) &\mapsto \frac{(1-t)\cdot f(x) - t \cdot x}{\|(1-t)\cdot f(x) - t \cdot x\|}. \end{aligned}$$

This map is a homotopy between the maps $x \mapsto f(x)$ and $x \mapsto -x$. Thus it follows that

$$\deg(f) \underset{\substack{\uparrow \\ \text{Sphere–Degree Lemma 43.1 (3)}}}{=} \deg(-\mathrm{id}) \underset{\substack{\uparrow \\ \text{Sphere–Degree–Matrix Lemma 43.4}}}{=} (-1)^{n+1}.$$

■

To formulate our next theorem we need to introduce or recall the following definition.

Definition. Let $M \subset \mathbb{R}^n$ be a submanifold. Given $P \in M \setminus \partial M$ we denote by $\mathrm{T}_P M \subset \mathbb{R}^n$ the tangent vector space. A vector field on M is a continuous map $v\colon M \to \mathbb{R}^n$ such that for any $P \in M$ we have $v(P) \in \mathrm{T}_P M$.

Examples.

(1) Let P be a point on S^n. As is explained in [Fri23], the tangent space is given by

$$\mathrm{T}_P S^n \;=\; \{v \in \mathbb{R}^{n+1} \mid \langle P, v\rangle = 0\}.$$

(2) For an odd-dimensional sphere S^{2k-1} a vector field is given by

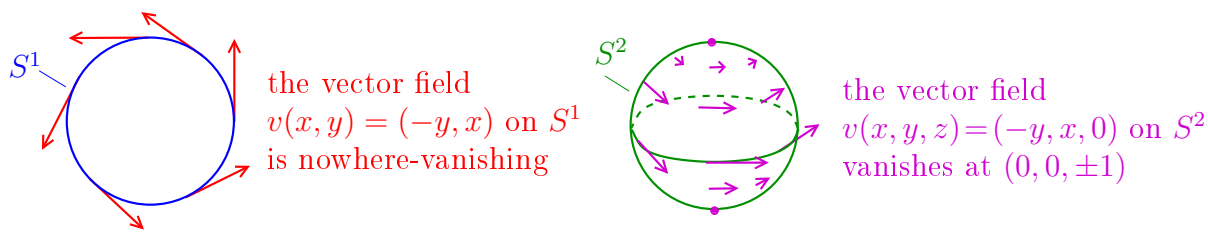
$$\begin{aligned} v\colon S^{2k-1} &\to \mathbb{R}^{2k} \\ (x_1, x_2, \ldots, x_{2k-1}, x_{2k}) &\mapsto (-x_2, x_1, \ldots, -x_{2k}, x_{2k-1}). \end{aligned}$$

This vector field has the property that it is nowhere-vanishing.

(3) For an even-dimensional sphere S^{2k} a vector field is similarly given by

$$\begin{array}{rcl} v\colon S^{2k} & \to & \mathbb{R}^{2k+1} \\ (x_1, x_2, \ldots, x_{2k-1}, x_{2k}, x_{2k+1}) & \mapsto & (-x_2, x_1, \ldots, -x_{2k}, 0). \end{array}$$

This vector field vanishes at the "North Pole" $(0, \ldots, 0, 1)$ and it also vanishes at the "South Pole" $(0, \ldots, 0, -1)$.



The above examples raise the question whether it is possible to find a nowhere vanishing vector field on even-dimensional spheres. The following theorem shows that this is *not* possible.

Theorem 43.6. (Hairy Ball Theorem) Every vector field on an even-dimensional sphere S^{2k} vanishes on at least one point.

Proof. Let v be a vector field on the sphere S^n which is nowhere-vanishing. We have to show that n is odd. Since v is nowhere vanishing we can consider the vector field w on S^n that is given by rescaling v to length one, i.e. that is given by

$$\begin{array}{rcl} S^n & \to & \mathbb{R}^{n+1} \\ x & \mapsto & w(x) := \frac{1}{\|v(x)\|} \cdot v(x). \end{array}$$

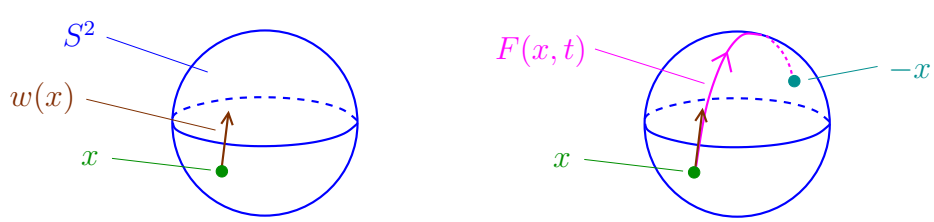
Furthermore we consider the map

$$\begin{array}{rcl} F\colon S^n \times [0,1] & \to & S^n \\ (x,t) & \mapsto & \cos(\pi t) \cdot x + \sin(\pi t) \cdot w(x). \end{array}$$

The map $F_0 = F(-, 0)$ is the identity and the map $F_1 = F(-, 1)$ equals F is $-\mathrm{id}$. Put differently, the map F is a homotopy between the two maps id and $-\mathrm{id}$. It follows that

$$\begin{array}{ccccccccc} 1 & \underset{\uparrow}{=} & \deg(\mathrm{id}) & \underset{\uparrow}{=} & \deg(-\mathrm{id}) & \underset{\uparrow}{=} & \det(-\mathrm{id}_{n+1}) & = & (-1)^{n+1}. \\ & \text{by the Sphere–Degree} & & \text{Lemma 43.1 (3)} & & \text{by the Sphere–Degree–Matrix} & & & \\ & \text{Lemma 43.1 (1)} & & & & \text{Lemma 43.4} & & & \end{array}$$

But this means that n is odd. ■



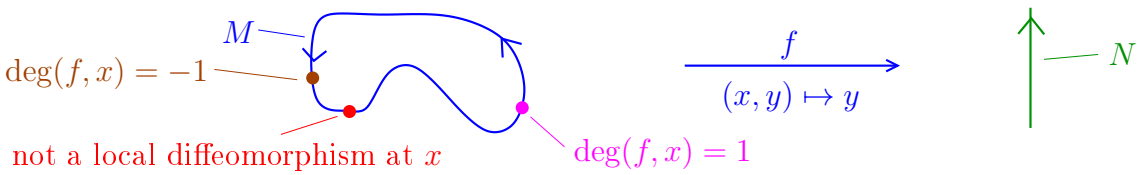
44. Local degrees of self-maps of spheres

44.1. Degrees and local orientations.

We introduce the following harmless definition.

Definition. Let $f\colon M \to N$ be a continuous map between two oriented n-dimensional smooth manifolds and let $x \in M \setminus \partial M$. If f is a local diffeomorphism at x, then we define the local degree of f at the point x as

$$\deg(f,x) \;:=\; \begin{cases} +1, & \text{if } f \text{ is orientation-preserving at } x \\ -1, & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$



Proposition 44.7. (Degree-via-Local Orientations Proposition) Let $n \in \mathbb{N}$. Let $f\colon S^n \to S^n$ be a continuous map and let $y \in S^n$ such that the following conditions are satisfied:

(1) $f^{-1}(y)$ consists of finitely many points $x_1, \dots, x_m$.
(2) The map f is a local diffeomorphism at each x_i.

Then

$$\deg(f) \;=\; \sum_{i=1}^{m} \deg(f,x_i) \;=\; \sum_{i=1}^{m} \begin{cases} 1, & \text{if } f \text{ is orientation-preserving at } x_i, \\ -1, & \text{if } f \text{ is orientation-reversing at } x_i. \end{cases}$$

Examples.

(1) Let $m \in \mathbb{Z}$. We consider the map

$$\begin{aligned} f\colon S^1 &\to S^1 \\ z &\mapsto z^m \end{aligned}$$

In the Degree-of-Power Lemma 43.2 we already proved that $\deg(f) = m$. Now we will show, just for fun, that the degree of f can also easily be determined using the Degree-via-Local Orientations Proposition 44.7. First let $m \neq 0$. Then we choose $y = 1$. The preimages of y are given by $x_j = \exp(2\pi \mathrm{i} j/m)$ where $j = 1, \dots, m$. It follows easily from the definitions that for each $j \in \{1, \dots, m\}$ we have

$$\deg(f,x_j) \;=\; \begin{cases} 1, & \text{if } m > 0, \\ -1, & \text{if } m < 0. \end{cases}$$

It follows from this calculation and the Degree-via-Local Orientations Proposition 44.7 that $\deg(f) = |m| \cdot \operatorname{sign}(m) = m$. If $m = 0$, then we take $y = -1$. Note that for $m = 0$ we have $f^{-1}(y) = \varnothing$, thus we see that $\deg(f) = 0$.

(2) If $f\colon S^n \to S^n$ is a diffeomorphism, then we can apply the Degree-via-Local Orientations Proposition 44.7 with any $y \in S^n$ and we obtain that

$$\deg(f) \;=\; \begin{cases} 1, & \text{if } f \text{ is orientation-preserving,} \\ -1, & \text{if } f \text{ is orientation-reversing.} \end{cases}$$

44.2. The local degree.

Definition. We say a map $f\colon X \to Y$ between two topological spaces is nice at a point $x \in X$ if there exists a neighborhood U of x such that for all $x' \neq x$ in U we have $f(x') \neq f(x)$. We refer to such a neighborhood as a nice neighborhood of x.

Examples.

(1) Let $f\colon X \to Y$ be a continuous map between two topological spaces and let $x \in X$. If U is a neighborhood such that $f|_U$ is injective, then U is evidently a nice neighborhood of x.

(2) We consider the maps

$$\begin{array}{rcl} f\colon \mathbb{R} & \to & \mathbb{R} \\ t & \mapsto & t^2 \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon \mathbb{R} & \to & \mathbb{R} \\ t & \mapsto & \begin{cases} 0, & \text{if } t = 0, \\ t \cdot \sin(\frac{1}{t}), & \text{if } t \neq 0. \end{cases} \end{array}$$

For the map f every point on $\mathbb{R}$ is nice, in fact for $x < 0$ we can take $U = (-\infty, 0)$, for $x > 0$ we can take $U = (0, \infty)$ and for $x = 0$ we can take $U = \mathbb{R}$. On the other hand it is straightforward to see that for the map g the point $x = 0$ is not nice.

Definition. Let $n \geq 1$, let $f\colon S^n \to S^n$ be a continuous map and let $x \in S^n$ be a nice point. Let U be a nice neighborhood of x. The local degree of f at the point x is defined as the degree of the homomorphism

$$\begin{tikzcd} & & \text{isomorphism by the Excision Theorem 40.2} \arrow[d, "\cong"] & & \\ \mathrm{H}_n(S^n) \arrow[r, "\text{я}", "\cong"'] & \mathrm{H}_n(S^n, S^n \setminus \{x\}) & \mathrm{H}_n(U, U \setminus \{x\}) \arrow[l, "\cong"'] \arrow[r, "f_*"] & \mathrm{H}_n(S^n, S^n \setminus \{f(x)\}) & \mathrm{H}_n(S^n). \arrow[l, "\text{я}", "\cong"'] \\ & & \text{the long exact sequence of the pair } (S^n, S^n \setminus \{y\}) \text{ shows that these maps are isomorphisms} & & \end{tikzcd}$$

We denote this local degree by $\deg(f, x)$.

Lemma 44.1. (Local Degree–Well-Defined Lemma) The local degree $\deg(f, x)$ of a map $f\colon S^n \to S^n$ at a nice point $x \in S^n$ is well-defined.

Proof. Let $n \in \mathbb{N}$, let $f\colon S^n \to S^n$ be a continuous map and let $x \in S^n$ be a nice point. Let U, U' be nice neighborhoods of x. After replacing U' by $U \cap U'$ we can without loss of generality assume that $U' \subset U$. We consider the following diagram

$$\begin{tikzcd} & & \mathrm{H}_n(U', U' \setminus \{x\}) \arrow[r, "f_*"] \arrow[dl, "\cong"'] \arrow[dd, "\cong"] & \mathrm{H}_n(S^n, S^n \setminus \{f(x)\}) \arrow[dd, "\mathrm{id}"] & \\ \mathrm{H}_n(S^n) \arrow[r] & \mathrm{H}_n(S^n, S^n \setminus \{x\}) & & & \mathrm{H}_n(S^n). \arrow[ull, "\cong"'] \arrow[dll, "\cong"] \\ & & \mathrm{H}_n(U, U \setminus \{x\}) \arrow[r, "f_*"] \arrow[ul, "\cong"] & \mathrm{H}_n(S^n, S^n \setminus \{f(x)\}). & \end{tikzcd}$$

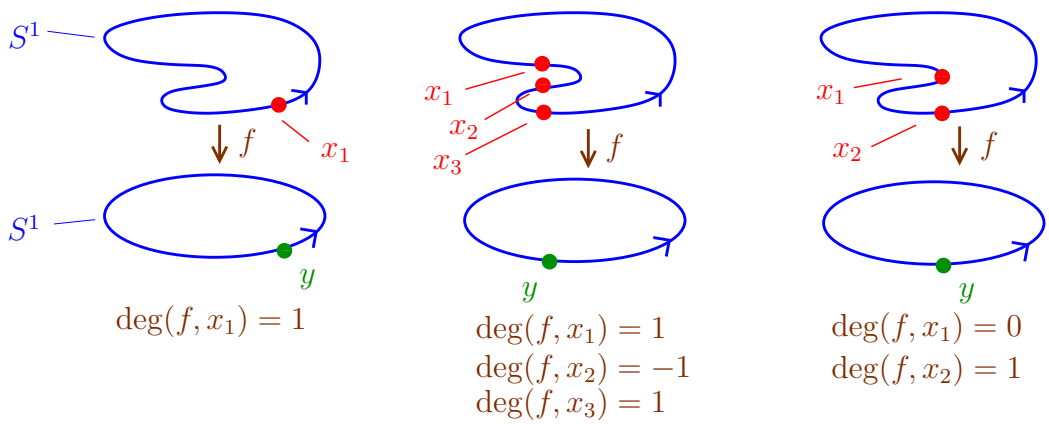
All the non-decorated maps are induced by inclusions. It follows from the functoriality of relative homology that the diagram commutes. Also note that the left vertical map is an isomorphism by the Excision Theorem 40.2. It follows immediately from these observations that the local degree defined using U' agrees with the local degree defined using U. ■

Proposition 44.2. (Local-Global Degree Proposition) Let $n \in \mathbb{N}$, let $f\colon S^n \to S^n$ be a continuous map and let $y \in S^n$ such that $f^{-1}(y)$ consists of finitely many points

$x_1, \ldots, x_m$. Then all the points $x_1, \ldots, x_m$ are nice and we obtain the equality

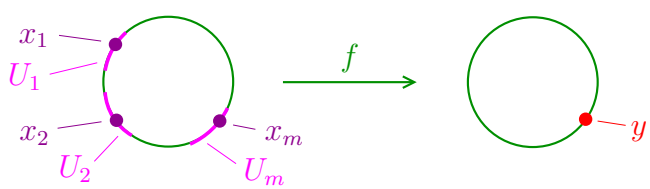
$$\deg(f) \;=\; \sum_{i=1}^{m} \deg(f, x_i).$$

Example. In the figure below we sketch a map $f\colon S^1 \to S^1$ and we choose several points



$y \in S^1$. We see that the number of preimages and the local degrees at the points in the preimages vary, but in all three cases the sum of the local degrees at the points in the preimage is the same.

Proof. Let $n \in \mathbb{N}$, let $f\colon S^n \to S^n$ be a continuous map and let $y \in S^n$ such that $f^{-1}(y)$ consists of finitely many points $x_1, \ldots, x_m$. It follows easily from the fact that S^n is Hausdorff that we can choose pairwise disjoint open neighborhoods $U_1, \ldots, U_m$ for $x_1, \ldots, x_m$. Note that this observation implies already that all the x_i are nice.



In the following we adopt the following notation:

(1) Given homomorphisms $\varphi_i\colon A_i \to B$ we write $\bigoplus_{i=1}^{m} \varphi_i\colon \bigoplus_{i=1}^{m} A_i \to B$.

(2) Given homomorphisms $\varphi_i\colon A \to B_i$ we write $(\varphi_1, \ldots, \varphi_m)\colon A \to \bigoplus_{i=1}^{m} B_i$.

(3) Given homomorphisms $\varphi_i\colon A_i \to B_i$ we write $\operatorname{diag}(\varphi_1, \ldots, \varphi_m)\colon \bigoplus_{i=1}^{m} A_i \to \bigoplus_{i=1}^{m} B_i$.

We consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}_n(S^n) \arrow[rr, "f_*"] \arrow[d, "(\text{я}_1,\ldots,\text{я}_m)"'] \arrow[dr, "\text{я}"] & & \mathrm{H}_n(S^n) \arrow[dd, "\text{я}"', "\cong"] \\
\bigoplus_{i=1}^m \mathrm{H}_n(S^n, S^n\setminus\{x_i\}) & \mathrm{H}_n(S^n, S^n\setminus f^{-1}(y)) \arrow[l, "(p_1,\ldots,p_m)"', "\cong"] \arrow[dr, "f_*"] & \\
\mathrm{H}_n\Big(\bigcup_{i=1}^m U_i, \bigcup_{i=1}^m U_i\setminus\{x_i\}\Big) \arrow[u, "\iota_*"', "\cong"] & & \\
\bigoplus_{i=1}^m \mathrm{H}_n(U_i, U_i\setminus\{x_i\}) \arrow[uu, "\mathrm{diag}(r_{1*},\ldots,r_{m*})"', "\cong"] \arrow[ur, "\oplus s_i"', "\cong"] \arrow[rr, "\oplus f_{i*}"] & & \mathrm{H}_n(S^n, S^n\setminus\{y\}).
\end{tikzcd}$$

We make the following clarifications and observations:

(1) The maps r_i, s_i, p_i and ι are the maps induced by the obvious inclusions of pairs of topological spaces. Furthermore the maps $f_i\colon (U_i, U_i\setminus\{x_i\}) \to (S^n, S^n\setminus\{y\})$ are given by the restriction of f to U_i.

(2) It follows pretty easily from the functoriality of relative homology groups and elementary considerations that the diagram commutes.

(3) It follows easily from the fact that the U_i are disjoint open subsets of S^n and the Path-Component–H_*–Lemma 35.8 that the diagonal map $\bigoplus_{i=1}^m s_i$ to the bottom left is an isomorphism.

(4) It follows from the Excision Theorem 40.2 that the vertical map ι_* and the maps r_{i*} are isomorphisms.

(5) It follows from the fact that the diagram commutes together with (3) and (4) that the horizontal map $(p_1,\ldots,p_m)$ is also an isomorphism.

(6) As we pointed out on page 363, it follows from the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5 that the reduced homology groups of $S^n\setminus\{y\}$ vanish. Since $n\geq 1$ it follows from the LES-of-Pair Proposition 38.3 that $\text{я}\colon \mathrm{H}_n(S^n)\to \mathrm{H}_n(S^n, S^n\setminus\{y\})$ is an isomorphism.

By (6) it suffices to prove the following claim.

Claim. We have
$$\text{я}(f_*([S^n])) \;=\; \sum_{i=1}^m \deg(f,x_i)\cdot \text{я}([S^n]) \;\in\; \mathrm{H}_n(S^n, S^n\setminus\{y\}).$$

Proof. We verify the claim by performing the following calculation:

$$\begin{aligned}
\text{я}(f_*([S^n])) &\overset{\substack{\text{since the diagram commutes and since the}\\ \text{map } \mathrm{diag}(r_{1*},\ldots,r_{m*}) \text{ is an isomorphism}}}{=} \Big(\bigoplus_{i=1}^m f_{i*}\Big)\circ \mathrm{diag}(r_{1*},\ldots,r_{m*})^{-1}\circ(\text{я}_1,\ldots,\text{я}_m)([S^n]) \overset{\substack{\text{since each } r_{i*} \text{ is an isomorphism}}}{=} \\
&= \Big(\bigoplus_{i=1}^m f_{i*}\Big)\circ(r_{1*}^{-1}\circ\text{я}_1,\ldots,r_{m*}^{-1}\circ\text{я}_m)([S^n]) \\
&\underset{\substack{\text{since we always have the equality}\\ \left(\left(\bigoplus_{i=1}^m \varphi_i\right)\circ(\psi_1,\ldots,\psi_m)\right)(x) = \sum_{i=1}^m (\varphi_i\circ\psi_i)(x)}}{=} \sum_{i=1}^m (f_i\circ r_i^{-1}\circ\text{я}_i)([S^n]) \underset{\substack{\text{by the definition of the local degree, see page 370}}}{=} \sum_{i=1}^m \deg(f,x_i)\cdot\text{я}([S^n]).
\end{aligned}$$

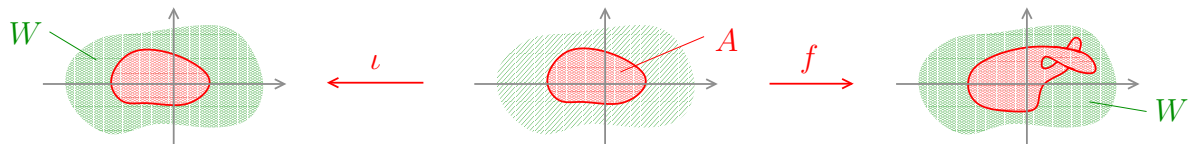
■

44.3. The local degree and local diffeomorphisms.

Proposition 44.3. (Local Diffeomorphism-Degree Proposition) Let W be a subset of $\mathbb{R}^n$ and let A be a subset of W such the origin 0 is contained in the interior of W. We denote by $\iota\colon A \to W$ the inclusion. Let $f\colon A \to W$ be an injective continuous map with $f(0) = 0$. We suppose that f is a local diffeomorphism at 0. Then the degree of the homomorphism

$$\mathrm{H}_n(W, W \setminus \{0\}) \xleftarrow[\cong]{\iota_*} \mathrm{H}_n(A, A \setminus \{0\}) \xrightarrow{f_*} \mathrm{H}_n(W, W \setminus \{0\})$$

equals $+1$ if f is orientation-preserving at 0 and -1 if f is orientation-reversing at 0.



Lemma 44.4. (Linearization–Homotopy Lemma) Let $f\colon U \to V$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$. Let $P \in U$. We denote by

$$\begin{array}{rcl} \lambda\colon \mathbb{R}^n & \to & \mathbb{R}^n \\ x & \mapsto & \mathrm{D}f_P \cdot x + f(P) \end{array}$$

the linearization of f at P. There exists an open neighborhood U' of P such that the restrictions of f and λ to maps $(U', U' \setminus \{P\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{f(P)\})$ are homotopic.

Proof. Let $f\colon U \to V$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$. To simplify the notation we assume that $P = 0$ and that $f(P) = 0$. Furthermore, by restriction to an open ball we might as well assume that U is convex. We write $D := \mathrm{D}f_0$. Thus the linearization of f at $P = 0$ is given by the map $\lambda(x) = D \cdot x$. Since f is in particular a local diffeomorphism a $P = 0$ we know by the Chain Rule that D is an invertible matrix.

Since $U \subset \mathbb{R}^n$ is convex we obtain from the Map-as-Matrix Multiplication Proposition [Fri23] that there exists a smooth map $A\colon U \to \mathrm{M}(n \times n, \mathbb{R})$ such that $A(0) = \mathrm{D}f_0$ and such that $f(x) = A(x) \cdot x$ for all $x \in U$. Since $\det(A(0)) \neq 0$ we see that there exists an $r > 0$ with $B^n_r(0) \subset U$ such that $\det(A(x)) \neq 0$ for all $x \in B^n_r(0)$. This implies that the map

$$\begin{array}{rcl} B^n_r(0) \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & A(x \cdot (1-t)) \cdot x \end{array}$$

is a homotopy of the maps f and λ as maps $(B^n_r(0), B^n_r(0) \setminus \{0\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ of pairs of topological spaces. We now see that $U' = B^n_r(0)$ has the desired properties. ■

Notation.
(1) Given a matrix $P \in \mathrm{GL}(n, \mathbb{R})$ we denote by $\rho(P)\colon \mathbb{R}^n \to \mathbb{R}^n$ the map given by $x \mapsto P \cdot x$.
(2) Given a non-zero real number r we denote its sign by $\mathrm{sign}(r) \in \{-1, +1\}$.

Lemma 44.5. (Matrix Multiplication–Degree Lemma) Let $n \in \mathbb{N}$. Given any invertible matrix $A \in \mathrm{GL}(n, \mathbb{R})$ we have

$$\deg\left(\rho(A)_*\colon \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \to \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})\right) = \mathrm{sign}(\det(A)).$$

Proof of the Matrix Multiplication–Degree Lemma 44.5. Let $A \in \mathrm{GL}(n,\mathbb{R})$. We start out by picking an orthogonal matrix $B \in \mathrm{O}(n)$ such that $\mathrm{sign}(\det(B)) = \mathrm{sign}(\det(A))$. Next we consider the following diagram

$$\begin{tikzcd}
\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \arrow[r, "\mathrm{id}"] \arrow[d, "\rho(A)_*"] & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \arrow[r, "\partial_n", "\cong"'] \arrow[d, "\rho(B)_*"] & \widetilde{\mathrm{H}}_{n-1}(\mathbb{R}^n \setminus \{0\}) \arrow[d, "\rho(B)_*"] & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}) \arrow[l, "\cong"] \arrow[d, "\rho(B)_*"] \\
\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \arrow[r, "\mathrm{id}"] & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \arrow[r, "\partial_n", "\cong"'] & \widetilde{\mathrm{H}}_{n-1}(\mathbb{R}^n \setminus \{0\}) & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}). \arrow[l, "\cong"]
\end{tikzcd}$$

We make the following observations:

(1) Since $\mathrm{sign}(\det(B)) = \mathrm{sign}(\det(A))$ we obtain from the Matrix Group-π_0-Proposition 2.23 (1) that there exists a path $[0,1] \to \mathrm{GL}(n,\mathbb{R})$ from A to B. Similar to the discussion on page 103 this path gives rise to a homotopy between $\rho(A)$ and $\rho(B)$ as maps of pairs of topological spaces $(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. We obtain from the Homotopy-H_*-Proposition 38.6 that $\rho(A)_* = \rho(B)_*$, in other words, the square to the left commutes.

(2) The square in the middle commutes by the naturality of the connecting homomorphism, see the LES-of-Pair Proposition 38.3.

(3) The horizontal maps to the right are induced by the inclusion $S^{n-1} \to \mathbb{R}^n \setminus \{0\}$. Since S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{0\}$ we obtain from the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 39.4 that these horizontal maps are isomorphisms.

(4) Since $B \in \mathrm{O}(n)$ we see that $\rho(B)\colon \mathbb{R}^n \to \mathbb{R}^n$ restricts to a self-map of S^{n-1}. It follows from the functoriality of reduced homology groups that the square to the right commutes.

Now we see that

$$\begin{aligned}
\deg\left(\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \xrightarrow{\rho(A)_*} \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})\right) &\overset{\text{by the above commutative diagram}}{=} \deg\left(S^{n-1} \xrightarrow{\rho(B)} S^{n-1}\right) \overset{\text{by the Sphere–Degree–Matrix Lemma 43.4}}{=} \det(B) \\
&\underset{\substack{\text{since } B \in \mathrm{O}(n) \text{ we} \\ \text{have } \det(B) \in \{\pm 1\}}}{=} \mathrm{sign}(\det(B)) \underset{\text{choice of } B}{=} \mathrm{sign}(\det(A)).
\end{aligned}$$

■

Proof of the Local Diffeomorphism-Degree Proposition 44.3. Let W be a subset of $\mathbb{R}^n$ and let A be a subset of W such that the origin 0 is contained in the interior. Let $f\colon A \to W$ be an injective continuous map with $f(0) = 0$. We suppose that f is a local diffeomorphism at 0. Recall that this means that there exists an open neighborhood U of 0 that is contained in A and an open neighborhood V of 0 such that the map $f\colon U \to V$ is a diffeomorphism.

We start out with the following preparations.

(1) We set
$$\epsilon := \begin{cases} +1, & \text{if } f \text{ is orientation-preserving at } 0, \\ -1, & \text{if } f \text{ is orientation-reversing at } 0. \end{cases}$$
Note that $\epsilon = \mathrm{sign}(\det(\mathrm{D}f_0))$.

(2) Note that by the Linearization–Homotopy Lemma 44.4 there exists an open neighborhood U' of 0 that is contained in U such that the restrictions of f and $\rho(\mathrm{D}f_0)$ to maps $(U', U' \setminus \{0\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ are homotopic.

Next we consider the following diagram

$$\begin{tikzcd}
\mathrm{H}_n(W, W\setminus\{0\}) \arrow[dd, "\cong"] & \mathrm{H}_n(A, A\setminus\{0\}) \arrow[l, "\cong"'] \arrow[rr, "f_*"] & & \mathrm{H}_n(W, W\setminus\{0\}) \arrow[dl, "\cong"] \arrow[dd, "\cong"] \\
& \mathrm{H}_n(U', U'\setminus\{0\}) \arrow[u, "\cong"] \arrow[r, "f_*"] \arrow[d, "\mathrm{id}"'] & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \arrow[d, "\mathrm{id}"] & \\
& \mathrm{H}_n(U', U'\setminus\{0\}) \arrow[r, "\rho(\mathrm{D}f_0)_*"] \arrow[dl, "\cong"'] & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \arrow[dr, "\cong"] & \\
\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \arrow[rrr, "\rho(\mathrm{D}f_0)_*"] & & & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}).
\end{tikzcd}$$

We continue with the following clarifications and observations:

(3) The undecorated maps are induced by inclusions. All of them are isomorphisms by the Excision Theorem 40.2.

(4) It follows from (2) and the Homotopy-H_*-Proposition 38.6 that the central square commutes. All other regions of the diagram commute by the functoriality of relative homology groups.

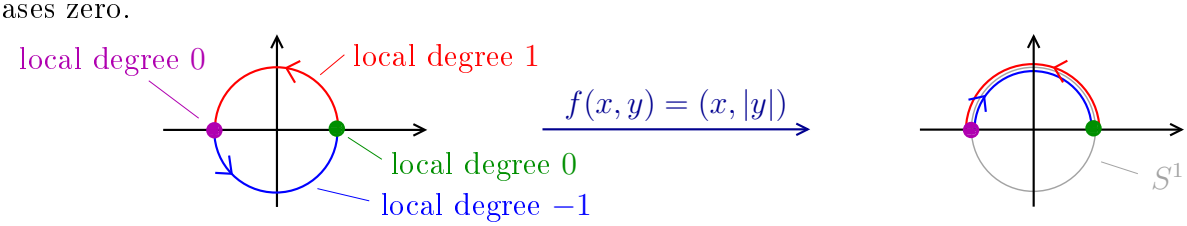
(5) By the Matrix Multiplication–Degree Lemma 44.5 the bottom horizontal map equals multiplication by $\mathrm{sign}(\det(\mathrm{D}f_0))$.

(6) It follows from all of the above that the top map is given by multiplication by ϵ. But that is precisely what we wanted to show. ■

Proposition 44.6. (Local Degree-Calculation Proposition) Let $f\colon S^n \to S^n$ be a continuous map with $n \geq 1$ and let $x \in S^n$. If f is a local diffeomorphism at x, then x is nice and

$$\deg(f, x) \;=\; \begin{cases} 1, & \text{if } f \text{ is orientation-preserving at } x, \\ -1, & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$

Example. We consider the map $f\colon S^1 \to S^1$ that is given by $(x, y) \mapsto (x, |y|)$. It follows from the Local Degree-Calculation Proposition 44.6 that for every point on the open upper semicircle the local degree is $+1$ and that for every point on the open lower semicircle the local degree is -1. One can easily show that the local degree at the points ± 1 is in both cases zero.



Proof. Let $f\colon S^n \to S^n$ be a continuous map with $n \geq 1$ and let $x \in S^n$. We assume that f is a local diffeomorphism at x. By elementary linear algebra we can pick an $A \in \mathrm{SO}(n+1)$ with $A \cdot f(x) = x$. Note that the self-map of S^n given by $z \mapsto A \cdot z$ is orientation-preserving. Furthermore by the Sphere–Degree–Matrix Lemma 43.4 we know that $\deg(z \mapsto A \cdot z) = 1$. It follows from these observations, together with the Sphere–Degree Lemma 43.1 (4), that after replacing f by the map $z \mapsto A \cdot f(z)$, we might as well assume that $f(x) = x$.

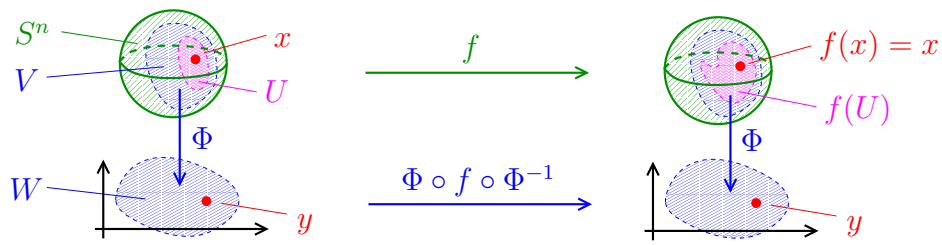
We make a few further preparations.

(1) We pick a chart $\Phi\colon V \to W$, from the smooth atlas, around $x \in S^n$.

(2) Since f is a local diffeomorphism at x there exists an open neighborhood U of x such that $f\colon U \to f(U)$ is a diffeomorphism. Note that after possibly replacing U by the subset $U \cap V \cap f^{-1}(V)$ we can assume that $U \subset V$ and $f(U) \subset V$.

(3) We set $y := \Phi(x)$.

(4) We set
$$\epsilon := \begin{cases} 1, & \text{if } f \text{ is orientation-preserving at } x, \\ -1, & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$



Next we consider the following diagram

$$\begin{tikzcd}
\mathrm{H}_n(S^n) \arrow[rr, "\cdot \deg(f,x)"] \arrow[d, "\text{я}", "\cong"'] & & \mathrm{H}_n(S^n) \arrow[d, "\cong"', "\text{я}"] \\
\mathrm{H}_n(S^n, S^n \setminus \{x\}) & \mathrm{H}_n(U, U \setminus \{x\}) \arrow[l, "\cong"'] \arrow[rr, "f_*"] \arrow[dr, "f_*"] \arrow[dd, "\Phi_*"] & & \mathrm{H}_n(S^n, S^n \setminus \{x\}) \\
& & \mathrm{H}_n(f(U), f(U) \setminus \{x\}) \arrow[dr, "\Phi_*"] & \\
\mathrm{H}_n(W, W \setminus \{y\}) \arrow[uu, "\Phi_*^{-1}", "\cong"'] \arrow[rrr, bend right, "\cdot\epsilon"'] & \mathrm{H}_n(\Phi(U), \Phi(U) \setminus \{y\}) \arrow[l, "\cong"'] \arrow[rr, "(\Phi \circ f \circ \Phi^{-1})_*"] & & \mathrm{H}_n(W, W \setminus \{y\}). \arrow[uu, "\Phi_*^{-1}", "\cong"']
\end{tikzcd}$$

We make the following observations and clarifications.

(1) As usual the undecorated maps are induced by inclusions.

(2) The top rectangle commutes by the definition of $\deg(f, x)$. The middle regions of the diagram commute by the functoriality of relative homology. Finally, note that $\Phi \circ f \circ \Phi^{-1}\colon \Phi^{-1}(U) \to W$ is orientation-preserving at y if and only if f is orientation-preserving at x. Thus we see that the bottom part of the diagram commutes by the Local Diffeomorphism-Degree Proposition 44.3.

(3) By the Excision Theorem 40.2 the map $\Phi_*\colon \mathrm{H}_n(W, W \setminus \{y\}) \to \mathrm{H}_n(S^n, S^n \setminus \{x\})$ is an isomorphism.

(4) It follows from the above that $\deg(f, x) = \epsilon$. ■

Proposition 44.7. (Degree-via-Local Orientations Proposition) Let $n \in \mathbb{N}_0$. Let $f\colon S^n \to S^n$ be a continuous map and let $y \in S^n$ such that the following conditions are satisfied:

(1) $f^{-1}(y)$ consists of finitely many points $x_1, \ldots, x_m$.

(2) The map f is a local diffeomorphism at each x_i.

Then
$$\deg(f) \;=\; \sum_{i=1}^{m} \deg(f, x_i) \;=\; \sum_{i=1}^{m} \begin{cases} 1, & \text{if } f \text{ is orientation-preserving at } x_i, \\ -1, & \text{if } f \text{ is orientation-reversing at } x_i. \end{cases}$$

Proof. The proposition is an immediate consequence of the Local-Global Degree Proposition 44.2 together with the Local Degree-Calculation Proposition 44.6. ■

45. The Mayer–Vietoris sequence and its applications

45.1. The Mayer–Vietoris sequence.

Notation. Given a topological space D and a subspace $C \subset D$ we denote the corresponding inclusion map $C \to D$ by i_C^D. To avoid pandemonium with subscripts we often drop the "$*$" when we deal with inclusion induced maps on homology, e.g. we will write $i_C^D \colon \mathrm{H}_n(C) \to \mathrm{H}_n(D)$ instead of the more accurate $(i_C^D)_* \colon \mathrm{H}_n(C) \to \mathrm{H}_n(D)$. We use similar notation for inclusions of pairs of topological spaces.

Theorem 45.1. (Mayer–Vietoris Theorem) Let X be a topological space and let $A, B \subset X$ be subspaces such that such that one of the following three conditions holds:

(1) X is the union of the interiors of A and B, i.e. $X = \mathring{A} \cup \mathring{B}$.
(2) X is a CW-complex and A and B are subcomplexes with $X = A \cup B$.
(3) X is a topological manifold and $A, B \subset X$ are two codimension-zero submanifolds such that the following conditions are satisfied:
   - $X = A \cup B$.
   - $A \cap B$ is a union of components of ∂A and it is a union of components of ∂B.
   - A and B are closed subsets of X.

By (1) the Excision Theorem 40.1, (2) the CW-Excision Theorem 40.4 or (3) the Manifold–Excision Theorem 40.3 each inclusion induced map $i_{(A,A\cap B)}^{(X,B)} \colon \mathrm{H}_n(A, A\cap B) \to \mathrm{H}_n(X, B)$ is an isomorphism. We refer to the homomorphism

$$\mathrm{H}_n(X) \xrightarrow{\text{я}} \mathrm{H}_n(X,B) \xrightarrow{\left(i_{(A,A\cap B)}^{(X,B)}\right)^{-1}} \mathrm{H}_n(A, A\cap B) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A\cap B)$$

connecting homomorphism of the long exact sequence corresponding to the pair $(A, A\cap B)$, provided by the LES-of-Pair Proposition 38.3

as the connecting homomorphism $\partial_n \colon \mathrm{H}_n(X) \to \mathrm{H}_{n-1}(A\cap B)$ corresponding to (X, A, B). With this notation the following sequence is exact:

$$\ldots \to \mathrm{H}_n(A\cap B) \xrightarrow{(i_{A\cap B}^A, -i_{A\cap B}^B)} \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) \xrightarrow{i_A^X \oplus i_B^X} \mathrm{H}_n(X) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A\cap B) \to \ldots$$

Totally analogous statements also hold if we replace the homology groups of $A\cap B, A, B$ and X by the corresponding reduced homology groups.

Proof. There are two approaches to proving the Mayer–Vietoris Theorem 45.1:

(1) One can deduce the Mayer–Vietoris Theorem 45.1 directly from the Excision Theorem 40.1 using an algebraic manipulation. We will carry out this approach in Section 45.2.
(2) One can prove the Mayer–Vietoris Theorem 45.1 using the Comfortable Covering Proposition 40.6. We will carry out this approach in Section 45.3.

The first proof is slicker and it has the advantage that it applies to other functors, besides singular homology, that also satisfy a suitable version of the Excision Theorem 40.1. On the other hand the second proof makes it arguably clearer how the long exact sequence arises. ■

45.2. First proof of the Mayer–Vietoris Theorem 45.1.

Lemma 45.2. (Barratt-Whitehead Lemma) We consider the following commutative diagram of homomorphisms between abelian groups:

$$\begin{tikzcd}
\dots \arrow[r, "\partial_{n+1}"] & C'_n \arrow[r, "i_n"] \arrow[d, "f'_n"] & C_n \arrow[r, "p_n"] \arrow[d, "f_n"] & C''_n \arrow[r, "\partial_n"] \arrow[d, "f''_n"] & C'_{n-1} \arrow[r] \arrow[d, "f'_{n-1}"] & \dots \\
\dots \arrow[r, "\partial_{n+1}"] & D'_n \arrow[r, "j_n"] & D_n \arrow[r, "q_n"] & D''_n \arrow[r, "\partial_n"] & D'_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

We suppose that the horizontal sequences are exact and we suppose that each $f''_n \colon C''_n \to D''_n$ is an isomorphism. For each $n \in \mathbb{N}_0$ we set $\Delta_n := \partial_n \circ (f''_n)^{-1} \circ q_n \colon D_n \to C'_{n-1}$. The following sequence is exact:

$$\dots \longrightarrow C'_n \xrightarrow{(i_n, -f'_n)} C_n \oplus D'_n \xrightarrow{f_n \oplus j_n} D_n \xrightarrow{\Delta_n} C'_{n-1} \longrightarrow \dots$$

Proof of the Barratt-Whitehead Lemma 45.2. The lemma is proved via an elementary diagram chase. The proof of this lemma is the content of Exercise **??**. ■

Proof of the Mayer–Vietoris Theorem 45.1. Let X be a topological space. Furthermore let $A, B \subset X$ be two subspaces. We assume that one of the following three conditions holds:

(1) $X = \mathring{A} \cup \mathring{B}$.
(2) X is a CW-complex and A and B are subcomplexes with $X = A \cup B$.
(3) X is a topological manifold and $A, B \subset X$ are two codimension-zero submanifolds such that the following conditions are satisfied:
   - $X = A \cup B$.
   - $A \cap B$ is a union of components of ∂A and it is a union of components of ∂B.
   - A and B are closed subsets of X.

We consider the following commutative diagram of continuous maps between pairs of topological spaces:

$$\begin{tikzcd}
(A \cap B, \varnothing) \arrow[r, "i^A_{A\cap B}"] \arrow[d, "i^B_{A\cap B}"] & (A, \varnothing) \arrow[r] \arrow[d, "i^X_A"] & (A, A \cap B) \arrow[d, "i^{(X,B)}_{(A,A\cap B)}"] \\
(B, \varnothing) \arrow[r, "i^X_B"] & (X, \varnothing) \arrow[r] & (X, B).
\end{tikzcd}$$

From the LES-of-Pair Proposition 38.3 we get two long exact sequences which, together with the vertical maps, form the following commutative diagram:

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}_n(A \cap B) \arrow[r, "i^A_{A\cap B}"] \arrow[d, "i^B_{A\cap B}"] & \mathrm{H}_n(A) \arrow[r, "я"] \arrow[d, "i^X_A"] & \mathrm{H}_n(A, A \cap B) \arrow[r, "\partial_n"] \arrow[d, "i^{(X,B)}_{(A,A\cap B)}"] & \mathrm{H}_{n-1}(A \cap B) \arrow[r] \arrow[d, "i^B_{A\cap B}"] & \dots \\
\dots \arrow[r] & \mathrm{H}_n(B) \arrow[r, "i^X_B"] & \mathrm{H}_n(X) \arrow[r, "я"] & \mathrm{H}_n(X, B) \arrow[r, "\partial_n"] & \mathrm{H}_{n-1}(B) \arrow[r] & \dots .
\end{tikzcd}$$

Depending on the setting we are in we obtain from

(1) the Excision Theorem 40.1 (applied with $Z = X \setminus A$), or
(2) the Manifold–Excision Theorem 40.3, or
(3) the CW-Excision Theorem 40.4

that the inclusion induced map $i^{(X,B)}_{(A,A\cap B)}\colon \mathrm{H}_n(A, A\cap B) \to \mathrm{H}_n(X,B)$ is an isomorphism. We apply the Barratt-Whitehead Lemma 45.2 to the above setup and we obtain that the following sequence is exact:

$$\ldots \to \mathrm{H}_n(A\cap B) \xrightarrow{(i^A_{A\cap B}, -i^B_{A\cap B})} \mathrm{H}_n(A)\oplus \mathrm{H}_n(B) \xrightarrow{i^X_A \oplus i^X_B} \mathrm{H}_n(X) \xrightarrow{\partial_n \circ (i^{(X,B)}_{(A,A\cap B)})^{-1} \circ \text{я}} \mathrm{H}_{n-1}(A\cap B) \to \ldots$$

But that was exactly our goal. Finally note that by replacing the LES-of-Pair Proposition 38.3 by the Reduced LES-of-Pair Proposition 39.6 we get the analogous result for reduced homology groups. ■

45.3. **Second proof of the Mayer–Vietoris Theorem 45.1.** In this section we give our second proof of the Mayer–Vietoris Theorem 45.1. Here it is:

Proof of the Mayer–Vietoris Theorem 45.1. Let X be a topological space.

(1) First we suppose we are given two subsets $A, B \subset X$ such that $X = \mathring{A} \cup \mathring{B}$. We recall the following notation from page 337:

$$\mathrm{C}_n^{\{A,B\}}(X) \;:=\; \{a + b \,|\, a \in \mathrm{C}_n(A) \text{ and } b \in \mathrm{C}_n(B)\}.$$

The inclusion maps induce the following sequence of chain maps:

$$0 \to \mathrm{C}_*(A\cap B) \xrightarrow{(i^A_{A\cap B}, -i^B_{A\cap B})} \mathrm{C}_*(A) \oplus \mathrm{C}_*(B) \xrightarrow{i^X_A \oplus i^X_B} \mathrm{C}_*^{\{A,B\}}(X) \to 0.$$

Claim. The above sequence of chain maps is exact.

Proof. We prove the claim in the following four easy steps:

(a) It is clear that each of the maps $\mathrm{C}_*(A\cap B) \to \mathrm{C}_*(A)$ and $\mathrm{C}_*(A\cap B) \to \mathrm{C}_*(B)$ is already injective, in particular the map $(i^A_{A\cap B}, -i^B_{A\cap B})$ is injective.

(b) It is an immediate consequence of the definition of $\mathrm{C}_*^{\{A,B\}}(X)$ that the map $i^X_A \oplus i^X_B$ is surjective.

(c) It follows immediately from the definitions that the composition of both maps is the zero map, i.e. $\mathrm{im}((i^A_{A\cap B}, -i^B_{A\cap B})) \subset \ker(i^X_A \oplus i^X_B)$.

(d) It remains to show that $\ker(i^X_A \oplus i^X_B) \subset \mathrm{im}((i^A_{A\cap B}, -i^B_{A\cap B}))$. Thus let $x \in \mathrm{C}_n(A)$ and $y \in \mathrm{C}_n(B)$ such that $(i^X_A \oplus i^X_B)((x,y)) = 0$. By definition this means that x and y, viewed as elements in $\mathrm{C}_n(X)$, satisfy $x = -y$. This means that x lies in $\mathrm{C}_n(A)$ and also in $\mathrm{C}_n(B)$. Put differently, $x \in \mathrm{C}_n(A) \cap \mathrm{C}_n(B) = \mathrm{C}_n(A\cap B)$. But then we have $((i^A_{A\cap B}, -i^B_{A\cap B}))(x) = x + y$. ⊞

Next we consider the following maps:

$$\begin{tikzcd}
\ldots \arrow[r] & \mathrm{H}_n(A\cap B) \arrow[r, "(i^A_{A\cap B}, -i^B_{A\cap B})"] & \mathrm{H}_n(A)\oplus \mathrm{H}_n(B) \arrow[r, "i^X_A \oplus i^X_B"] \arrow[dr, "i^X_A \oplus i^X_B"'] & \mathrm{H}_n^{\{A,B\}}(X) \arrow[r, "p_n"] \arrow[d, "\Phi_n"', "\cong"] & \mathrm{H}_{n-1}(A\cap B) \arrow[r] & \ldots \\
 & & & \mathrm{H}_n(X) \arrow[ur, ":=\partial_n"'] & &
\end{tikzcd}$$

We make the following clarifications and observations:

(a) Given $n \in \mathbb{N}_0$ we denote by $\mathrm{H}_n^{\{A,B\}}(X)$ the n-th homology group of the chain complex $\mathrm{C}_*^{\{A,B\}}(X)$ and we denote by $p_n\colon \mathrm{H}_n^{\{A,B\}}(X) \to \mathrm{H}_{n-1}(A\cap B)$ the connecting homomorphism of the above short exact sequence of chain complexes, as defined on page 319.

(b) By the LES-Proposition 37.4 the top horizontal sequence is exact.

(c) It follows from the Comfortable Covering Proposition 40.6, together with our hypothesis that $X = \mathring{A} \cup \mathring{B}$, that for every $n \in \mathbb{N}_0$ the inclusion map

$$\begin{array}{rcl} \mathrm{C}_*^{\{A,B\}}(X) & \to & \mathrm{C}_*(X) \end{array}$$

induces an isomorphism

$$\Phi_n \colon \mathrm{H}_n^{\{A,B\}}(X) \xrightarrow{\cong} \mathrm{H}_n(X).$$

(d) For each $n \in \mathbb{N}_0$ we write $\partial_n := p_n \circ \Phi_n^{-1} \colon \mathrm{H}_n(X) \to \mathrm{H}_{n-1}(A \cap B)$. Using the explicit description of the connecting homomorphisms on page 319 and in the LES-of-Pair Proposition 38.3 one can show easily that $\partial_n := p_n \circ \Phi_n^{-1} \colon \mathrm{H}_n(X) \to \mathrm{H}_{n-1}(A \cap B)$ equals the connecting homomorphism given in the theorem.

(2) Now we suppose that we are dealing with a CW-complex X and two subcomplexes A and B with $X = A \cup B$. One can use the Subcomplex–Neighborhood Proposition 31.8 together with the Homotopy Equivalence-H_*-Corollary 36.7 to reduce the case of CW-complexes and subcomplexes to the case of an open cover. We will not carry out the technical details for CW-complexes, since we already provided a complete proof of the Mayer–Vietoris Theorem 45.1 (2) in Section 45.2.

(3) Similarly to (2), as in the proof of the Manifold-Seifert–van Kampen-Theorem 23.1, we can use the Topological Collar Neighborhood Theorem [Fri23] to reduce the manifold case to the case of an open cover. As in (2) we leave the details to the reader.

Finally note that the statements for reduced homology can be proved as above, we just need to replace $\mathrm{C}_*^{\{A,B\}}(X)$ by its reduced analogue. Alternatively one can deduce the reduced long exact sequence from the above unreduced long exact sequence and the Reduced Homology Lemma 39.2. Again there is no point in providing the details. ■

45.4. The homology groups of a mapping torus.

Definition. Let X be a topological space and let $f \colon X \to X$ be a continuous map. We refer to

$$\mathrm{Tor}(X, f) \ := \ (X \times [0,1])/(x,0) \sim (f(x),1)$$

as the mapping torus of (X, f).

Proposition 45.3. (Mapping Torus-H_*-Proposition) Let X be a topological space and let $f \colon X \to X$ be a continuous map. We denote by $i \colon X \to \mathrm{Tor}(X, f)$ the natural map that is given by $x \mapsto [(x, 0)]$. For every $n \in \mathbb{N}_0$ there exists a natural homomorphism

$$\partial_n \colon \mathrm{H}_n(\mathrm{Tor}(X, f)) \ \to \ \mathrm{H}_{n-1}(X)$$

(which is described in greater detail in the proof of the proposition) such that the following sequence is exact

$$\cdots \to \ \mathrm{H}_n(X) \xrightarrow{f_* - \mathrm{id}} \mathrm{H}_n(X) \xrightarrow{i_*} \mathrm{H}_n(\mathrm{Tor}(X, f)) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(X) \xrightarrow{f_* - \mathrm{id}} \mathrm{H}_{n-1}(X) \ \to \ldots$$

Proof. Let X be a topological space and let $f \colon X \to X$ be a continuous map. For notational convenience we work in this proof with the following topological space which is clearly naturally homeomorphic to the mapping torus:

$$M \ := \ (X \times [0,10])/(x,0) \sim (f(x),10).$$

We introduce the following notation:

(1) We denote by $p \colon X \times [0,10] \to M = (X \times [0,1])/\sim$ the natural projection.

(2) We consider the subsets $A := p(X \times ([0,4) \cup (6,10])) \subset M$ and $B := p(X \times (2,8)) \subset M$.

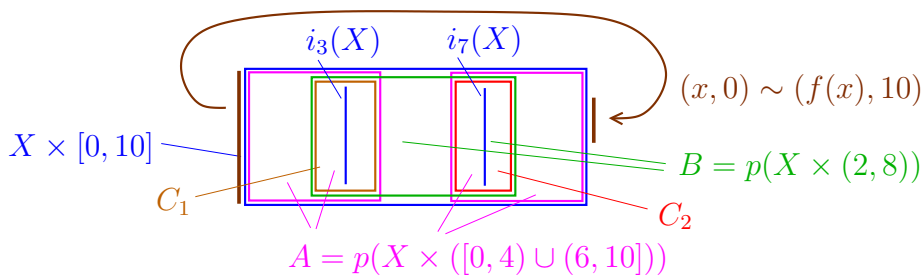
(3) We consider the two subsets $C_1 = p(X \times (2,4))$ and $C_2 = p(X \times (6,8))$. Note that $A \cap B = C_1 \cup C_2$.

(4) Given $t \in [0,10]$ we consider the map

$$\begin{array}{rcl} i_t \colon X & \to & p(X \times \{t\}) \\ x & \mapsto & p(x,t). \end{array}$$

(5) We write $i_3^A \colon X \to A$ for the composition of $i_3 \colon A \to p(X \times \{3\})$ with the inclusion map $p(X \times \{3\}) \to A$. We denote analogous compositions by the obvious variations in the notation.

(6) To simplify the notation we drop the "$*$" when we write down the corresponding induced maps on homology.



We can now consider the following diagram:

$$\begin{tikzcd}
\to & \begin{matrix} \mathrm{H}_n(C_1) \\ \mathrm{H}_n(C_2) \end{matrix} \arrow[r, "\begin{pmatrix} i^A_{C_1} & i^A_{C_2} \\ -i^B_{C_1} & -i^B_{C_2} \end{pmatrix}"] & \begin{matrix} \mathrm{H}_n(A) \\ \mathrm{H}_n(B) \end{matrix} \arrow[r, "(i^M_A \;\; i^M_B)"] & \mathrm{H}_n(M) \arrow[r, "\partial_n"] & \begin{matrix} \mathrm{H}_{n-1}(C_1) \\ \mathrm{H}_{n-1}(C_2) \end{matrix} \arrow[r, "\begin{pmatrix} i^A_{C_1} & i^A_{C_2} \\ -i^B_{C_1} & -i^B_{C_2} \end{pmatrix}"] & \begin{matrix} \mathrm{H}_{n-1}(A) \\ \mathrm{H}_{n-1}(B) \end{matrix} \arrow[r] & {} \\
& \begin{matrix} \mathrm{H}_n(X) \\ \mathrm{H}_n(X) \end{matrix} \arrow[u, "\begin{pmatrix} i_3 & 0 \\ 0 & i_7 \end{pmatrix}", "\cong"'] \arrow[r, "\begin{pmatrix} f_* & \mathrm{id} \\ -\mathrm{id} & -\mathrm{id} \end{pmatrix}"] & \begin{matrix} \mathrm{H}_n(X) \\ \mathrm{H}_n(X) \end{matrix} \arrow[u, "\cong", "\begin{pmatrix} i_7 & 0 \\ 0 & i_7 \end{pmatrix}"'] \arrow[r, "(i_7 \;\; i_7)"] & \mathrm{H}_n(M) \arrow[u, "\mathrm{id}"] \arrow[r, "=:\partial_n"] & \begin{matrix} \mathrm{H}_{n-1}(X) \\ \mathrm{H}_{n-1}(X) \end{matrix} \arrow[u, "\begin{pmatrix} i_3 & 0 \\ 0 & i_7 \end{pmatrix}", "\cong"'] \arrow[r, "\begin{pmatrix} f_* & \mathrm{id} \\ -\mathrm{id} & -\mathrm{id} \end{pmatrix}"] & \begin{matrix} \mathrm{H}_{n-1}(X) \\ \mathrm{H}_{n-1}(X) \end{matrix} \arrow[u, "\cong", "\begin{pmatrix} i_7 & 0 \\ 0 & i_7 \end{pmatrix}"'] \arrow[r] & {} \\
\to & \mathrm{H}_n(X) \arrow[u, "\begin{pmatrix} -\mathrm{id} \\ \mathrm{id} \end{pmatrix}"] \arrow[r, "\mathrm{id} - f_*"] & \mathrm{H}_n(X) \arrow[u, "\begin{pmatrix} \mathrm{id} \\ 0 \end{pmatrix}"] \arrow[r, "i_7"] & \mathrm{H}_n(M) \arrow[u, "\mathrm{id}"] \arrow[r, "=:\partial_n"] & \mathrm{H}_{n-1}(X) \arrow[u, "\begin{pmatrix} -\mathrm{id} \\ \mathrm{id} \end{pmatrix}"] \arrow[r, "\mathrm{id} - f_*"] & \mathrm{H}_{n-1}(X) \arrow[u, "\begin{pmatrix} \mathrm{id} \\ 0 \end{pmatrix}"] \arrow[r] & {}
\end{tikzcd}$$

We make the following clarifications and observations:

(1) Note that $A = p(X \times ([0,4) \cup (6,10])) \subset M$ and $B = p(X \times (2,8)) \subset M$ are open subsets of M that cover all of M. Furthermore note that we have $A \cap B = C_1 \sqcup C_2$. On the top of the diagram we wrote down the corresponding exact Mayer–Vietoris sequence from the Mayer–Vietoris Theorem 45.1 (1).

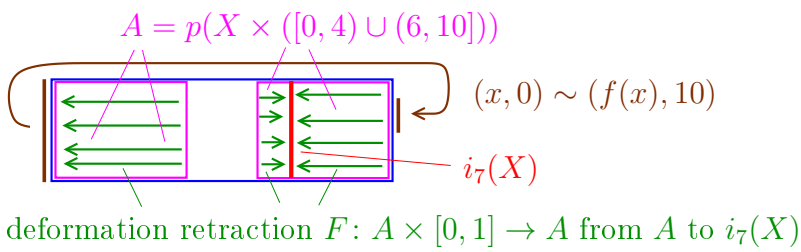
(2) We want to show that the vertical maps from the second to the first row are isomorphisms. In fact this statement follows immediately from the following claim together with the Homotopy Equivalence-H_*-Corollary 36.7 (1a).

Claim. The maps $i_3 \colon X \to C_1$, $i_7 \colon X \to C_2$, $i_7 \colon X \to B$ and $i_7 \colon X \to A$ are homotopy equivalences.

Proof. One can easily verify that the maps $i_t\colon X \to p(X \times \{t\})$, $p\colon X \times (2,4) \to C_1$, $p\colon X \times (6,8) \to C_2$ and $p\colon X \times (2,8) \to B$ are homeomorphisms. Using this observation one can easily verify that the first three maps are homotopy equivalences.

If f is not a homeomorphism, then A is not necessarily homeomorphic to a product. Thus we need to work somewhat harder. Let us consider the following map:

$$\begin{aligned} F\colon A \times [0,1] &\to A \\ ([(x,t)],s) &\mapsto \begin{cases} [(x,t)], & \text{if } s \in [0,\frac{1}{2}] \text{ and } t \in (6,10], \\ [(x, t\cdot(1-2s))], & \text{if } s \in [0,\frac{1}{2}] \text{ and } t \in [0,4), \\ [(x, t\cdot(2-2s)+7\cdot(2s-1))], & \text{if } s \in [\frac{1}{2},1] \text{ and } t \in (6,10], \\ [(f(x), 10\cdot(2-2s)+7\cdot(2s-1))], & \text{if } s \in [\frac{1}{2},1] \text{ and } t \in [0,4). \end{cases} \end{aligned}$$



First note that it follows from $(x,0) \sim (f(x),10)$ that this map F is well-defined. One can now easily verify, say using the Pasting Proposition 2.4, that the map F is continuous. It is now clear that the above map F is a deformation retraction from A to $i_7(X)$. This implies, by the Deformation Retract–Homotopy Equivalence Lemma 9.3, that $i_7\colon X \to A$ is a homotopy equivalence. ⊞

(3) Next we want to show that the top part of the diagram commutes.

(a) We start with the square to the left. We need to show that the following equality holds:

$$\underbrace{\begin{pmatrix} i^A_{C_1} & i^A_{C_2} \\ -i^B_{C_1} & -i^B_{C_2} \end{pmatrix} \begin{pmatrix} i_3^{C_1} & 0 \\ 0 & i_7^{C_2} \end{pmatrix}}_{=\begin{pmatrix} i^A_3 & i^A_7 \\ -i^B_3 & -i^B_7 \end{pmatrix}} = \underbrace{\begin{pmatrix} i^A_7 & 0 \\ 0 & i^B_7 \end{pmatrix} \begin{pmatrix} f_* & \mathrm{id} \\ -\mathrm{id} & -\mathrm{id} \end{pmatrix}}_{=\begin{pmatrix} i^A_7 \circ f_* & i^A_7 \circ \mathrm{id} \\ i^B_7 \circ (-\mathrm{id}) & i^B_7 \circ (-\mathrm{id}) \end{pmatrix}}.$$

If we compare the two matrices we see that it remains to prove the following claim.

Claim. For each $n \in \mathbb{N}_0$ we have $i^A_3 = i^A_7 \circ f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(A)$ and we also have $i^B_3 = i^B_7\colon \mathrm{H}_n(X) \to \mathrm{H}_n(B)$.

Proof. We consider the maps

$$\begin{aligned} X \times [0,1] &\to A \\ (x,t) &\mapsto \begin{cases} p(x, 3-6t), & \text{if } t \in [0,\frac{1}{2}], \\ p(f(x), 13-6t) & \text{if } t \in [\frac{1}{2},1] \end{cases} \end{aligned} \quad \& \quad \begin{aligned} X \times [0,1] &\to B \\ (x,t) &\mapsto p(x, 3\cdot(1-t)+7\cdot t). \end{aligned}$$

These two maps are homotopies between the maps $i^A_3, i^A_7 \circ f\colon X \to A$ and the maps $i^B_3, i^B_7\colon X \to B$. The claim now follows from the Homotopy-H_*-Proposition 36.5. ⊞

(b) The second square commutes by the functoriality of homology groups.

(c) Recall that in (1) we just showed that the vertical maps from the second to the first row are isomorphisms. Therefore we can define the connecting homomorphism $\partial_n \colon \mathrm{H}_n(M) \to \begin{smallmatrix}\mathrm{H}_{n-1}(X)\\ \mathrm{H}_{n-1}(X)\end{smallmatrix}$ in the second row in such a way that the third square commutes.

(4) It is straightforward to see that the bottom squares in the above diagram commute. We know that the middle sequence is exact. The following claim implies that the bottom sequence is also exact:

Claim. Suppose we are given the following commutative diagram of homomorphisms of abelian groups:

$$\begin{tikzcd}
 & \begin{smallmatrix}A_n\\ A_n\end{smallmatrix} \arrow[r, "\begin{pmatrix}\varphi_n & \mathrm{id}\\ -\mathrm{id} & -\mathrm{id}\end{pmatrix}"] & \begin{smallmatrix}A_n\\ A_n\end{smallmatrix} \arrow[r, "(\alpha_n\ \alpha_n)"] & B_n \arrow[r, "\delta_n"] & \begin{smallmatrix}A_{n-1}\\ A_{n-1}\end{smallmatrix} \arrow[r, "\begin{pmatrix}\varphi_{n-1} & \mathrm{id}\\ -\mathrm{id} & -\mathrm{id}\end{pmatrix}"] & \begin{smallmatrix}A_{n-1}\\ A_{n-1}\end{smallmatrix} \arrow[r] & {} \\
{} \arrow[r] & A_n \arrow[u, "\begin{pmatrix}-\mathrm{id}\\ \mathrm{id}\end{pmatrix}"] \arrow[r, "\mathrm{id}-\varphi_n"] & A_n \arrow[u, "\begin{pmatrix}\mathrm{id}\\ 0\end{pmatrix}"] \arrow[r, "\alpha_n"] & B_n \arrow[u, "\mathrm{id}"] \arrow[r, "\delta_n"] & A_{n-1} \arrow[u, "\begin{pmatrix}-\mathrm{id}\\ \mathrm{id}\end{pmatrix}"] \arrow[r, "\mathrm{id}-\varphi_{n-1}"] & A_{n-1} \arrow[u, "\begin{pmatrix}\mathrm{id}\\ 0\end{pmatrix}"] \arrow[r] & {}
\end{tikzcd}$$

If the top sequence is exact, then the bottom sequence is also exact.

Proof. We consider the following vertically expanded diagram:

$$\begin{tikzcd}
 & A_n \arrow[r, "-\mathrm{id}"] & A_n \arrow[r] & 0 \arrow[r] & A_{n-1} \arrow[r, "-\mathrm{id}"] & A_{n-1} \arrow[r] & {} \\
 & \begin{smallmatrix}A_n\\ A_n\end{smallmatrix} \arrow[u, "(\mathrm{id}\ \ \mathrm{id})"] \arrow[r, "\begin{pmatrix}\varphi_n & \mathrm{id}\\ -\mathrm{id} & -\mathrm{id}\end{pmatrix}"] & \begin{smallmatrix}A_n\\ A_n\end{smallmatrix} \arrow[u, "(0\ \ \mathrm{id})"'] \arrow[r, "(\alpha_n\ \alpha_n)"] & B_n \arrow[u] \arrow[r, "\delta_n"] & \begin{smallmatrix}A_{n-1}\\ A_{n-1}\end{smallmatrix} \arrow[u, "(\mathrm{id}\ \ \mathrm{id})"] \arrow[r, "\begin{pmatrix}\varphi_{n-1} & \mathrm{id}\\ -\mathrm{id} & -\mathrm{id}\end{pmatrix}"] & \begin{smallmatrix}A_{n-1}\\ A_{n-1}\end{smallmatrix} \arrow[u, "(0\ \ \mathrm{id})"'] \arrow[r] & {} \\
{} \arrow[r] & A_n \arrow[u, "\begin{pmatrix}-\mathrm{id}\\ \mathrm{id}\end{pmatrix}"] \arrow[r, "\mathrm{id}-\varphi_n"] & A_n \arrow[u, "\begin{pmatrix}\mathrm{id}\\ 0\end{pmatrix}"'] \arrow[r, "\alpha_n"] & B_n \arrow[u, "\mathrm{id}"] \arrow[r, "\delta_n"] & A_{n-1} \arrow[u, "\begin{pmatrix}-\mathrm{id}\\ \mathrm{id}\end{pmatrix}"] \arrow[r, "\mathrm{id}-\varphi_{n-1}"] & A_{n-1} \arrow[u, "\begin{pmatrix}\mathrm{id}\\ 0\end{pmatrix}"'] \arrow[r] & {}
\end{tikzcd}$$

By hypothesis the bottom part of the diagram commutes. It follows easily from the definitions that the top part also commutes. Furthermore note that the vertical sequences, with 0 added on top and bottom, are exact.

Next note that evidently the top sequence is exact. Furthermore by hypothesis the middle sequence is exact. Since the bottom vertical maps are monomorphisms we see that the bottom sequence is a chain complex.

It follows from the above discussion that all these maps form a short exact sequence of chain complexes. Clearly the homology groups of the top chain complex are zero. By hypothesis the homology groups of the middle chain complex are also zero. It follows from the LES-Proposition 37.4 that the homology groups of the bottom chain complex are also zero. But this means that the bottom sequence is exact.

We have thus completed the proof that the bottom horizontal sequence of the diagram on page 382 is exact. If one goes carefully through the definitions one sees that the connecting homomorphism $\partial_n \colon \mathrm{H}_n(\mathrm{Tor}(X, f)) \to \mathrm{H}_{n-1}(X)$ is given by the composition of the

homomorphisms in the following unreadable sequence of homomorphisms:

$$\begin{aligned}\mathrm{H}_n(\mathrm{Tor}(X,f)) &\xrightarrow{\text{я}} \mathrm{H}_n(\mathrm{Tor}(X,f),A) \xleftarrow{\cong} \mathrm{H}_n(B, C_1\cup C_2)\\ &\xrightarrow{\partial_n} \mathrm{H}_{n-1}(C_1\cup C_2) \quad \xleftarrow{\cong} \mathrm{H}_{n-1}(C_1)\oplus \mathrm{H}_{n-1}(C_2) \to \mathrm{H}_{n-1}(C_2) \underset{i_7^{C_2}}{\xleftarrow{\cong}} \mathrm{H}_{n-1}(X).\end{aligned}$$

■

Proposition 45.4. (Product-with-S^1-H_*-Proposition) Let X be a topological space. We denote by $p\colon X\times S^1\to X$ the natural projection. For every $n\in\mathbb{N}_0$ there exists a natural homomorphism $\psi_X\colon \mathrm{H}_n(X\times S^1)\to \mathrm{H}_{n-1}(X)$ (which will be described in greater detail in the proof) such that the map

$$\Xi_X := p_*\oplus\psi_X\colon \mathrm{H}_n(X\times S^1) \xrightarrow{\cong} \mathrm{H}_n(X)\oplus \mathrm{H}_{n-1}(X)$$

is an isomorphism.

Proof. Let $n\in\mathbb{N}_0$. It is pretty elementary to show that the map

$$\begin{aligned}\Theta\colon\ \mathrm{Tor}(X,\mathrm{id}) &\to X\times S^1\\ [(x,t)] &\mapsto (x,\exp(2\pi\,\mathrm{i}t))\end{aligned}$$

is a natural homeomorphism. We denote by $i\colon X\to \mathrm{Tor}(X,\mathrm{id})$ the natural map that is given by $x\mapsto[(x,0)]$ and we denote by $j\colon X\to X\times S^1$ the natural embedding that is given by $x\mapsto(x,1)$. Next we consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}_n(X) \arrow[r, "\mathrm{id}_*-\mathrm{id}_*"] & \mathrm{H}_n(X) \arrow[r, "i_*"] \arrow[d, "\mathrm{id}"] & \mathrm{H}_n(\mathrm{Tor}(X,\mathrm{id})) \arrow[r, "\partial_n"] \arrow[d, "\Theta_*", "\cong"'] & \mathrm{H}_{n-1}(X) \arrow[r, "\mathrm{id}_*-\mathrm{id}_*"] \arrow[d, "\mathrm{id}"] & \mathrm{H}_{n-1}(X) \\
0 \arrow[r] & \mathrm{H}_n(X) \arrow[r, bend left, "j_*"] & \mathrm{H}_n(X\times S^1) \arrow[r, "\partial_n\circ\Theta_*^{-1}"] \arrow[l, bend left, "p_*"] & \mathrm{H}_{n-1}(X) \arrow[r] & 0.
\end{tikzcd}$$

We make the following clarifications and observations:

(1) The top sequence is the short exact sequence that we obtain from the Mapping Torus-H_*-Proposition 45.3.

(2) Since $\mathrm{id}_*-\mathrm{id}_*=0$ we see that the homomorphisms to the top left and top right are the zero map.

(3) Since Θ is a homeomorphism and since $\Theta\circ i=j$ we obtain from (1) and (2) that the bottom sequence is also exact.

(4) The map $p\colon X\times S^1\to X$ is the natural map that is given by projection onto the first factor. Note that $p_*\circ j_*=(p\circ j)_*=\mathrm{id}$.

(5) We define
$$\begin{aligned}\psi_X\colon \mathrm{H}_n(X\times S^1) &\to \mathrm{H}_{n-1}(X)\\ \sigma &\mapsto (\partial_n\circ\Theta_*^{-1})(\sigma-(j_*\circ p_*)(\sigma)).\end{aligned}$$

It is straightforward to verify that this homomorphism is natural.

Finally note that as in the proof of the Splitting Lemma 37.2 (3)⇒(2) one can show that the map

$$\begin{aligned}\Xi_X\colon \mathrm{H}_n(X\times S^1) &\to \mathrm{H}_n(X)\oplus \mathrm{H}_{n-1}(X)\\ \sigma &\mapsto p_*(\sigma)\oplus \underbrace{(\partial_n\circ\Theta_*^{-1})(\sigma-(j_*\circ p_*)(\sigma))}_{=\psi_X(\sigma)}\end{aligned}$$

is an isomorphism. ■

Proposition 45.5. (Torus-H_*-Proposition) Let $n \in \mathbb{N}_0$.

(1) For $k = 0, \dots, n$ we have

$$\mathrm{H}_k(n\text{-dimensional torus } (S^1)^n) \;=\; \text{free abelian group of rank } \binom{n}{k}$$

and all other homology groups of $(S^1)^n$ vanish.

(2) For $i = 1, \dots, n$ we consider the singular 1-cycles

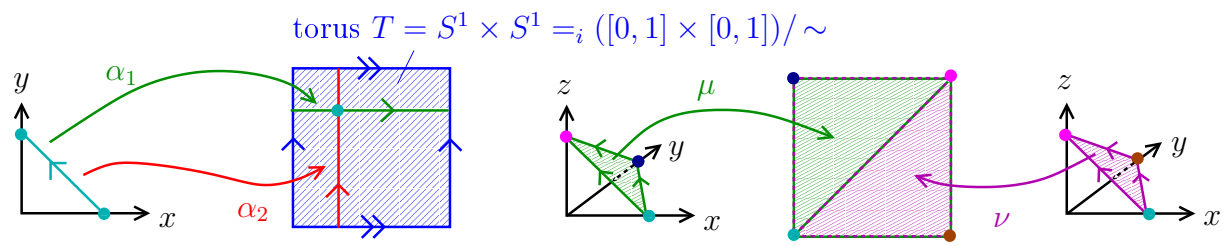
$$\begin{array}{rcl} \alpha_i \colon \Delta^1 & \to & (S^1)^n \\ (t, 1-t) & \mapsto & (1, \dots, 1, \underbrace{\exp(2\pi \mathrm{i} t)}_{\substack{\uparrow \\ i\text{-th coordinate}}}, 1, \dots, 1). \end{array}$$

These singular 1-cycles represent a basis for $\mathrm{H}_1((S^1)^n) \cong \mathbb{Z}^n$.

Example. Let $T = S^1 \times S^1 = ([0,1] \times [0,1])/\sim$ be the 2-dimensional torus. It follows from the Torus-H_*-Proposition 45.5 and the Detect-Cycle Proposition 42.3 that

$$\mathrm{H}_n(T) \;=\; \begin{cases} \mathbb{Z} \cdot [P], & \text{if } n = 0, \\ \mathbb{Z} \cdot [\alpha_1] \oplus \mathbb{Z} \cdot [\alpha_2], & \text{if } n = 1, \\ \mathbb{Z} \cdot [\mu - \nu], & \text{if } n = 2, \\ 0, & \text{otherwise,} \end{cases}$$

where $P \in T$ is any point and where $\alpha_1, \alpha_2 \colon \Delta^1 \to T$ and $\mu, \nu \colon \Delta^2 \to T$ are indicated in the figure below.



Proof.

(1) We prove this statement by induction on n. For $n = 0$ this statement follows from the Singleton-H_*-Lemma 35.11. The induction step is a straightforward consequence of the Product-with-S^1-H_*-Proposition 45.4.

(2) For $i = 1, \dots, n$ let $p_i \colon (S^1)^n \to S^1$ be the projection onto the i-th factor. We consider the homomorphism

$$\begin{array}{rcl} \Psi \colon \mathrm{H}_1((S^1)^n) & \to & \mathrm{H}_1(S^1) \times \cdots \times \mathrm{H}_1(S^1) \\ \alpha & \mapsto & (p_{1*}(\alpha_1), \dots, p_{n*}(\alpha_n)) \end{array}$$

We make the following observations:

(a) By Statement (1) we know that Ψ is a homomorphism between abelian groups that are both isomorphic to $\mathbb{Z}^n$.

(b) On page 362 we showed that the singular 1-cycle $\alpha \colon \Delta^1 \to S^1$ that is given by $(t, 1-t) \mapsto \exp(2\pi \mathrm{i} t)$ represents a generator of $\mathrm{H}_1(S^1)$.

(c) It follows easily from (b) and the discussion on page **??** that Ψ is an epimorphism that sends $[\alpha_1], \dots, [\alpha_n]$ to a basis of $\mathrm{H}_1(S^1) \times \cdots \times \mathrm{H}_1(S^1)$.

(d) It follows from (a) and (c)and some elementary algebra that Ψ is an isomorphism.
(e) It follows from (c) and (d) that $[\alpha_1], \dots, [\alpha_n]$ is a basis of $\mathrm{H}_1((S^1)^n)$. ■

Lemma 45.6. (Klein Bottle-H_*-Lemma) Let

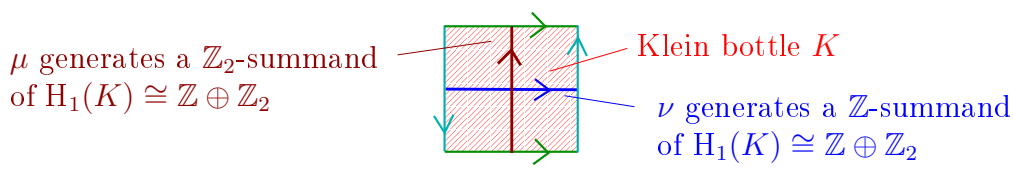
$$K = ([0,1] \times [0,1])/\sim \qquad \text{where } (x,0) \sim (x,1) \text{ and } (0,y) \sim (1,1-y)$$

be the Klein bottle as defined on page 82. We consider the two singular 1-cycles

$$\begin{array}{rcl} \mu\colon \Delta^1 & \to & K \\ (t,1-t) & \mapsto & [(\frac{1}{2},t)] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \nu\colon \Delta^1 & \to & K \\ (t,1-t) & \mapsto & [(\frac{1}{2},t)] \end{array}$$

For any $P \in K$ we have

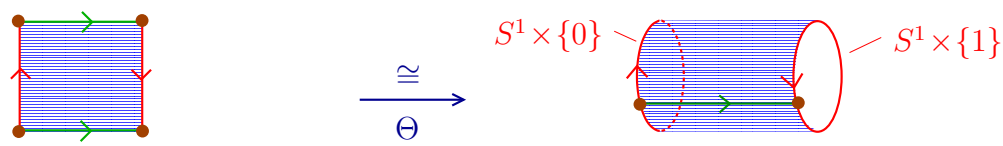
$$\mathrm{H}_n(K) = \begin{cases} \mathbb{Z} \cdot [P], & \text{if } n = 0, \\ \mathbb{Z}_2 \cdot [\mu] \oplus \mathbb{Z} \cdot [\nu], & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$



Proof. We consider $X = S^1$ and we consider the self-homeomorphism $f\colon S^1 \to S^1$ that is given by $f(z) = \overline{z} = z^{-1}$. It is not difficult to show that the map

$$\begin{array}{rcl} \Theta\colon ([0,1] \times [0,1])/\sim & \to & \mathrm{Tor}(S^1, z \mapsto \overline{z}) \\ [(x,y)] & \mapsto & [(\exp(2\pi \mathrm{i} x), y)] \end{array}$$

is a homeomorphism. We will use this homeomorphism as an excuse to study the homology groups of the mapping torus $\mathrm{Tor}(S^1, z \mapsto \overline{z})$.



Klein bottle $([0,1] \times [0,1])/\sim$ mapping torus $\mathrm{Tor}(S^1, z \mapsto \overline{z})$

First note that it follows from the Mapping Torus-H_*-Proposition 45.3 that we get the long exact sequence

$$\begin{array}{ccccccc} & & 0 & \longrightarrow & \mathrm{H}_2(\mathrm{Tor}(S^1,f)) & \xrightarrow{\partial_2} & \\ \to & \mathrm{H}_1(S^1) & \xrightarrow{f_*-\mathrm{id}} & \mathrm{H}_1(S^1) & \xrightarrow{i_*} & \mathrm{H}_1(\mathrm{Tor}(S^1,f)) & \xrightarrow{\partial_1} \\ \to & \mathrm{H}_0(S^1) & \xrightarrow{f_*-\mathrm{id}} & \mathrm{H}_0(S^1) & \xrightarrow{i_*} & \mathrm{H}_0(\mathrm{Tor}(S^1,f)) & \longrightarrow \ 0. \end{array}$$

To get our hands on the homology groups of $K \cong \mathrm{Tor}(S^1, f)$ we make the following observations:

(1) By the Sphere-H_n-Proposition 41.2 we know that $\mathrm{H}_0(S^1) \cong \mathbb{Z}$, $\mathrm{H}_1(S^1) \cong \mathbb{Z} \cdot [\alpha]$ where $\alpha(1-t,t) = \exp(2\pi \mathrm{i} t)$, and that $\mathrm{H}_k(S^1) = 0$ for $k \geq 2$.
(2) It follows from the H_0-Proposition 35.10 (2) that the induced map $f_*\colon \mathrm{H}_0(S^1) \to \mathrm{H}_0(S^1)$ is the identity map.
(3) It follows from the Sphere–Degree–Matrix Lemma 43.4 that the induced homomorphism $f_*\colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(S^1)$ is given by multiplication by -1.

It follows easily from these observations that $\mathrm{H}_2(K) \cong \mathrm{H}_2(\mathrm{Tor}(S^1, f)) = 0$ and that $\mathrm{H}_0(K) \cong \mathrm{H}_0(\mathrm{Tor}(S^1, f)) \cong \mathbb{Z}$. Furthermore it follows that we have a short exact sequence

$$0 \;\to\; \underbrace{\mathrm{coker}(\mathrm{H}_1(S^1) \xrightarrow{\cdot(-2)} \mathrm{H}_1(S^1))}_{\cong \mathbb{Z}_2} \;\xrightarrow{i_*}\; \mathrm{H}_1(\mathrm{Tor}(S^1, f)) \;\xrightarrow{\partial_1}\; \underbrace{\mathrm{H}_0(S^1)}_{\cong \mathbb{Z}} \;\to\; 0.$$

Note that $i_*([\alpha]) = [\Theta \circ \mu]$. Furthermore, using the explicit description of the connecting homomorphisms ∂_1 on page 385 and using the Singular 1-Simplex Lemma 35.12 one can show that $\partial_1([\Theta \circ \nu])$ is a generator of $\mathrm{H}_0(S^1)$. Using the Free Abelian Group–Splitting Lemma 18.5 together with the Splitting Lemma 37.2 (1)⇒(3) we now deduce that

$$\mathrm{H}_1(\mathrm{Tor}(S^1, f)) \;=\; \mathbb{Z}_2 \cdot \underbrace{[\Theta \circ \nu]}_{=\Theta_*([\mu])} \oplus \mathbb{Z} \cdot \underbrace{[\Theta \circ \nu]}_{=\Theta_*([\nu])}.$$

Finally note that by the isomorphism $\Theta_* \colon \mathrm{H}_1(K) \to \mathrm{H}_1(\mathrm{Tor}(S^1, f))$ this implies that $\mathrm{H}_1(K) = \mathbb{Z}_2 \cdot [\mu] \oplus \mathbb{Z} \cdot [\nu]$. ■

46. Homology groups and direct limits

46.1. **Direct limits of topological spaces and homology.** First let us remind the reader of the notion of a direct limit of a sequence of morphisms, as defined on page 218: Given a sequence $X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$ of morphisms the notion of a direct limit can be summarized in the following diagram:

$$\begin{tikzcd}
X_0 \arrow[r, "\varphi_0"] \arrow[drr] \arrow[ddrr] & X_1 \arrow[r, "\varphi_1"] \arrow[dr] & X_2 \arrow[r, "\varphi_2"] \arrow[d] & X_3 \arrow[r, "\varphi_3"] \arrow[dl] \arrow[ddl] & \dots \\
& & \varinjlim X_i \arrow[d, dashed, "\exists!"] & & \\
& & Y & &
\end{tikzcd}$$

More precisely, a direct limit $\varinjlim X_i$, if it exists, comes with morphisms $X_i \to \varinjlim X_i$ which form a commutative diagram; furthermore given morphisms $X_i \to Y$ which form commutative diagrams, we get a unique morphism $\varinjlim X_i \to Y$ which makes everything commute.

Next let us look at two of our favorite categories and types of sequences:

- First we consider the category $\mathcal{Top}$ of topological spaces. Let X be a topological space and let $X_0 \subset X_1 \subset X_2 \subset \dots$ be a sequence of subspaces of X with $X = \bigcup_{i \in \mathbb{N}_0} X_i$. We suppose that one of the following two conditions is satisfied:
  (1) Each X_i is contained in the interior of X_{i+1}.
  (2) X is a CW-complex and each X_i is a subcomplex of X.
  It follows from (1) the Union-equals-Direct Limit Criterion 25.6 and (2) the Direct Limit–CW Lemma 33.6 that in these two settings we have $X = \varinjlim X_i$.
- Next we consider the category $\mathcal{AbGroup}$ of abelian groups. If we are given a sequence $A_0 \xrightarrow{\varphi_0} A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} \dots$ of homomorphisms between abelian groups, then it follows easily from the Algebraic–Direct Limit Existence Proposition 25.3 (1b), that
$$\varinjlim A_i \;=\; \Big(\bigoplus_{i \in \mathbb{N}_0} A_i\Big) / \langle \varphi_i(a) - a \mid \; i \in \mathbb{N}_0 \text{ and } a \in A_i \rangle.$$

The following lemma, together with the above discussion, shows that in reasonable circumstances for each $n \in \mathbb{N}_0$ the singular chain group functor $X \mapsto \mathrm{C}_n(X)$ from the category $\mathcal{Top}$ of topological spaces to the category $\mathcal{AbGroup}$ of abelian groups preserves direct limits.

Lemma 46.1. (Exhaustion-C_*-Lemma) Let X be a topological space. Furthermore let $X_1 \subset X_2 \subset X_3 \subset \dots$ be a sequence of subspaces of X with $\bigcup_{i \in \mathbb{N}_0} X_i = X$. Let $n \in \mathbb{N}_0$. By the universal property of direct limits the inclusion induced maps $\mathrm{C}_n(X_i) \to \mathrm{C}_n(X)$ induce a natural homomorphism
$$\varinjlim_i \mathrm{C}_n(X_i) \;\to\; \mathrm{C}_n(X).$$
If one of the following conditions is satisfied
(1) each X_i is contained in the interior of X_{i+1}, or

(2) X is a CW-complex and each X_i is a subcomplex of X,
then the above map $\varinjlim_i \mathrm{C}_n(X_i) \to \mathrm{C}_n(X)$ is an isomorphism.

Proof. Let X be a topological space and let $X_1 \subset X_2 \subset X_3 \subset \ldots$ be a sequence of subspaces with $\bigcup_{i\in\mathbb{N}_0} X_i = X$ such that one of the following conditions is satisfied:

(1) each X_i is contained in the interior of X_{i+1}, or
(2) X is a CW-complex and each X_i is a subcomplex of X.

We need to show that for each $n \in \mathbb{N}_0$ the natural map $\varinjlim \mathrm{C}_n(X_i) \to \mathrm{C}_n(X)$ is an isomorphism. We start out with the following claim:

Claim. For each compact subset $K \subset X$ there exists an $i \in \mathbb{N}_0$ with $K \subset X_i$.

Proof. Under the hypothesis (1) the claim follows immediately from the elementary Compact Subspace–Exhaustion Lemma 20.11, whereas under the hypothesis (2) the claim follows immediately from the CW-Complex-Finiteness Theorem 30.10. ⊞

In the following, given a topological space Y we denote by $\mathrm{S}_n(Y)$ the set of singular n-simplices in Y. We consider the following diagram:

$$\begin{tikzcd}
\varinjlim_i \mathrm{C}_n(X_i) \arrow[rr] \arrow[d, "\mathrm{id}"] & & \mathrm{C}_n(X) \arrow[d, "\mathrm{id}"] \\
\varinjlim_i \mathbb{Z}^{(\mathrm{S}_n(X_i))} \arrow[r, "\cong"] & \mathbb{Z}^{\left(\bigcup_{i\in\mathbb{N}_0} \mathrm{S}_n(X_i)\right)} \arrow[r, "\mathrm{id}"] & \mathbb{Z}^{(\mathrm{S}_n(X))}.
\end{tikzcd}$$

The lemma follows from the following observations:

(a) The top map and the bottom left map are induced by the inclusions $X_i \to X$.
(b) It follows easily from (a) and the definitions that the diagram commutes.
(c) On page 218 we saw that the bottom left homomorphism is an isomorphism.
(d) It follows immediately from the fact that Δ^n is compact, the Compact Image Lemma 2.8 and the above claim that the image of any continuous map $\Delta^n \to X$ is contained in some X_i. It follows from this observation and the hypothesis that $X = \bigcup_{i\in\mathbb{N}_0} X_i$ that $\mathrm{S}_n(X) = \bigcup_{i\in\mathbb{N}_0} \mathrm{S}_n(X_i)$. In other words, the bottom right map is indeed the identity.
(e) It follows from (b), (c) and (d) that the top homomorphism is also an isomorphism. ■

Lemma 46.5. (Direct Limit-Homology Lemma) Suppose that for each $i \in \mathbb{N}_0$ we are given a chain complex $(C_{i*}, \partial_*) = (\{C_{in}\}_{n\in\mathbb{N}_0}, \{\partial_n\}_{n\in\mathbb{N}_0})$. Furthermore suppose that we are given a sequence $C_{0*} \to C_{1*} \to C_{2*} \to \ldots$ of chain maps.

(1) The groups $\varinjlim_i C_{in}$ with $n \in \mathbb{N}_0$ naturally form a chain complex such that the natural maps $C_{j*} \to \varinjlim_i C_{i*}$ are chain maps.
(2) For each $n \in \mathbb{N}_0$ the homomorphisms $\mathrm{H}_n(C_{j*}) \to \mathrm{H}_n\big(\varinjlim_i C_{i*}\big)$ from (1) induce an isomorphism
$$\varinjlim_j \mathrm{H}_n(C_{j*}) \xrightarrow{\cong} \mathrm{H}_n\big(\varinjlim_i C_{i*}\big).$$

Proof. To keep the conversation going we outsource the proof of this algebraic lemma to Section 46.2. In fact in that section we will formulate and prove a slightly more general statement which we will need later on in a different setting. ■

Proposition 46.2. (Exhaustions-H_*-Proposition) Let X be a topological space and let $X_0 \subset X_1 \subset X_2 \subset \dots$ be a sequence of subspaces such that $X = \bigcup_{i \in \mathbb{N}_0} X_i$ and such that one of the following conditions holds:

(1) each X_i is contained in the interior of X_{i+1}, or
(2) X is a CW-complex and each X_i is a subcomplex of X,

then for any $n \in \mathbb{N}_0$ the inclusion induced maps $\mathrm{H}_n(X_i) \to \mathrm{H}_n(X)$ induce an isomorphism

$$\varinjlim \mathrm{H}_n(X_i) \xrightarrow{\cong} \mathrm{H}_n(X).$$

Proof. For each $j \in \mathbb{N}_0$ we consider the following diagram:

$$\begin{tikzcd}
\varinjlim_i \mathrm{H}_n(X_i) \arrow[rrr] \arrow[ddd, "\mathrm{id}"'] & & & \mathrm{H}_n(X) \\
& \mathrm{H}_n(X_j) = \mathrm{H}_n(\mathrm{C}_*(X_j)) \arrow[ul] \arrow[ur] \arrow[d] & & \\
\varinjlim_i \mathrm{H}_n(\mathrm{C}_*(X_i)) \arrow[r] & \mathrm{H}_n\big(\varinjlim_i \mathrm{C}_*(X_i)\big) \arrow[r] & \mathrm{H}_n(\mathrm{C}_*(X)) \arrow[uu, "\mathrm{id}"'] &
\end{tikzcd}$$

One can easily verify that each segment of the diagram involving $\mathrm{H}_n(X_j)$ commutes. It follows from the universal property of direct limits that the total diagram commutes. Next note that the bottom left map is an isomorphism by the purely algebraic Direct Limit-Homology Lemma 46.5 and that the bottom right map is an isomorphism by the Exhaustion-C_*-Lemma 46.1 (here we use our hypothesis on the sequence of subspaces). It follows that the top map is also an isomorphism. ■

Example. For any $n \in \mathbb{N}_0$ we have

$$\mathrm{H}_n(S^\infty) \underset{\uparrow}{=} \varinjlim_k \mathrm{H}_n(S^k) \underset{\uparrow}{=} \varinjlim_k \begin{cases} \mathbb{Z}, & \text{if } n = k, 0, \\ 0, & \text{else} \end{cases} \underset{\uparrow}{=} \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{else.} \end{cases}$$

by the Homology-via-Exhaustions Proposition 46.2, where we view S^∞ as a CW-complex as on page 263

by the Sphere-H_n-Proposition 41.2

by the Direct Limit Stabilization Lemma 25.1

46.2. **Direct limits of chain complexes.** First we recall the following definition from pages **??** and 217.

Definition.

(1) A directed set is a set I together with a partial order $\leq$ that has the extra property that for any $i, j \in I$ there exists a $k \in I$ with $i \leq k$ and $j \leq k$.
(2) Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be a category. A direct system in the category $\mathcal{C}$ over I is a family of objects $\{X_i\}_{i \in I}$ in $\mathcal{C}$, together with a family of morphisms

$\{f_{ji}\colon X_i \to X_j\}_{i\le j}$ such that the following two conditions are satisfied:

(a) $\quad f_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$,

(b) $\quad f_{ki} = f_{kj} \circ f_{ji}$ for all $i, j, k \in I$ with $i \le j \le k$.

$$\begin{tikzcd}
 & X_i \arrow[dr, "f_{ki}"] & \\
 & & X_k \arrow[r] & {} \\
 & X_j \arrow[ur, "f_{kj}"'] &
\end{tikzcd}$$

Example. Let $\mathcal{C}$ be a category. Any sequence $X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$ of morphisms in $\mathcal{C}$ gives rise to a direct system over the directed set $(\mathbb{N}_0, \le)$ by considering for $i \le j$ the morphism $f_{ji} := \varphi_{j-1} \circ \dots \circ \varphi_{i+1} \circ \varphi_i\colon X_i \to X_j$.

Definition. Let $(I, \le)$ be a directed set and let $(\{K_i\}_{i\in I}, \{\kappa_{ji}\}_{i\le j})$ and $(\{L_i\}_{i\in I}, \{\lambda_{ji}\}_{i\le j})$ be direct systems in some category $\mathcal{C}$.

(1) A morphism between the above two direct systems is a family $\{f_i\colon K_i \to L_i\}_{i\in I}$ of morphisms such that for any $i \le j$ the following diagram commutes:

$$\begin{tikzcd}
K_i \arrow[r, "f_i"] \arrow[d, "\kappa_{ji}"'] & L_i \arrow[d, "\lambda_{ji}"] \\
K_j \arrow[r, "f_j"] & L_j.
\end{tikzcd}$$

(2) By the universal property of direct limits the morphisms $K_i \to L_i \to \varinjlim L_i$ give rise to a natural morphism

$$\varinjlim f_i\colon \varinjlim K_i \to \varinjlim L_i.$$

Proposition 46.3. (Direct Limit-Exactness Proposition) Let $(I, \le)$ be a directed set. Also let $(\{K_i\}_{i\in I}, \{\kappa_{ji}\}_{i\le j})$, $(\{L_i\}_{i\in I}, \{\lambda_{ji}\}_{i\le j})$ and $(\{M_i\}_{i\in I}, \{\mu_{ji}\}_{i\le j})$ be direct systems of abelian groups. Finally suppose that we are given morphisms $f_i\colon K_i \to L_i$ and $g_i\colon L_i \to M_i$ of the direct systems.

(1) If for all $i \in I$ we have $g_i \circ f_i = 0$, then $\varinjlim g_i \circ \varinjlim f_i = 0$.

(2) If for all $i \in I$ the sequence

$$K_i \xrightarrow{f_i} L_i \xrightarrow{g_i} M_i$$

is exact, then the induced sequence

$$\varinjlim K_i \xrightarrow{\varinjlim f_i} \varinjlim L_i \xrightarrow{\varinjlim g_i} \varinjlim M_i$$

is also exact.

Proof. Let $(I, \le)$ be a directed set. We start out with two general observations regarding direct limits:

(a) Let $(\{A_i\}_{i\in I}, \{\alpha_{ji}\}_{i\le j})$ be a direct system of abelian groups. By the Algebraic–Direct Limit Existence Proposition 25.3 (1b) we know that

$$\varinjlim A_i = \Big(\bigoplus_{i\in I} A_i\Big) / \langle \alpha_{ji}(x) - x \mid i \le j \text{ and } x \in A_i \rangle.$$

We adopt the following notation: Given $a_j \in A_j \subset \bigoplus_{i\in I} A_i$ we denote by $[a_j] \in \varinjlim A_j$ the element represented by a_j.

Claim.

(i) Any element in $\varinjlim A_i$ is of the form $[a_i]$ for some $a_i \in A_i$.

(ii) If we are given $a_j \in A_j$ with $[a_j] = 0 \in \varinjlim A_i$, then there exists a $k \geq j$ such that $\alpha_{kj}(a) = 0 \in A_k$.

Proof.

(i) By the above construction any element in $\varinjlim A_i$ is of the form $[a_{i_1}] + \cdots + [a_{i_m}]$ where $a_{i_j} \in A_{i_j}$. Since $(I, \leq)$ is a directed set we see easily, by considering an $i \in I$ with $i \geq i_1, \ldots, i_m$, that any element of $\varinjlim A_i$ is in fact contained in the image of a *single* A_i, i.e. is of the form $[a_i]$ for some $a_i \in A_i$.

(ii) If we are given $a_j \in A_j$ with $[a_j] = 0 \in \varinjlim A_i$, then it follows from the above description of $\varinjlim A_i$ that in $\bigoplus_{i\in I} A_i$ we can write $a_j = \sum_{r=1}^{m} c_r \cdot (\alpha_{t_r,s_r}(x_r) - x_r) \in \bigoplus_{i\in I} A_i$ where $c_r \in \mathbb{Z}$, $x_r \in A_{s_r}$ and $t_r \geq s_r$. Since we are dealing with a directed set there exists a $k \in I$ with $k \geq j, t_{r_1}, \ldots, t_{r_m}$. Then we obtain the following equality in A_k:

$$\begin{aligned}\alpha_{kj}(a_j) &= \Big(\bigoplus_{i\leq k} \alpha_{ki}\Big)(a_j) = \Big(\bigoplus_{i\leq k} \alpha_{ki}\Big)\Big(\sum_{r=1}^{m} c_r \cdot (\alpha_{t_r,s_r}(x_r) - x_r)\Big) \\ &= \sum_{r=1}^{m} c_r \cdot \Big(\underbrace{\Big(\bigoplus_{i\leq k} \alpha_{ki}\Big)(\alpha_{t_r,s_r}(x_r))}_{=\alpha_{k,t_r}(\alpha_{t_r,s_r}(x_r))=\alpha_{k,s_r}(x_r)} - \underbrace{\Big(\bigoplus_{i\leq k} \alpha_{ki}\Big)(x_r)}_{=\alpha_{k,s_r}(x_r)}\Big) = 0.\end{aligned}$$

⊞

(b) If $(\{A_i\}_{i\in I}, \{\alpha_{ji}\}_{i\leq j})$ and $(\{B_i\}_{i\in I}, \{\beta_{ji}\}_{i\leq j})$ are two direct systems of abelian groups and if $\{f_i \colon A_i \to B_i\}_{i\in I}$ is a morphism between these two direct systems, then it follows easily from the definitions and (a) that the induced map of direct systems is given by

$$\begin{array}{ccc} \varinjlim A_i & \to & \varinjlim B_i \\ [a_i] & \mapsto & [f_i(a_i)] \end{array}$$

Now we turn to the actual proof of the two statements of the proposition. We write $f = \varinjlim f_i$ and $g = \varinjlim g_i$.

(1) We assume that for all $i \in I$ we have $g_i \circ f_i = 0$. We need to show that $g \circ f = 0$. Suppose we are given an element of $\varinjlim K_i$. By (a-i) we know that it is of the form $[k_i]$ for some $i \in I$ and some $k_i \in K_i$. We now see that

$$(g \circ f)([k_i]) = g(f([k_i])) \underset{\substack{\uparrow \\ \text{by (b)}}}{=} g_i([f_i(k_i)]) \underset{\substack{\uparrow \\ \text{by (b)}}}{=} [g_i(f_i(k_i))] = [(g_i \circ f_i)(k_i)] \underset{\substack{\uparrow \\ \text{since } g_i \circ f_i = 0}}{=} 0.$$

(2) Now we assume that each $K_j \xrightarrow{f_j} L_j \xrightarrow{g_j} M_j$ is exact. Note that by (1) it remains to prove that $\ker(g) \subset \operatorname{im}(f)$. Thus suppose we are given an element of $\ker(g) \in \varinjlim L_i$. By (a-i) we know that it is of the form $[l_i]$ for some $i \in I$ and some $l_i \in L_i$. Since $g([l_i]) = 0$ it follows from (a-ii) and (b) that there exists a $j \geq i$ such that $\mu_{ji}(g_i(l_i)) = 0$. Next

we consider the following diagram

$$\begin{tikzcd} K_i \arrow[r, "f_i"] \arrow[d, "\kappa_{ji}"] & L_i \arrow[r, "g_i"] \arrow[d, "\lambda_{ji}"] & M_i \arrow[d, "\mu_{ji}"] \\ K_j \arrow[r, "f_j"] & L_j \arrow[r, "g_j"] & M_j. \end{tikzcd}$$

By hypothesis the diagram commutes, thus we see that $g_j(\lambda_{ji}(l_i)) = \mu_{ji}(g_i(l_i)) = 0$. Since $K_j \xrightarrow{f_j} L_j \xrightarrow{g_j} M_j$ is exact we know that there exists a $k_j \in K_j$ with $f_j(k_j) = \lambda_{ji}(l_i)$. It follows from (b) that $f([k_j]) = [f_j(k_j)] = [l_j] = [l_i] \in \varinjlim L_i$. We have thus shown that $[l_i] \in \operatorname{im}(f)$. ■

Corollary 46.4. (Direct Limit-(Co-)-Kernel Corollary) Let $(I, \leq)$ be a directed set and let $(\{K_i\}_{i\in I}, \{\kappa_{ji}\}_{i\leq j})$ and $(\{L_i\}_{i\in I}, \{\lambda_{ji}\}_{i\leq j})$ be direct systems of abelian groups. Furthermore suppose that we are given a morphism $f_i\colon K_i \to L_i$ between the direct systems. There exist natural isomorphisms

$$\varinjlim \ker(f_i\colon K_i \to L_i) \xrightarrow{\cong} \ker\big(\varinjlim f_i\colon \varinjlim K_i \to \varinjlim L_i\big)$$

and

$$\varinjlim \operatorname{coker}(f_i\colon K_i \to L_i) \xrightarrow{\cong} \operatorname{coker}\big(\varinjlim f_i\colon \varinjlim K_i \to \varinjlim L_i\big).$$

Proof. By the elementary Exactness Lemma 37.1 we know that for each $i \in I$ the sequence

$$0 \to \ker(f_i) \to K_i \xrightarrow{f_i} L_i \to \operatorname{coker}(f_i) \to 0.$$

is exact. One can easily verify that these homomorphisms form morphisms between the various direct systems. It follows from the Direct Limit-Exactness Proposition 46.3 (2) (applied four times!) that the sequence

$$0 \to \varinjlim \ker(f_i) \to \varinjlim K_i \xrightarrow{\varinjlim f_i} \varinjlim L_i \to \varinjlim \operatorname{coker}(f_i) \to 0$$

is also exact. By the Exactness Lemma 37.1 this observation implies the desired statements. ■

Lemma 46.5. (Direct Limit-Homology Lemma) Let $(I, \leq)$ be a directed set and let $(\{C_i\}_{i\in I}, \{f_{ji}\}_{i\leq j})$ be a direct system of chain complexes.

(1) The groups $\varinjlim_i C_{i*}$ naturally form a chain complex such that the natural maps $C_{j*} \to \varinjlim_i C_{i*}$ are chain maps.

(2) For each $n \in \mathbb{N}_0$ the homomorphisms $\mathrm{H}_n(C_{j*}) \to \mathrm{H}_n\big(\varinjlim_i C_{j*}\big)$ from (1) induce an isomorphism

$$\varinjlim_j \mathrm{H}_n(C_{j*}) \xrightarrow{\cong} \mathrm{H}_n\big(\varinjlim_i C_{i*}\big).$$

Proof.

(1) It follows immediately from the Direct Limit-Exactness Proposition 46.3 (1), applied to the maps $C_{i,n+1} \xrightarrow{\partial_{n+1}} C_{i,n} \xrightarrow{\partial_n} C_{i,n-1}$, that the direct limits $\varinjlim_i C_{i*}$ form a chain complex.

Furthermore note that it follows immediately from the universal properties of direct limits that the natural maps $C_{j*} \to \varinjlim_i C_{i*}$ are chain maps.

(2) Given any $n \in \mathbb{N}_0$ we see that

$$\begin{aligned}
\varinjlim_i \mathrm{H}_n(C_{i*}) &= \varinjlim_i \overbrace{\operatorname{coker}\left(C_{i,n+1} \xrightarrow{\partial_{n+1}} \ker(C_{i,n} \xrightarrow{\partial_n} C_{i-1,n})\right)}^{=\mathrm{H}_n(C_{i*})} \\
&\underset{\uparrow}{\cong} \operatorname{coker}\left(\varinjlim_i C_{i,n+1} \xrightarrow{\varinjlim \partial_{n+1}} \ker\left(\varinjlim_i C_{i,n} \xrightarrow{\varinjlim \partial_n} \varinjlim_i C_{i,n-1}\right)\right) \\
&\qquad \text{follows from applying the Direct Limit-(Co-)-Kernel Corollary 46.4 twice} \\
&\underset{\uparrow}{=} \mathrm{H}_n\big(\varinjlim_i C_{i*}\big). \\
&\qquad \text{by definition of the } n\text{-th homology group}
\end{aligned}$$

The fact that this isomorphism is induced by the maps $\mathrm{H}_n(C_{i*}) \to \mathrm{H}_n(\varinjlim C_{i*})$ follows easily from the various definitions. We leave the details to the reader. ■

47. Cellular homology

47.1. Relative homology groups of CW-complexes.

Notation. Let X be a CW-complex

(1) As usual, we denote the filtration of X by $\varnothing = X^{-1} \subset X^0 \subset X^1 \subset X^2 \subset \dots$ and we refer to X^n as the n-skeleton of X.

(2) We denote by $\mathrm{Cell}_n(X)$ the set of n-cells of X.

Lemma 47.1. (Skeleton-H_*-Lemma) Let X be a CW-complex and let $n \in \mathbb{N}_0$.

(1) Let $\{\Phi_c \colon \overline{B}^n \to X^n\}_{c \in \mathrm{Cell}_n(X)}$ be a full set of characteristic maps for the n-cells of X. For every $k \in \mathbb{N}_0$ the following map is an isomorphism:

$$\bigoplus_{c \in \mathrm{Cell}_n(X)} \Phi_{c*} \colon \bigoplus_{c \in \mathrm{Cell}_n(X)} \mathrm{H}_k(\overline{B}^n, S^{n-1}) \;\to\; \mathrm{H}_k(X^n, X^{n-1}).$$

In particular the following statements hold:

(a) For $k \neq n$ we have $\mathrm{H}_k(X^n, X^{n-1}) = 0$.

(b) The group $\mathrm{H}_n(X^n, X^{n-1})$ is a free abelian group where the rank equals the cardinality of the set $\mathrm{Cell}_n(X)$ of n-cells in the CW-structure.

(2) For $k > n$ we have $\mathrm{H}_k(X^n) = 0$.

(3) For $k < n$ the inclusion $i \colon X^n \to X$ induces an isomorphism $i_* \colon \mathrm{H}_k(X^n) \xrightarrow{\cong} \mathrm{H}_k(X)$.

(4) The inclusion $i \colon X^n \to X$ induces an epimorphism $i_* \colon \mathrm{H}_n(X^n) \twoheadrightarrow \mathrm{H}_n(X)$.

Proof. Let X be a CW-complex and let $\{\Phi_c \colon \overline{B}^n \to X^n\}_{c \in \mathrm{Cell}_n(X)}$ be a full set of characteristic maps for the n-cells of X.

(1) We consider the following diagram:

$$\begin{tikzcd}
\bigoplus_{c \in \mathrm{Cell}_n(X)} \mathrm{H}_k(\overline{B}^n, S^{n-1}) \arrow[rr, "\bigoplus \Phi_{c*}"] \arrow[d, "\text{(i)}"', "\mathrm{diag}(\text{ю})"] & & \mathrm{H}_k(X^n, X^{n-1}) \arrow[d, "\text{ю}"', "\text{(iv)}"] \\
\bigoplus_{c \in \mathrm{Cell}_n(X)} \widetilde{\mathrm{H}}_k(\overline{B}^n / S^{n-1}) \arrow[rr, "\bigoplus \overline{\Phi}_{c*}"] \arrow[dr, "\text{(ii)}", "\bigoplus \iota_{c*}"'] & & \widetilde{\mathrm{H}}_k(X^n / X^{n-1}). \\
& \widetilde{\mathrm{H}}_k\Big(\bigvee_{c \in \mathrm{Cell}_n(X)} \overline{B}^n / S^{n-1}\Big) \arrow[ur, "\text{(iii)}", "(\vee \overline{\Phi}_c)_*"'] &
\end{tikzcd}$$

We make the following clarifications and observations:

- Given a pair (Y, A) of topological space we consider the natural homomorphism $\text{ю} \colon \mathrm{H}_n(Y, A) \to \widetilde{\mathrm{H}}_n(Y/A)$ from page 352.
- We use (obvious generalizations of) the notation from page 371 regarding maps out of direct sums of abelian groups.
- Given $c \in \mathrm{Cell}_n(X)$ we denote by $\overline{\Phi}_c \colon \overline{B}^n / S^{n-1} \to X^n / X^{n-1}$ the map that is induced by the characteristic map $\Phi_c \colon \overline{B}^n \to X^n$.
- It follows almost immediately from the naturality of the maps ю that the rectangle commutes.
- We use the points $[S^{n-1}] \in \overline{B}^n / S^{n-1}$ to form the wedge $\bigvee_{c \in \mathrm{Cell}_n(X)} \overline{B}^n / S^{n-1}$.

- For $j \in \mathrm{Cell}_n(X)$ let $\iota_j \colon \overline{B}^n/S^{n-1} \to \bigvee_{c \in \mathrm{Cell}_n(X)} \overline{B}^n/S^{n-1}$ be the natural inclusion.
- For each $j \in \mathrm{Cell}_n(X)$ we have $\overline{\Phi}_j = \big(\bigvee_{c \in \mathrm{Cell}_n(X)} \overline{\Phi}_c\big) \circ \iota_j$. It follows from this observation and the definition of $\bigoplus_{c \in \mathrm{Cell}_n(X)} \overline{\Phi}_{c*}$ that the triangle commutes.

We want to show that the top horizontal map is an isomorphism. By the above it remains to show that the maps (i)–(iv) are isomorphisms.

(i) This map is an isomorphism by the Relative-to-Quotient-H_*-Proposition 41.8.
(ii) This map is an isomorphism by the Wedge-H_*-Proposition 41.10.
(iii) This map is an isomorphism by the Skeleton-Quotient Lemma 33.5.
(iv) It follows from the Relative-to-Quotient-H_*-Proposition 41.8 together with the Subcomplex–Neighborhood Proposition 31.8 that this map is an isomorphism.

The remaining statements of (1) are now an immediate consequence of the Ball-rel-Sphere-H_*-Lemma 41.4.

(2) We fix $k \in \mathbb{N}$. We need to prove the following claim.

Claim. For $n = 0, \dots, k-1$ we have $\mathrm{H}_k(X^n) = 0$.

Proof. We prove the claim by induction on $n \in \mathbb{N}_0$. First we consider the case $n = 0$. By the discussion on page 261 we know that the 0-skeleton X^0 has the discrete topology. It follows from the Singleton-H_*-Lemma 35.11, the Path-Component–H_*–Lemma 35.8 and the hypothesis that $k > 0$ that $\mathrm{H}_k(X^0) = 0$.

Now assume that the statement holds for some $n \in \{0, \dots, k-2\}$. We consider the long exact sequence of the pair (X^{n+1}, X^n):

$$\dots \to \mathrm{H}_{k+1}(X^{n+1}, X^n) \to \underbrace{\mathrm{H}_k(X^n)}_{=0 \text{ by induction}} \to \mathrm{H}_k(X^{n+1}) \to \underbrace{\mathrm{H}_k(X^{n+1}, X^n)}_{\substack{=0 \text{ by (1) since } n < k-1 \\ \text{which implies that } k \neq n+1}} \to \dots$$

It follows that we have an epimorphism $\mathrm{H}_k(X^n) \to \mathrm{H}_k(X^{n+1})$. Since $\mathrm{H}_k(X^n) = 0$ we conclude that $\mathrm{H}_k(X^{n+1}) = 0$. ⊞

(3) Let $k \in \mathbb{N}_0$ and $n > k$. Recall that we need to show that the inclusion induced map $\mathrm{H}_k(X^n) \to \mathrm{H}_k(X)$ is an isomorphism. We consider the following diagram of inclusion induced homomorphisms:

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}_k(X^{n-1}) \arrow[r] \arrow[dr] & \mathrm{H}_k(X^n) \arrow[r, "\cong"] \arrow[d] & \mathrm{H}_k(X^{n+1}) \arrow[r, "\cong"] \arrow[dl] & \mathrm{H}_k(X^{n+2}) \arrow[r, "\cong"] \arrow[dll] & \dots \\
& & \mathrm{H}_k(X). & & &
\end{tikzcd}$$

We make the following observations:

(a) The diagram commutes by the functoriality of homology.
(b) We just want to prove the following claim:

Claim. For any $m \geq n$ the inclusion induced homomorphism $\mathrm{H}_k(X^m) \to \mathrm{H}_k(X^{m+1})$ is an isomorphism.

Proof. This claim follows by considering the following excerpt from the long exact sequence of the pair (X^{m+1}, X^m):

$$\ldots \to \underbrace{\mathrm{H}_{k+1}(X^{m+1}, X^m)}_{=0 \text{ by (1) since } k<n\leq m} \to \mathrm{H}_k(X^m) \to \mathrm{H}_k(X^{m+1}) \to \underbrace{\mathrm{H}_k(X^{m+1}, X^m)}_{=0 \text{ by (1) since } k<n\leq m} \to \ldots$$

⊞

(c) By the Exhaustions-H_*-Proposition 46.2 (2) we know that $\mathrm{H}_k(X) = \varinjlim \mathrm{H}_k(X^n)$.

(d) It follows from (a)–(c) and the Direct Limit–Stabilization Lemma 25.1 that the vertical inclusion induced homomorphism $\mathrm{H}_k(X^n) \to \mathrm{H}_k(X)$ is an isomorphism.

(4) We consider the following excerpt from the long exact sequence of the pair (X^{n+1}, X^n):

$$\begin{tikzcd}
\ldots \arrow[r] & \mathrm{H}_{n+1}(X^{n+1}, X^n) \arrow[r] & \mathrm{H}_n(X^n) \arrow[r] \arrow[dr] & \mathrm{H}_n(X^{n+1}) \arrow[r] \arrow[d, "\cong \text{ by (3)}"] & \underbrace{\mathrm{H}_n(X^{n+1}, X^n)}_{=0 \text{ by (1)}} \arrow[r] & \ldots \\
& & & \mathrm{H}_n(X) & &
\end{tikzcd}$$

We see that the inclusion induced map $\mathrm{H}_n(X^n) \to \mathrm{H}_n(X)$ is an epimorphism. ■

47.2. The definition of cellular homology.

Definition. Given a CW-complex X and given $n \in \mathbb{N}_0$ we denote by d_n the homomorphism

$$\begin{tikzcd}
\mathrm{H}_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[rr, bend right, "=:d_n"'] & \mathrm{H}_{n-1}(X^{n-1}) \arrow[r, "я_{n-1}"] & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2})
\end{tikzcd}$$

where ∂_n is the connecting map in the long exact sequence of the pair (X^n, X^{n-1}) and where $я_{n-1} \colon \mathrm{H}_{n-1}(X^{n-1}) \to \mathrm{H}_{n-1}(X^{n-1}, X^{n-2})$ is the natural map introduced on page 325. We refer to d_n as the cellular boundary map.

Lemma 47.2. (Cellular Boundary Map Lemma) For all $n \in \mathbb{N}_0$ we have $d_n \circ d_{n+1} = 0$.

Proof. We consider the following diagram of homomorphisms:

$$\begin{tikzcd}
& \mathrm{H}_n(X^n) \arrow[dr, "я_n"] & & & \\
\mathrm{H}_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & \mathrm{H}_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"] & & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}) \\
& & & \mathrm{H}_{n-1}(X^{n-1}) \arrow[ur, "я_{n-1}"'] &
\end{tikzcd}$$

Note that the diagram commutes by the definition of the cellular boundary maps. The map $\partial_n \circ я_n \colon \mathrm{H}_n(X^n) \to \mathrm{H}_{n-1}(X^{n-1})$ is the composition of two successive maps in the long exact sequence of the pair (X^n, X^{n-1}). This implies that the map $\partial_n \circ я_n$ is the zero map. Since the diagram commutes we obtain that $d_n \circ d_{n+1} = 0$. ■

Definition. Let X be a CW-complex.

(1) Given $n \in \mathbb{N}_0$ we write $\mathrm{C}_n^{\mathrm{CW}}(X) := \mathrm{H}_n(X^n, X^{n-1})$.

(2) We refer to $(\mathrm{C}_*^{\mathrm{CW}}(X), d_*)$ as the cellular chain complex of the CW-complex X.

(3) We denote the homology groups of the cellular chain complex by $\mathrm{H}_n^{\mathrm{CW}}(X)$ and we refer to these groups as the cellular homology groups of the CW-complex X.

Example. For $n \geq 2$ we consider the n-dimensional sphere S^n. On page 261 we showed that we can view S^n as a CW-complex with one 0-cell and one n-cell. It follows from the Skeleton-H_*-Lemma 47.1 (1) that the cellular chain complex $(\mathrm{C}_*^{\mathrm{CW}}(S^n), d_*)$ is of the form

$$0 \to \underbrace{\mathrm{C}_n^{\mathrm{CW}}(S^n)}_{\cong\mathbb{Z}} \to 0 \to \ldots \to 0 \to \underbrace{\mathrm{C}_0^{\mathrm{CW}}(S^n)}_{\cong\mathbb{Z}} \to 0.$$

Since $n \geq 2$ the two $\mathbb{Z}$'s are separated by at least one zero group. The cellular boundary maps are therefore necessarily the zero maps. This implies that

$$\mathrm{H}_i^{\mathrm{CW}}(S^n) \cong \begin{cases} \mathbb{Z}, & \text{if } i = 0, n, \\ 0, & \text{otherwise.} \end{cases}$$

These cellular homology groups are of course isomorphic to the "usual" homology groups of S^n. In the next section we will see that this is not a coincidence.

47.3. The relationship between cellular and singular homology.

Proposition 47.3. (Singular-Cellular-H_*-Proposition) Let $n \in \mathbb{N}_0$.

(1) For each CW-complex X there exists a uniquely determined *isomorphism*

$$\Theta_X \colon \mathrm{H}_n(X) \xrightarrow{\cong} \mathrm{H}_n^{\mathrm{CW}}(X)$$

with the property that the following diagram commutes:

$$\begin{tikzcd} \mathrm{H}_n(X^n) \arrow[r, two heads] \arrow[d, "\text{я}_n"'] & \mathrm{H}_n(X) \arrow[d, "\Theta_X"] \\ \ker(d_n) \arrow[r, two heads, "a \mapsto [a]"] & \mathrm{H}_n^{\mathrm{CW}}(X). \end{tikzcd}$$

(2) The homomorphisms from (1) define a natural isomorphism between the functors $X \mapsto \mathrm{H}_n(X)$ and the functor $X \mapsto \mathrm{H}_n^{\mathrm{CW}}(X)$.

Proof of Proposition 47.3 (1). We add a couple of maps to the commutative diagram that we had already encountered in the proof of the Cellular Boundary Map Lemma 47.2:

$$\begin{tikzcd} & & & 0 \\ 0 \arrow[dr] & & \mathrm{H}_n(X^{n+1}) \arrow[ur] & \\ & \mathrm{H}_n(X^n) \arrow[ur, two heads] \arrow[dr, hook, "\text{я}_n"] & & \\ \mathrm{H}_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & \mathrm{H}_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"'] & & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}). \\ & & & \mathrm{H}_{n-1}(X^{n-1}) \arrow[ur, hook, "\text{я}_{n-1}"'] & \\ & & 0 \arrow[ur] & & \end{tikzcd}$$

Claim 1. All the diagonal sequences in the above commutative diagram are exact.

Proof. By the Skeleton-H_*-Lemma 47.1 (1) and (2) we know that

$$\mathrm{H}_n(X^{n+1}, X^n) = 0, \quad \mathrm{H}_n(X^{n-1}) = 0 \quad \text{and} \quad \mathrm{H}_{n-1}(X^{n-2}) = 0.$$

It thus follows that the diagonal sequences in the above commutative diagram are just excerpts from the long exact sequences of the corresponding pairs of topological spaces. In particular all the diagonal sequences are exact. ⊞

Claim 2. The map $\text{я}_n\colon \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^n, X^{n-1})$ induces an isomorphism

$$\mathrm{H}_n(X^n)/\operatorname{im}(\partial_{n+1}) \;\to\; \underbrace{\ker(d_n)/\operatorname{im}(d_{n+1})}_{=\mathrm{H}_n^{\mathrm{CW}}(X)}.$$

Proof.

(1) The given map is well-defined since it follows immediately from $\partial_n \circ \text{я}_n = 0$ that $\text{я}_n(\mathrm{H}_n(X^n)) \subset \ker(d_n)$.

(2) The given map is an epimorphism. This can be seen as follows:
   (a) It follows from Claim 1 that the map я_{n-1} is injective.
   (b) It follows from (a) and Claim 1 that $\ker(d_n) = \ker(\partial_n) = \operatorname{im}(\text{я}_n)$.
   (c) It follows from (b) that the map я_n induces an epimorphism $\mathrm{H}_n(X^n) \to \ker(d_n)$.
   (d) Since the left triangle commutes we obtain from (c) that я_n also induces an epimorphism $\mathrm{H}_n(X^n)/\operatorname{im}(\partial_{n+1}) \to \ker(d_n)/\operatorname{im}(d_{n+1})$.

(3) The given map is a monomorphism: We need to show that if we are given $c \in \mathrm{H}_n(X^n)$ such that $\text{я}_n(c) \in \mathrm{H}_n(X^n, X^{n-1})$ lies in the image of d_{n+1}, then $c \in \operatorname{im}(\partial_{n+1})$. Thus suppose that $\text{я}_n(c) \in \mathrm{H}_n(X^n, X^{n-1})$ lies in the image of d_{n+1}. This means that there exists an $e \in \mathrm{H}_{n+1}(X^{n+1}, X^n)$ with $\text{я}_n(c) = d_{n+1}(e) = \text{я}_n(\partial_{n+1}(e))$. Since by Claim 1 we know that я_n is injective we obtain that $c = \partial_{n+1}(e)$. ⊞

Next we consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}_n(X^n)/\operatorname{im}(\partial_{n+1}) \arrow[r, "\text{isomorphism since in the above diagram the left diagonal sequence is exact}"'] \arrow[d, "\text{by the claim we know that } \text{я}_n \text{ induces an isomorphism}"', "\cong"] & \mathrm{H}_n(X^{n+1}) \arrow[r, "\text{isomorphism by the Skeleton-H}_*\text{-Lemma 47.1 (3)}"'] & \mathrm{H}_n(X) \arrow[d, "=:\ \Theta_X"] \\
\ker(d_n)/\operatorname{im}(d_{n+1}) \arrow[rr, "\mathrm{id}"'] & & \mathrm{H}_n^{\mathrm{CW}}(X).
\end{tikzcd}$$

Since all the given maps are isomorphisms we can now define the right vertical map Θ_X as the unique map which makes the diagram commute. Evidently Θ_X is also an isomorphism.

Finally we note that by construction the isomorphism Θ_X has the property that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{H}_n(X^n) \arrow[r, two heads] \arrow[d, "\text{я}_n"'] & \mathrm{H}_n(X) \arrow[d, "\Theta_X"] \\
\ker(d_n) \arrow[r, two heads] & \mathrm{H}_n^{\mathrm{CW}}(X).
\end{tikzcd}$$

Note that by the Skeleton-H_*-Lemma 47.1 (4) the top horizontal map is an epimorphism. This implies immediately that Θ_X is the unique homomorphism that makes the diagram commute. ■

Proof of Proposition 47.3 (2). We need to show that the isomorphisms Θ_X that we just constructed are natural. Thus suppose that we are given a cellular map $f\colon X \to Y$

and let $n \in \mathbb{N}_0$. We consider the following diagram:

$$\begin{tikzcd}
 & & \mathrm{H}_n(X) \arrow[dr, "\Theta_X"] \arrow[dd, "f_*"] & \\
\mathrm{H}_n(X^n) \arrow[rrr, two heads] \arrow[dr, "\text{я}_n"] \arrow[dd, "f_*"] & & & \mathrm{H}_n^{\mathrm{CW}}(X) \arrow[dd, "f_*"] \\
 & \ker(d_n) \arrow[rrr, two heads] \arrow[dd, "f_*"] & & \\
 & & \mathrm{H}_n(Y) \arrow[dr, "\Theta_Y"] & \\
\mathrm{H}_n(Y^n) \arrow[rrr, two heads] \arrow[dr, "\text{я}_n"] & & & \mathrm{H}_n^{\mathrm{CW}}(Y). \\
 & \ker(d_n) \arrow[rrr, two heads] & &
\end{tikzcd}$$

We make the following observations:

(a) The sides of the hyperrectangle that do not involve Θ_X and Θ_Y commute basically by definition.
(b) The top and bottom parallelogram commute by definition of Θ_X and Θ_Y.
(c) By the Skeleton-H_*-Lemma 47.1 (4) the map $\mathrm{H}_n(X^n) \to \mathrm{H}_n(X)$ is an epimorphism.

It follows from these observation that also the parallelogram involving Θ_X and Θ_Y commutes. But that is exactly, what we had to show. ■

Proposition 47.4. ($\mathbb{C}\mathrm{P}^n$-H_*-Proposition)

(1) Given $n \in \mathbb{N}_0$ we have
$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \ldots, 2n \\ 0, & \text{otherwise.} \end{cases}$$
(2) We have
$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^\infty) \cong \begin{cases} \mathbb{Z}, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is } k \text{ is odd.} \end{cases}$$

Proof.

(1) Let $n \in \mathbb{N}_0$. On page 264 we showed that $\mathbb{C}\mathrm{P}^n$ admits a CW-structure with exactly one cell in dimensions $0, 2, \ldots, 2n$ and no other cells. It follows from the Skeleton-H_*-Lemma 47.1 (1) that the corresponding cellular chain complex is of the form
$$0 \to \underbrace{\mathrm{C}_{2n}^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n)}_{\cong\mathbb{Z}} \to \underbrace{\mathrm{C}_{2n-1}^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n)}_{=0} \to \ldots \to \underbrace{\mathrm{C}_{2}^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n)}_{\cong\mathbb{Z}} \to \underbrace{\mathrm{C}_{1}^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n)}_{=0} \to \underbrace{\mathrm{C}_{0}^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n)}_{\cong\mathbb{Z}} \to 0.$$
Since every other group is trivial it follows that all boundary maps are zero. In particular the cellular homology groups equal the cellular chain groups. It follows that
$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n) \underset{\substack{\uparrow \\ \text{by the Singular-Cellular} \\ \text{Homology Proposition 47.3}}}{\cong} \mathrm{H}_k^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n) \underset{\substack{\uparrow \\ \text{follows from the above discussion}}}{\cong} \mathrm{C}_k^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^n) \underset{\uparrow}{\cong} \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \ldots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

(2) Just for fun we provide two different ways to calculate $\mathrm{H}_k(\mathbb{C}\mathrm{P}^\infty)$:
  (a) On page 264 we saw that we can equip the infinite-dimensional complex projective space $\mathbb{C}\mathrm{P}^\infty$ with a CW-structure that has precisely one cell in each even dimension

and no cells in odd dimensions. From the same arguments as in (1) it follows that

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^\infty) \;\cong\; \mathrm{H}_k^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^\infty) \;\cong\; \mathrm{C}_k^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^\infty) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

(b) We start out with the following claim.

Claim. Let $j \geq i$. Let $\iota\colon \mathbb{C}\mathrm{P}^i \to \mathbb{C}\mathrm{P}^j$ be the usual embedding that is given by $[x] \mapsto [(x,0)]$. The induced map $\iota_*\colon \mathrm{H}_k^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^i) \to \mathrm{H}_k^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^j)$ on singular homology is an isomorphism for $k = 0, \ldots, 2i$.

Proof. Note that $\iota\colon \mathbb{C}\mathrm{P}^i \to \mathbb{C}\mathrm{P}^j$ is clearly cellular. Thus we obtain from the Singular-Cellular-H_*-Proposition 47.3 (2) the following commutative diagram:

$$\begin{tikzcd} \mathrm{H}_k(\mathbb{C}\mathrm{P}^i) \arrow[r, "\iota_*"] \arrow[d, "\Theta_{\mathbb{C}\mathrm{P}^i}"', "\cong"] & \mathrm{H}_k^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^j) \arrow[d, "\cong"', "\Theta_{\mathbb{C}\mathrm{P}^j}"] \\ \mathrm{H}_k^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^i) \arrow[r, "\iota_*"] & \mathrm{H}_k^{\mathrm{CW}}(\mathbb{C}\mathrm{P}^j) \end{tikzcd} .$$

Next note that it follows from inspecting the cellular chain complexes that for any $k \leq 2i$ the bottom horizontal map is an isomorphism. The claim follows immediately from these two observations. ⊞

It follows from the claim, the Exhaustions-H_*-Proposition 46.2 and the Direct Limit–Stabilization Lemma 25.1 that for any $k \leq 2i$ the inclusion induced map $\mathrm{H}_k(\mathbb{C}\mathrm{P}^i) \to \mathrm{H}_k(\mathbb{C}\mathrm{P}^\infty)$ is an isomorphism. Together with (1) we thus get the promised result. ■

47.4. The cellular boundary maps.

Definition. Let $n \in \mathbb{N}$.

(1) We denote by $[S^{n-1}] \in \widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ the standard generator of $\widetilde{\mathrm{H}}_n(S^{n-1})$, as defined on page 362.

(2) The standard generator $[\overline{B}^n]$ of $\mathrm{H}_n(\overline{B}^n, S^{n-1})$ is defined to be the unique element with $\partial_n([\overline{B}^n]) = [S^{n-1}] \in \widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ where $\partial_n\colon \mathrm{H}_n(\overline{B}^n, S^{n-1}) \to \widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ is the connecting homomorphism of the long exact sequence in reduced homology of the pair $(\overline{B}^n, S^{n-1})$.

Example. Recall that on page 362 we saw that the standard generator of $\widetilde{\mathrm{H}}_0(S^0)$ is given by $[S^0] = [(+1) - (-1)] \in \widetilde{\mathrm{H}}_0(S^0)$. It follows easily from this observation, the explicit description of the connecting homomorphism in the LES-of-Triple Proposition 38.2, and the explicit description of the boundary of a singular 1-simplex on page 302, that the standard generator of $\mathrm{H}_1(\overline{B}^1, S^0)$ is represented by the following cycle in $\mathrm{C}_1(\overline{B}^1, S^0)$:

$$\begin{array}{rcl} \Delta^1 & \to & \overline{B}^1 \\ (1-t, t) & \mapsto & 2t - 1. \end{array}$$

Definition.

(1) A charted CW-complex is a pair consisting of a CW-complex X together with a full set $\{\Phi_c\colon \overline{B}^{\dim(c)} \to X^{\dim(c)}\}_{c \in \mathrm{Cell}(X)}$ of characteristic maps of the cells of X.

(2) Let X be a charted CW-complex as in (1). For an n-cell α we write

$$\Xi_\alpha \;:=\; \Phi_{\alpha*}([\overline{B}^n]) \;\in\; \mathrm{H}_n(X^n, X^{n-1}) \;=\; \mathrm{C}_n^{\mathrm{CW}}(X).$$

With this notation the Skeleton-H_*-Lemma 47.1 (1) says that $\{\Xi_c\}_{c\in\mathrm{Cell}(X)}$ is a basis of the free abelian group $\mathrm{C}_n^{\mathrm{CW}}(X)$.

Lemma 47.5. (Cellular-First Boundary Map Lemma) Let X be a charted CW-complex. For a 1-cell α with characteristic map $\Phi_\alpha\colon \overline{B}^1 \to X^1$ we have

$$d_1(\Xi_\alpha) \;=\; \Xi_{\Phi_\alpha(1)} - \Xi_{\Phi_\alpha(-1)} \quad\in\; \mathrm{C}_0^{\mathrm{CW}}(X).$$

Proof. Recall that the cellular boundary map d_n is defined as follows:

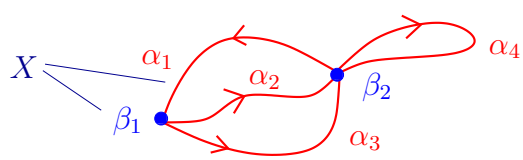
$$\begin{tikzcd}
\mathrm{H}_n(X^n, X^{n-1}) \arrow[rr, "=:d_n"] \arrow[dr, "\partial_n"'] & & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}). \\
& \mathrm{H}_{n-1}(X^{n-1}) \arrow[ur, "\text{я}"'] &
\end{tikzcd}$$

In our special case $n = 1$ we see that $X^{n-2} = \varnothing$, thus d_1 equals the connecting homomorphism ∂_1 of the pair (X^1, X^0) as defined in the LES-of-Pair Proposition 38.3. Now we see that

$$\begin{aligned}
d_1(\Xi_\alpha) &\overset{\text{by definition of } \Xi_\alpha \in \mathrm{C}_1^{\mathrm{CW}}(X)}{=} d_1(\Phi_{\alpha*}([\overline{B}^1])) \overset{\text{by the above discussion}}{=} \partial_1(\Phi_{\alpha*}([\overline{B}^1])) \overset{\text{see the description of } [\overline{B}^1] \text{ on page 402}}{=} \partial_1([(t, 1-t) \mapsto \Phi_\alpha(2t-1)]) \\
&\underset{\substack{\text{see the definition of the connecting homomorphism } \partial_1 \text{ in} \\ \text{the LES-of-Pair Proposition 38.3 and the discussion of } \partial_1 \text{ on page 302}}}{=} \Xi_{\Phi_\alpha(1)} - \Xi_{\Phi_\alpha(-1)}.
\end{aligned}$$

■

Example. We consider the charted 1-dimensional CW-complex X with two 0-cells β_1, β_2 and four 1-cells $\alpha_1, \ldots, \alpha_4$. that is given in the figure below:



By the Cellular-First Boundary Map Lemma 47.5 the cellular chain complex of X is of the form

$$0 \;\to\; \underbrace{\begin{matrix}\mathbb{Z}\cdot\Xi_{\alpha_1}\\ \mathbb{Z}\cdot\Xi_{\alpha_2}\\ \mathbb{Z}\cdot\Xi_{\alpha_3}\\ \mathbb{Z}\cdot\Xi_{\alpha_4}\end{matrix}}_{=\mathrm{C}_1^{\mathrm{CW}}(X)} \xrightarrow{\begin{matrix} & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & 1 & -1 & -1 & 0 \\ \beta_2 & -1 & 1 & 1 & 0 \end{matrix}} \underbrace{\begin{matrix}\mathbb{Z}\cdot\Xi_{\beta_1}\\ \mathbb{Z}\cdot\Xi_{\beta_2}\end{matrix}}_{=\mathrm{C}_0^{\mathrm{CW}}(X)} \;\to\; 0.$$

It follows that

$$\mathrm{H}_n(X) \;\underset{\substack{\uparrow \\ \text{by the Singular-Cellular-H}_*\text{-} \\ \text{Proposition 47.3}}}{\cong}\; \mathrm{H}_n^{\mathrm{CW}}(X) \;\underset{\substack{\uparrow \\ \text{by the above calculation} \\ \text{and elementary linear algebra}}}{=}\; \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z}^3, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Proposition 47.6. (Cellular Boundary–Degree Proposition) Let X be a charted CW-complex and let $n \in \mathbb{N}$. Given an n-cell α and an $(n-1)$-cell β we consider the following continuous self-map of S^{n-1}:

$$\begin{tikzcd} S^{n-1} \arrow[r, equal] & S^{n-1} \arrow[r, "\Phi_\alpha|_{S^{n-1}}"] & X^{n-1} \arrow[r, "q"] & X^{n-1}/X^{n-2} & \\ & & & \bigvee_{j\in\mathrm{Cell}_{n-1}(X)} \overline{B}^{n-1}/S^{n-2} \arrow[u, "\cong"', "\bigvee_{j\in\mathrm{Cell}_{n-1}(X)}\overline{\Phi}_j"] \arrow[r, "p_\beta"] & \overline{B}^{n-1}/S^{n-2} \arrow[r, "g", "\cong"'] & S^{n-1} \end{tikzcd}$$

where we adopt the following notation:

- For $c \in \mathrm{Cell}(X)$ we denote by Φ_c the corresponding characteristic map.
- $q\colon X^{n-1} \to X^{n-1}/X^{n-2}$ is the natural projection.
- Let $j \in \mathrm{Cell}_{n-1}(X)$. Since $\Phi_j(S^{n-2}) \subset X^{n-2}$ we see that the characteristic map $\Phi_j\colon \overline{B}^{n-1} \to X^{n-1}$ descends to a continuous map $\overline{\Phi}_j\colon \overline{B}^{n-1}/S^{n-2} \to X^{n-1}/X^{n-2}$.
- $\bigvee_{j\in\mathrm{Cell}_{n-1}(X)} \overline{\Phi}_j\colon \bigvee_{j\in\mathrm{Cell}_{n-1}(X)} \overline{B}^{n-1}/S^{n-2} \to X^{n-1}/X^{n-2}$ is the natural map provided by the Wedge Proposition 17.1. Note that it follows from the Skeleton-Quotient Lemma 33.5 that this map is a homeomorphism.
- $p_\beta\colon \bigvee_{j\in\mathrm{Cell}_n(X)} \overline{B}^{n-1}/S^{n-2} \to \overline{B}^{n-1}/S^{n-2}$ is the natural map that is induced by the identity on the β-copy of $\overline{B}^{n-1}/S^{n-2}$ and the constant map on all other copies of $\overline{B}^{n-1}/S^{n-2}$.
- $g\colon \overline{B}^{n-1}/S^{n-2} \xrightarrow{\cong} S^{n-1}$ denotes the homeomorphism from the Ball Quotient-Sphere Lemma 6.5.

For any n-cell α we have

$$d_n(\Xi_\alpha) \;=\; \sum_{\beta\in\mathrm{Cell}_{n-1}(X)} \deg(\Omega_{\beta,\alpha}) \cdot \Xi_\beta \;\in\; \mathrm{C}^{\mathrm{CW}}_{n-1}(X)$$

where

$$\Omega_{\beta,\alpha} \;:=\; g \circ p_\beta \circ \big(\bigvee_{j\in\mathrm{Cell}_{n-1}(X)} \overline{\Phi}_j\big)^{-1} \circ q \circ \Phi_\alpha|_{S^{n-1}}\colon S^{n-1} \;\to\; S^{n-1}.$$

Lemma 47.7. (Standard Generator-of-$\overline{B}^n$-Lemma) Let $n \in \mathbb{N}_0$. Furthermore let $ю\colon \mathrm{H}_n(\overline{B}^n, S^{n-1}) \to \widetilde{\mathrm{H}}_n(\overline{B}^n/S^{n-1})$ be the natural map and let $g\colon \overline{B}^n/S^{n-1} \to S^n$ be the homeomorphism from the Ball Quotient-Sphere Lemma 6.5. Then

$$(g_* \circ ю)([\overline{B}^n]) \;=\; [S^n] \quad \in \widetilde{\mathrm{H}}_n(S^n).$$

Proof of the Standard Generator-of-$\overline{B}^n$-Lemma 47.7. First note that it follows from the Relative-to-Quotient-H_*-Proposition refprop:homology-of-x-mod-a that the map $g_* \circ ю\colon \mathrm{H}_n(\overline{B}^n, S^{n-1}) \to \widetilde{\mathrm{H}}_n(S^n)$ is an isomorphism. Since $[\overline{B}^n] \in \mathrm{H}_n(\overline{B}^n, S^{n-1})$ and $[S^n] \in \widetilde{\mathrm{H}}_n(S^n)$ are generators we thus see that $(g_* \circ ю)([\overline{B}^n]) = \pm[S^n] \in \widetilde{\mathrm{H}}_n(S^n)$. Therefore it remains to determine the precise sign. This is a little tricky and we outsource the proof to [Fri23]. For most purposes the precise sign is not really relevant though. For example in the statement of the Cellular Boundary–Degree Proposition 47.6 a different sign in the equality $(g_* \circ ю)([\overline{B}^n]) = \pm[S^n] \in \widetilde{\mathrm{H}}_n(S^n)$ changes at worst all $\deg(\Omega_{\beta,\alpha})$ by the *same* sign, which has no effect later on when we need to determine (co-) kernels. ■

Proof of the Cellular Boundary–Degree Proposition 47.6. We adopt the following notation:

(1) We write $I := \mathrm{Cell}_n(X)$ and $J := \mathrm{Cell}_{n-1}(X)$.
(2) Given $k \in J$ we denote by $\iota_k \colon \overline{B}^{n-1}/S^{n-2} \to \bigvee_{j\in J} \overline{B}^{n-1}/S^{n-2}$ the natural injection.

For any n-cell α and any $(n-1)$-cell β we consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}_n(\overline{B}^n, S^{n-1}) \arrow[rrr, "\Phi_{\alpha *}", "{[\overline{B}^n] \mapsto \Xi_\alpha}"'] \arrow[d, "\cong"', "\partial_n \;\; [\overline{B}^n] \mapsto [S^{n-1}]"] & & & \mathrm{H}_n(X^n, X^{n-1}) \arrow[d, "\partial_n"] \arrow[ddd, bend left, "d_n"] \\
\widetilde{\mathrm{H}}_{n-1}(S^{n-1}) \arrow[rrr, "{(\Phi_\alpha|_{S^{n-1}})_*}"] \arrow[d, "\cdot \det(\Omega_{\alpha\beta})"] & & & \widetilde{\mathrm{H}}_{n-1}(X^{n-1}) \arrow[ddd, "\text{я}_{n-1}"] \arrow[dddl, "q_*"] \\
\widetilde{\mathrm{H}}_{n-1}(S^{n-1}) & \widetilde{\mathrm{H}}_{n-1}(\overline{B}^{n-1}/S^{n-2}) \arrow[l, "g_*"'] & & \\
\widetilde{\mathrm{H}}_{n-1}(\overline{B}^n/S^{n-1}) \arrow[u, "g_*"] & & \widetilde{\mathrm{H}}_{n-1}\Big(\bigvee_{j\in J} \overline{B}^{n-1}/S^{n-2}\Big) \arrow[ul, "p_{\beta *}"'] \arrow[dr, "\cong", "\bigoplus \iota_{j*}"'] \arrow[dr, "{\left(\bigvee_{j\in J} \overline{\Phi}_j\right)_*}"] & & \\
\mathrm{H}_{n-1}(\overline{B}^{n-1}, S^{n-2}) \arrow[u, "\text{ю}"] & \bigoplus_{j\in J} \widetilde{\mathrm{H}}_{n-1}(\overline{B}^{n-1}/S^{n-2}) \arrow[uu, two heads, "\text{projection to the } \beta\text{-th summand}"] & & \widetilde{\mathrm{H}}_{n-1}(X^{n-1}/X^{n-2}) \\
\bigoplus_{j\in J} \mathrm{H}_{n-1}(\overline{B}^{n-1}, S^{n-2}) \arrow[u, two heads, "\text{projection to the } \beta\text{-th summand}"] \arrow[ur, "\cong", "\mathrm{diag}(\text{ю})"'] \arrow[rrr, "\bigoplus_{j\in J} \Phi_{j*}", "{[\overline{B}^{n-1}]_j \mapsto \Xi_j}"'] & & & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}). \arrow[ul, "\text{ю}", "\cong"'] \arrow[uuu]
\end{tikzcd}$$

We make the following claim.

Claim. The diagram commutes and all the maps decorated with "$\cong$" are indeed isomorphisms.

Proof. We consider the various parts of the diagram.

(1) It follows from the naturality of the connecting homomorphism, see the Reduced LES-of-Pair Proposition 39.6 (2), that the upper rectangle in the diagram commutes.
(2) The red arrows trace out the map $\Omega_{\beta,\alpha}$ of which we take the degree. In other words, the oddly shaped hexagon in the center of the diagram commutes by definition.
(3) We consider the hexagon to the lower left of the diagram. Using the functoriality of relative homology one can easily verify that the hexagon commutes on each summand, hence it commutes for the direct sum.
(4) (a) Note that the map $p_\beta \circ \iota_j$ equals $\mathrm{id}_{\overline{B}^{n-1}/S^{n-2}}$ if $j = \beta$ and that it is constant otherwise. It follows easily from this observation that the triangle towards the center left, involving the map $p_{\beta *}$ commutes.
    (b) It follows from the Wedge-H_*-Proposition 41.10 that the map decorated with "$\cong$" in the triangle is indeed an isomorphism.
(5) In the proof of the Skeleton-H_*-Lemma 47.1 (1) we showed that the large triangle at the bottom commutes and we showed that all the maps involved in the large triangle are indeed isomorphisms.

(6) (a) It follows easily from the definitions that the triangle to the bottom right commutes.
(b) It follows from the Relative-to-Quotient-H_*-Proposition 41.8 that the diagonal map at the very bottom right is an isomorphism. ⊞

Next note that by the definition of the standard generator of $\mathrm{H}_n(\overline{B}^n, S^{n-1})$ on page 402 we have $\partial_n([\overline{B}^n]) = [S^{n-1}] \in \widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ and note that the Standard Generator-of-$\overline{B}^n$-Lemma 47.7 says that $g_*(\text{ю}([\overline{B}^{n-1}])) = [S^{n-1}] \in \widetilde{\mathrm{H}}_{n-1}(S^{n-1})$. The reader who is still awake will notice that the proposition follows immediately from the claim and this observation. ■

47.5. The homology groups of surfaces.

Notation. Given $g \in \mathbb{N}_0$ and given $k \in \mathbb{N}$ we write as usual

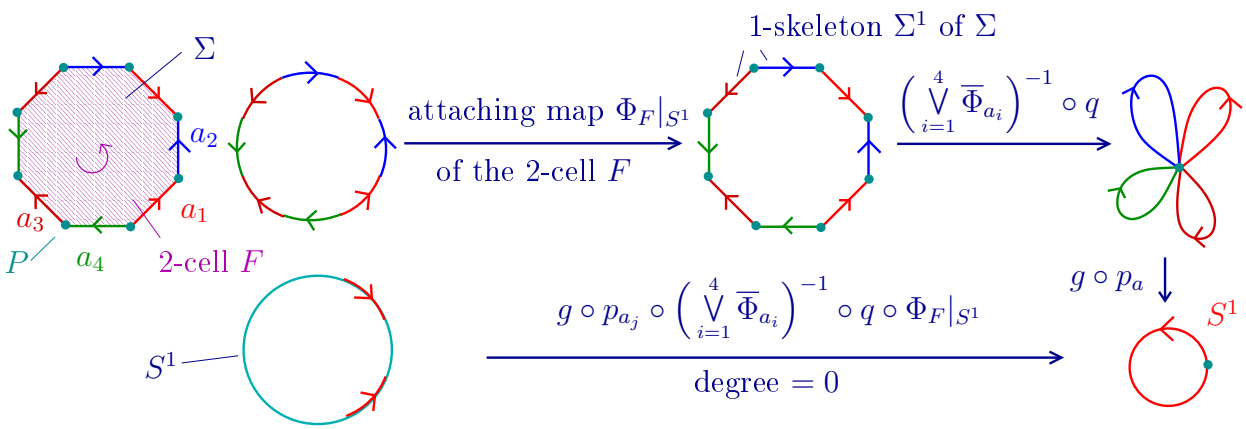
$$\begin{aligned} \Sigma_g &:= \text{ surface of genus } g \\ N_k &:= \text{ non-orientable surface of genus } k. \end{aligned}$$

Furthermore given $m \in \mathbb{N}_0$ we write

$$\begin{aligned} \Sigma_{g,m} &:= \text{ the surface of genus } g \text{ minus } m \text{ open disks} \\ N_{k,m} &:= \text{ the non-orientable surface of genus } k \text{ minus } m \text{ open disks.} \end{aligned}$$

Example.

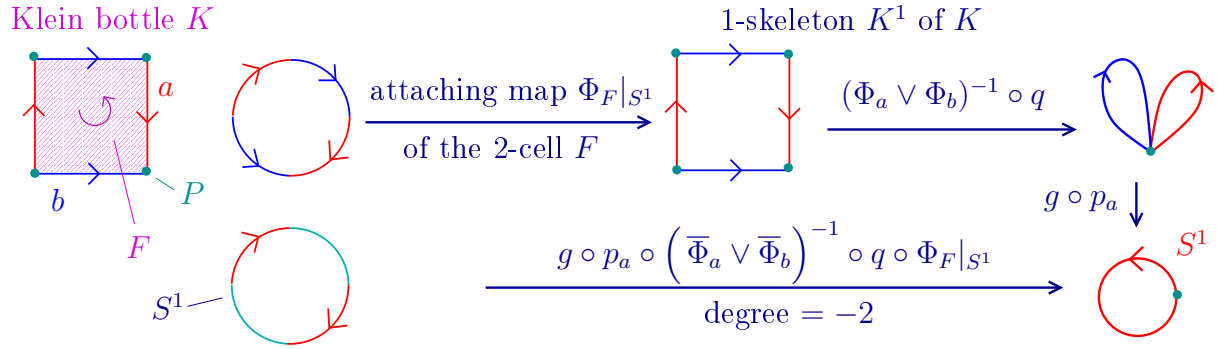
(1) We consider the surface Σ_2 of genus 2. As usual we view Σ_2 as a charted CW-complex with one 0-cell P, four 1-cells a_1, a_2, a_3, a_4 and one 2-cell F.



Note that it follows easily from the definitions, or alternatively from the Degree-via-Local Orientations Proposition 44.7, that for each $j \in \{1, \dots, 4\}$ the degree of the map $\Omega_{F,a_j} = g \circ p_a \circ \left(\bigvee_{i=1}^4 \overline{\Phi}_{a_i}\right)^{-1} \circ q \circ \Phi_F|_{S^1} \colon S^1 \to S^1$ is zero. It follows from the Cellular Boundary–Degree Proposition 47.6 that the cellular boundary map d_2 in the cellular chain complex of Σ_2 is zero. Since Σ_2 has only one 0-cell we know from the Cellular-First Boundary Map Lemma 47.5 that the cellular boundary map d_1 is also zero. Thus we see that

$$\mathrm{H}_n(\Sigma_2) \underset{\substack{\uparrow \\ \text{by the Singular-Cellular-H}_*\text{-} \\ \text{Proposition 47.3}}}{\cong} \mathrm{H}_n^{\mathrm{CW}}(\Sigma_2) \underset{\substack{\uparrow \\ \text{since all the cellular} \\ \text{boundary maps are zero}}}{=} \mathrm{C}_n^{\mathrm{CW}}(\Sigma_2) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ \mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z}^4, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

(2) As a reality check we consider once again the Klein bottle $K = N_2$. We view K as a charted CW-complex with one 0-cell P, two 1-cells a, b and one 2-cell F, as shown in the figure below.



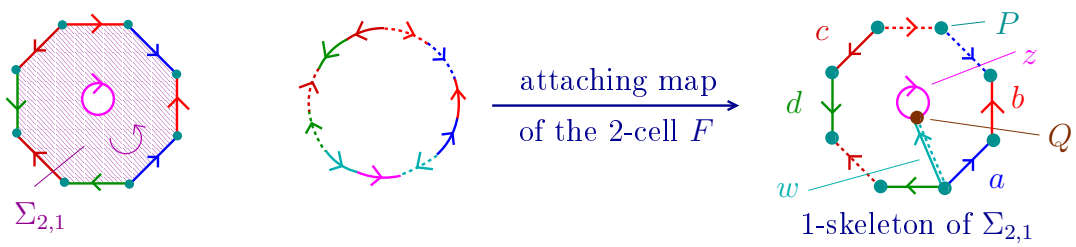
It follows from the Cellular Boundary–Degree Proposition 47.6, the Degree-via-Local Orientations Proposition 44.7 and the Cellular-First Boundary Map Lemma 47.5 that the cellular chain complex is given by

$$0 \to \mathbb{Z} \cdot \Xi_F \xrightarrow{\begin{pmatrix} -2 \\ 0 \end{pmatrix}} \begin{matrix} \mathbb{Z} \cdot \Xi_a \\ \mathbb{Z} \cdot \Xi_b \end{matrix} \xrightarrow{\begin{pmatrix} 0 & 0 \end{pmatrix}} \mathbb{Z} \cdot \Xi_P \to 0.$$

Therefore we see that

$$\mathrm{H}_n(K) \underset{\substack{\uparrow \\ \text{by the Singular-Cellular-H}_*\text{-} \\ \text{Proposition 47.3}}}{\cong} \mathrm{H}_n^{\mathrm{CW}}(K) \underset{\substack{\uparrow \\ \text{by the above calculation} \\ \text{and elementary linear algebra}}}{=} \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z} \oplus \mathbb{Z}_2, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

(3) We consider $\Sigma_{2,1}$ and we equip it with the charted CW-structure that is illustrated in the figure below.



This CW-structure has two 0-cells P, Q, six 1-cells a, b, c, d, w, z and one 2-cell F. Using the Cellular Boundary–Degree Proposition 47.6 and using the Cellular-First Boundary 47.5 one can easily show that the cellular chain complex is of the following form:

$$0 \to \mathbb{Z} \cdot \Xi_F \xrightarrow{\begin{array}{c} \\ a \\ b \\ c \\ d \\ w \\ z \end{array} \begin{array}{c} F \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{array}} \begin{matrix} \mathbb{Z} \cdot \Xi_a \\ \mathbb{Z} \cdot \Xi_b \\ \mathbb{Z} \cdot \Xi_c \\ \mathbb{Z} \cdot \Xi_d \\ \mathbb{Z} \cdot \Xi_w \\ \mathbb{Z} \cdot \Xi_z \end{matrix} \xrightarrow{\begin{array}{c} \\ P \\ Q \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & w & z \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}} \begin{matrix} \mathbb{Z} \cdot \Xi_P \\ \mathbb{Z} \cdot \Xi_Q \end{matrix} \to 0.$$

Now we see that

$$\mathrm{H}_n(\Sigma_{2,1}) \quad \underset{\substack{\uparrow \\ \text{by the Singular-Cellular-H}_*\text{-} \\ \text{Proposition 47.3}}}{\cong} \quad \mathrm{H}_n^{\mathrm{CW}}(\Sigma_{2,1}) \quad \underset{\substack{\uparrow \\ \text{by the above chain complex and} \\ \text{some elementary linear algebra}}}{\cong} \quad \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z}^4, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Proposition 47.8. (Surface-H$_*$-Proposition) For any $g \in \mathbb{N}_0$ we have

$$\mathrm{H}_n(\Sigma_g) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ \mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z}^{2g}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases} \quad \text{Furthermore for } m \geq 1 \text{ we have} \quad \mathrm{H}_n(\Sigma_{g,m}) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}^{2g+m-1}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

For any $k \in \mathbb{N}$ we have

$$\mathrm{H}_n(N_k) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}^{k-1} \oplus \mathbb{Z}_2, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases} \quad \text{Furthermore for } m \geq 1 \text{ we have} \quad \mathrm{H}_n(N_{k,m}) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}^{k+m-1}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Proof. The calculation of the homology groups Σ_g is a slight generalization of the arguments that we provided on page 406. Furthermore the case N_k is dealt with the same way as we dealt with the Klein bottle $K = N_2$ on page 407. Finally note that for $m \in \mathbb{N}$ the calculation of the homology groups of $\Sigma_{g,m}$ and $N_{k,m}$ is a slight generalization of the argument preceding the proposition. We leave the details to the reader. ■

47.6. The homology groups of real projective spaces.

Proposition 47.9. ($\mathbb{R}\mathrm{P}^n$-H$_*$-Proposition)

(1) For any $n \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}_2, & \text{if } k \text{ is odd and } k < n, \\ 0, & \text{if } k \text{ is even and } 0 < k \neq 0, \\ 0, & \text{if } k > n, \\ \mathbb{Z}, & \text{if } k = n \text{ and } n \text{ is odd.} \end{cases}$$

(2) We have

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^\infty) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}_2, & \text{if } k \text{ is odd,} \\ 0, & \text{if } k \text{ is even and } k > 0. \end{cases}$$

Proof.

(1) Let $n \in \mathbb{N}_0$. On page **??** we showed that for any $k \leq n$ the map $\mathbb{R}\mathrm{P}^k \to \mathbb{R}\mathrm{P}^n$ that is given by $[x] \mapsto [(x, 0)]$ is an embedding. For the duration of the proof we use this embedding as an excuse to view $\mathbb{R}\mathrm{P}^k$ as a subspace of $\mathbb{R}\mathrm{P}^n$.

Next recall we recall the CW-structure on $\mathbb{R}\mathrm{P}^n$ which we introduced on page 263: For each $k = 0, \ldots, n$ we define

$$\begin{aligned} \Phi_k \colon \overline{B}^k &\to \mathbb{R}\mathrm{P}^k \\ x &\mapsto [(x, \sqrt{1 - \|x\|^2})]. \end{aligned}$$

We saw in the proof of the Projective Space–Cell Attachment Lemma 29.2 that the map

$$\begin{array}{rcl} \mathbb{R}\mathrm{P}^{k-1} \cup_{\mathbb{R}\mathrm{P}^{k-1} \xleftarrow{\Phi_k|_{S^{k-1}}} S^{k-1}} \overline{B}^k & \xrightarrow{\cong} & \mathbb{R}\mathrm{P}^k \\ [x] & \mapsto & \begin{cases} [(x,0)], & \text{if } x \in \mathbb{R}\mathrm{P}^{k-1} \\ \Phi_k(x), & \text{if } x \in \overline{B}^k \end{cases} \end{array}$$

is a homeomorphism. Thus we see inductively that we can view $\mathbb{R}\mathrm{P}^n$ as a charted CW-complex with a single cell in the dimensions $0, \dots, n$ where the characteristic maps are given precisely by the maps $\Phi_0, \dots, \Phi_n$.

Claim. For each $k = 1, \dots, n$ the cellular boundary map

$$d_k \colon \mathbb{Z} \cong \mathrm{C}_k^{\mathrm{CW}}(\mathbb{R}\mathrm{P}^n) \ \to \ \mathrm{C}_{k-1}^{\mathrm{CW}}(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}$$

is given by multiplication by $\pm(1 + (-1)^k)$.

Proof. Let $k \in \{1, \dots, n\}$. We denote by $q \colon \mathbb{R}\mathrm{P}^{k-1} \to \mathbb{R}\mathrm{P}^{k-1}/\mathbb{R}\mathrm{P}^{k-2}$ the projection and we denote by $g \colon \overline{B}^{k-1}/S^{k-2} \to S^{k-1}$ the homeomorphism from the Ball Quotient-Sphere Lemma 6.5. By the Cellular Boundary–Degree Proposition 47.6 it suffices to show that the degree of the map

$$\begin{tikzcd} S^{k-1} \arrow[r, "\Phi_k|_{S^{k-1}}"] \arrow[rrr, bend right, "=:\Omega"'] & \mathbb{R}\mathrm{P}^{k-1} \arrow[r, "q"] & \mathbb{R}\mathrm{P}^{k-1}/\mathbb{R}\mathrm{P}^{k-2} \arrow[r, "(\overline{\Phi}_{k-1})^{-1}"] & \overline{B}^{k-1}/S^{k-2} \arrow[r, "g"] & S^{k-1} \end{tikzcd}$$

equals $\pm(1 + (-1)^k)$. We will do so using the Degree-via-Local Orientations Proposition 44.7. To apply this proposition we recall a few definitions and we make a few preparations:

(a) If $\Omega \colon S^{k-1} \to S^{k-1}$ is a local diffeomorphism at a point x, then as on page 369 we write

$$\deg(\Omega, x) \ := \ \begin{cases} +1, & \text{if } \Omega \text{ is orientation-preserving at } x \\ -1, & \text{if } \Omega \text{ is orientation-reversing at } x. \end{cases}$$

(b) If $f \colon S^{k-1} \to S^{k-1}$ is a diffeomorphism, then we write

$$\deg(f) \ := \ \begin{cases} +1, & \text{if } f \text{ is orientation-preserving} \\ -1, & \text{if } f \text{ is orientation-reversing.} \end{cases}$$

(c) If $\Omega \colon S^{k-1} \to S^{k-1}$ is a local diffeomorphism at a point x and if $f \colon S^{k-1} \to S^{k-1}$ is a diffeomorphism, then it follows immediately from $\mathrm{D}(\Omega \circ f)_{f^{-1}(x)} = \mathrm{D}\Omega_x \circ \mathrm{D}\Omega_{f^{-1}(x)}$ that $\deg(\Omega \circ f, f^{-1}(x)) = \deg(\Omega, x) \cdot \deg(f)$.

(d) We pick the point $y = (0, \dots, 0, 1) \in S^{k-1}$. The preimages of y under the map Ω are $x_+ = (0, \dots, 0, 1)$ and $x_- = (0, \dots, 0, -1)$. One can easily verify that the restrictions of Ω to the open hemispheres $B^n_{>0}$ and $B^n_{<0}$ are smooth embeddings. Since $x_+ \in B^n_{>0}$ and $x_- \in B^n_{<0}$ we see that Ω is a local diffeomorphism around x_+ and x_-.

(e) Let $\rho \colon S^{k-1} \to S^{k-1}$ be the reflection in the origin, i.e. ρ is given by $z \mapsto -z$.

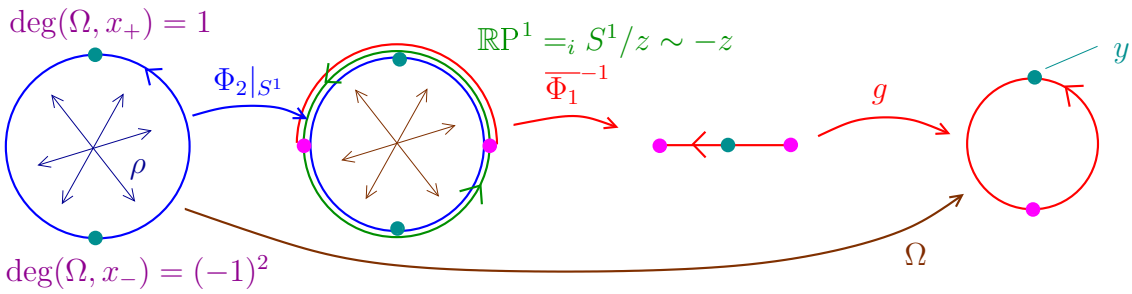
(f) One can easily show that $\deg(\rho) = \det(-\operatorname{id}_k) = (-1)^k$.

(g) Note that $(\Phi_k|_{S^{k-1}}) \circ \rho = \Phi_k|_{S^{k-1}} \colon S^{k-1} \to \mathbb{R}\mathrm{P}^{k-1}$. This implies that $\Omega \circ \rho = \Omega$.

We now see that

$$\begin{aligned}
\deg(\Omega) &\overset{\substack{\text{by the Degree-via-Local Orientations}\\ \text{Proposition 44.7 and by (d)}}}{=} \deg(\Omega, x_+) + \deg(\Omega, x_-) &&\overset{\text{since by (g) we know that } \Omega\circ\rho=\Omega}{=} \deg(\Omega, x_+) + \deg(\Omega\circ\rho, x_-)\\
&\underset{\substack{\uparrow\\ \text{follows from (c), since } \rho(x_+)=x_-}}{=} \deg(\Omega, x_+) + \deg(\Omega, x_+)\cdot\deg(\rho) &&= \underbrace{\deg(\Omega, x_+)}_{=\pm 1}\cdot(1+\underbrace{\deg(\rho)}_{=(-1)^k \text{ by (f)}}).
\end{aligned}$$

⊞



It follows from the claim that the cellular chain complex of $\mathbb{R}\mathrm{P}^n$ is of the form

$$0 \to \underset{\substack{\uparrow\\ \text{degree } n}}{\mathbb{Z}} \xrightarrow{\pm(1+(-1)^n)} \mathbb{Z} \to \dots \xrightarrow{\cdot(\pm 2)} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot(\pm 2)} \mathbb{Z} \xrightarrow{0} \underset{\substack{\uparrow\\ \text{degree } 0}}{\mathbb{Z}} \to 0.$$

It is now straightforward to determine the cellular homology groups which, by the Singular-Cellular-H_*-Proposition 47.3, are isomorphic to the singular homology groups of $\mathbb{R}\mathrm{P}^n$.

(2) On page 264 we equipped $\mathbb{R}\mathrm{P}^\infty = \varinjlim \mathbb{R}\mathrm{P}^n$ with a CW-structure with precisely one cell in each dimension. Basically the same calculation as in (1) gives the desired result. ■

48. The Euler characteristic

48.1. The Euler characteristic of a CW-complex.

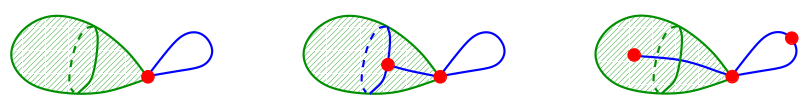
Definition. Given a finite CW-complex X we define its Euler characteristic $\chi(X)$ as follows:

$$\chi(X) \;:=\; \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \text{number of } n\text{-cells}.$$

Examples.

(1) If G is a finite topological graph, then, according to the Graph–CW-Complex Lemma 30.1 we can view G as a 1-dimensional CW-complex where the number of 0-cells equals the number of vertices and the number of 1-cells equals the number of edges of G. Thus it follows immediately from the definitions that in this case the above Euler characteristic agrees with the definition of the Euler characteristic of a finite topological graph that we gave on page 100.

(2) In the figure below we show the topological space $S^2 \vee S^1$ with three different CW-structures. We notice that even though the numbers of cells in the various dimensions differ, the Euler characteristic is always the same.



number of 0-cells	1	2	3
number of 1-cells	1	3	3
number of 2-cells	1	2	1
Euler characteristic	1	1	1

Proposition 48.1. (Homology-χ-Proposition) Given any finite CW-complex X we have

$$\chi(X) \;=\; \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \operatorname{rank}(\mathrm{H}_n(X)).$$

Proof. We will provide the proof of Homology-χ-Proposition 48.1 in the next section. ■

Corollary 48.2. (Homotopy Type-χ-Corollary) Let X and Y be finite CW-complexes. If X and Y are homotopy equivalent, then $\chi(X) = \chi(Y)$.

Proof of the Homotopy Type-χ-Corollary 48.2. The corollary is an immediate consequence of the Homology-χ-Proposition 48.1 together with the Homotopy Equivalence-H_*-Corollary 36.7 (2). ■

Examples.

(1) Let $n \in \mathbb{N}_0$. On page 261 we showed that the n-dimensional sphere S^n has a CW-structure of S^n with one 0-cell and one extra cell in dimension n. It follows that

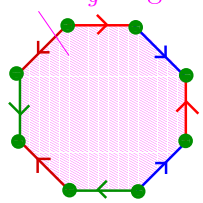
$$\chi(S^n) \;=\; 1 + (-1)^n \;=\; \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

(2) Given $g \in \mathbb{N}_0$ we consider the surface Σ_g of genus g and given $k \in \mathbb{N}$ we consider the non-orientable surface N_k of genus k. On page 412 we equipped Σ_g and N_k with

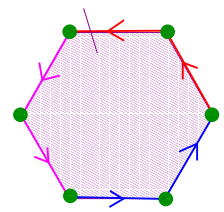
CW-structures that are also shown (for $g = 2$ and $k = 3$) in the figure below. One can now easily calculate that

$$\chi(\Sigma_g) \;=\; 2 - 2g \qquad \text{and} \qquad \chi(N_k) \;=\; 2 - k.$$

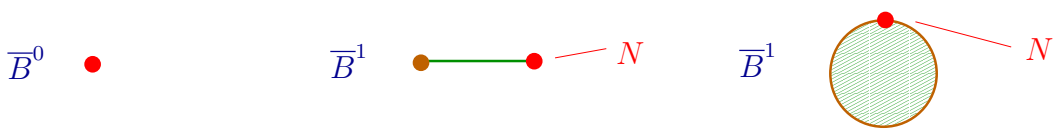
surface Σ_g of genus $g{=}2$ — non-orientable surface N_k of genus $k{=}3$



1	number of 0-cells	1
$2 \cdot g$	number of 1-cells	k
1	number of 2-cells	1
$2{-}2{\cdot}g$	Euler characteristic	$2{-}k$



(3) Let $n \in \mathbb{N}_0$. For $n = 0$ we have $\overline{B}^0 = \{0\}$, thus $\chi(\overline{B}^0) = 1$. For $n \geq 1$ a CW-structure for the ball $\overline{B}^n$ is given by the filtration $\varnothing \subset \{N\} \subset S^{n-1} \subset \overline{B}^n$, thus we have a CW-structure with one 0-cell $\{N\}$, one extra cell in dimension $n-1$ and one extra cell of dimension n. Thus again we obtain that $\chi(\overline{B}^n) = 1 + (-1)^{n-1} + (-1)^n = 1$.



(4) On page 263 we showed that $\mathbb{R}\mathrm{P}^n$ admits a CW-structure with precisely one cell of dimensions $0, 1, \ldots, n$ and no other cell. It follows that

$$\chi(\mathbb{R}\mathrm{P}^n) \;=\; \sum_{i=0}^{n} (-1)^n \;=\; \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

(5) On page 264 we showed that $\mathbb{C}\mathrm{P}^n$ admits a CW-structure with precisely one cell of dimensions $0, 2, \ldots, 2n$ and no other cell. It follows that $\chi(\mathbb{C}\mathrm{P}^n) = n + 1$.

48.2. Proof of the Homology-χ-Proposition 48.1.

Lemma 48.3. (Chain Complex–H_*–χ-Lemma) Given any chain complex

$$0 \;\to\; C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \ldots\; C_1 \xrightarrow{\partial_1} C_0 \;\to\; 0$$

of finitely generated abelian groups the following equality holds:

$$\sum_{n=0}^{k} (-1)^n \cdot \mathrm{rank}(C_n) \;=\; \sum_{n=0}^{k} (-1)^n \cdot \mathrm{rank}(\mathrm{H}_n(C_*)).$$

Proof. For the above chain complex (C_*, ∂_*) we write for each $n \in \mathbb{N}_0$

$$Z_n \;:=\; \ker(\partial_n), \quad B_n \;:=\; \mathrm{im}(\partial_{n+1}) \quad \text{and} \quad H_n \;=\; Z_n / B_n.$$

Note that these groups form the following two types of short exact sequences

$$\begin{array}{lccccccccc} \text{(a)} & 0 & \hookrightarrow & Z_n & \to & C_n & \xrightarrow{\partial_n} & B_{n-1} & \to & 0, \\ \text{(b)} & 0 & \hookrightarrow & B_n & \to & Z_n & \to & H_n & \to & 0. \end{array}$$

We see that

$$\begin{aligned}
\sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank}C_n &\overset{\substack{\text{by (a) we have } \operatorname{rank}C_n=\operatorname{rank}Z_n+\operatorname{rank}B_{n-1}\\ \downarrow}}{=} \sum_{n=0}^{k}(-1)^n\cdot(\operatorname{rank}Z_n+\operatorname{rank}B_{n-1})\\
&\underset{\substack{\uparrow\\ \text{by (b) we have } \operatorname{rank}Z_n=\operatorname{rank}B_n+\operatorname{rank}H_n}}{=} \sum_{n=0}^{k}(-1)^n\cdot(\operatorname{rank}B_n+\operatorname{rank}H_n+\operatorname{rank}B_{n-1})\\
&= \sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank}H_n+\sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank}B_n+\sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank}B_{n-1}\\
&\underset{\substack{\uparrow\\ \text{we do the substitution } m=n-1 \text{ and we use that } B_k=0 \text{ and } B_{-1}=0}}{=} \sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank}H_n+\sum_{n=0}^{k-1}(-1)^n\cdot\operatorname{rank}B_n-\sum_{m=0}^{k-1}(-1)^m\cdot\operatorname{rank}B_m\\
&= \sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank}H_n.
\end{aligned}$$

■

Proof of the Homology-χ-Proposition 48.1. Let X be a finite k-dimensional CW-complex. We perform the following calculation:

$$\begin{aligned}
\chi(X) &\overset{\substack{\text{by definition of } \chi(X)\\ \downarrow}}{=} \sum_{n=0}^{k}(-1)^n\cdot\text{number of } n\text{-cells} &&\overset{\substack{\text{by the Skeleton-H}_*\text{-Lemma 47.1 (1)}\\ \downarrow}}{=} \sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank}(\mathrm{C}_n^{\mathrm{CW}}(X))\\
&\underset{\substack{\uparrow\\ \text{by the Chain Complex–H}_*\text{–}\chi\text{-Lemma 48.3}}}{=} \sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank}(\mathrm{H}_n^{\mathrm{CW}}(X)) &&\underset{\substack{\uparrow\\ \text{by the Singular-Cellular-H}_*\text{-Proposition 47.3}}}{=} \sum_{n=0}^{k}(-1)^n\cdot\operatorname{rank}(\mathrm{H}_n(X)).
\end{aligned}$$

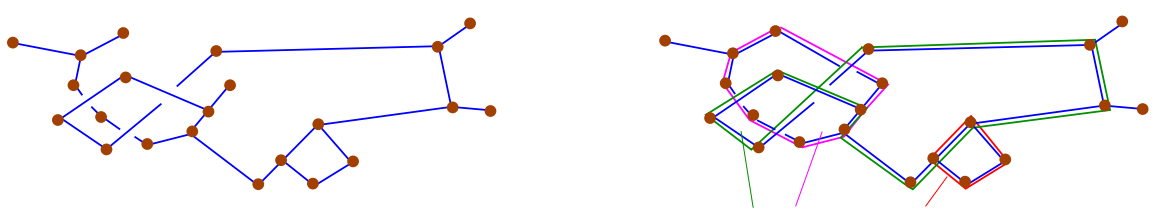
■

48.3. Homology groups of topological graphs.

Lemma 48.4. (Graph-H$_*$-Lemma) Let G be a connected non-empty topological graph with v vertices and e edges. Then

$$\mathrm{H}_0(G)\cong\mathbb{Z},\quad \mathrm{H}_1(G)\cong\mathbb{Z}^{e-v+1}\quad\text{and}\quad \mathrm{H}_i(G)=0 \text{ for } i\geq 2.$$

Example. In the figure below we see a connected topological graph G with 22 vertices and 24 edges. It follows from the Graph-H$_*$-Lemma 48.4 that its first homology group is isomorphic to $\mathbb{Z}^3$. Using the observation that an epimorphism of $\mathbb{Z}^3$ is an isomorphism one can show fairly easily that the three cycles which are shown in the figure below represent a basis for $\mathrm{H}_1(G)\cong\mathbb{Z}^3$.



topological graph G with 22 vertices and 24 edges basis for $\mathrm{H}_1(G)\cong\mathbb{Z}^3$

Proof. Let G be a connected topological graph with v vertices and e edges. Recall that by the Graph–CW-Complex Lemma 30.1 we can view G as a 1-dimensional CW-complex with v 0-cells and e 1-cells. From the H$_0$-Proposition 35.10 (1) together with the CW-Complex–Local Properties Proposition 31.7 (2) and the Locally Path-Connected Lemma 2.24 it follows immediately that $\mathrm{H}_0(G)\cong\mathbb{Z}$. Furthermorefrom considering the cellular chain

complex it follows that $\mathrm{H}_i(G) = 0$ for $i \geq 2$ and that $\mathrm{H}_1(G)$ is torsion-free. We now see that

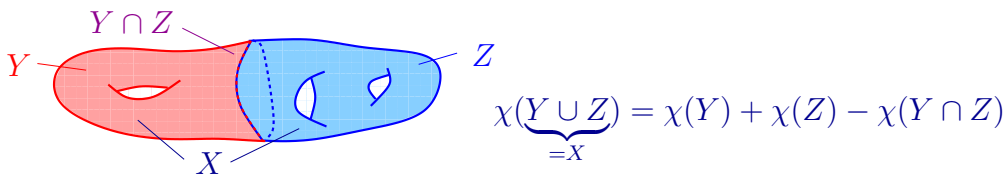
$$1 - \mathrm{rank}(\mathrm{H}_1(G)) \underset{\substack{\uparrow \\ \text{by the above}}}{=} \sum_i (-1)^i \cdot \mathrm{rank}(\mathrm{H}_i(X)) \underset{\substack{\uparrow \\ \text{by by the Homology-}\chi\text{-Proposition 48.1}}}{=} \chi(X) = \#0\text{-cells} - \#1\text{-cells} = v - e.$$

Since $\mathrm{H}_1(G)$ is torsion-free we obtain that $\mathrm{H}_1(G) \cong \mathbb{Z}^{e-k+1}$. ■

48.4. Properties of the Euler characteristic.

Proposition 48.5. (Union-χ-Proposition) Let $X = Y \cup Z$ be a decomposition of a finite CW-complex X into two subcomplexes Y and Z. The following equality holds

$$\chi(X) = \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$



Proof. Given $n \in \mathbb{N}_0$ and given a subcomplex W of X we denote by $n(W)$ the number of n-cells that are contained in W. In our setting we have

$$n(X) \underset{\substack{\uparrow \\ \text{since } X = Y \cup Z \text{ and since } Y \text{ and } Z \text{ are subcomplexes we obtain from the} \\ \text{Subcomplex–Characterization Proposition 30.7 together with the} \\ \text{Subcomplex–Closed Lemma 30.6 that every } n\text{-cell of } X \\ \text{lies in } Y \text{ or } Z\text{, the } n\text{-cells that lie in the intersection } Y \cap Z \text{ get} \\ \text{counted twice in } n(Y) + n(Z)\text{, so we have to subtract } n(Y \cap Z) \text{ to obtain } n(X)}}{=} n(Y) + n(Z) - n(Y \cap Z).$$

By performing the alternating sum over all $n \in \mathbb{N}_0$ we obtain immediately that

$$\chi(X) = \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$

■

Proposition 48.6. (Quotient-χ-Proposition) Let X be a finite CW-complex and let A be a subcomplex of X. We equip the quotient X/A with the CW-structure given by the CW-Quotient Lemma 33.3. Then

$$\chi(X/A) = \chi(X) - \chi(A) + 1.$$

Proof. Recall that in the CW-Quotient Lemma 33.3 we equipped X/A with a CW-structure such that for each $k \in \mathbb{N}_0$ we have the following equalities:

$$\#k\text{-cells of } X/A = \begin{cases} \#k\text{-cells of } X \text{ not in } A, & \text{if } k > 0, \\ \#k\text{-cells of } X \text{ not in } A + 1, & \text{if } k = 0. \end{cases}$$

Evidently these equalities imply the equality $\chi(X/A) = \chi(X) - \chi(A) + 1$. ■

Proposition 48.7. (Product-χ-Proposition) Let X and Y be two finite CW-complexes. If we equip $X \times Y$ with the CW-structure from the CW-Category–Product Proposition 33.7 and the CW-Complex Product Proposition 33.9, then

$$\chi(X \times Y) = \chi(X) \cdot \chi(Y).$$

Example. Let X be a finite CW-complex.

(1) It follows from the Product-χ-Proposition 48.7 that $\chi(S^1 \times X) = \chi(S^1) \cdot \chi(X) = 0$.

(2) We consider the suspension $\Sigma(X) = ((X \times [-1,1])/(X \times \{-1\}))/(X \times \{1\})$. We see that

$$\begin{aligned} \chi(\Sigma(X)) &\overset{\substack{\text{follows from applying the Quotient-}\chi\text{-Proposition 48.6 twice}\\ \downarrow}}{=} \chi((X \times [-1,1]) - \chi(X \times \{-1\}) + 1 - \chi(X \times \{1\}) + 1 \\ &\underset{\substack{\uparrow\\ \text{follows from the Product-}\chi\text{-Proposition 48.7 and the observation that } \chi([-1,1]) = 1}}{=} 2 + \chi(X) - \chi(X) - \chi(X) \quad = \quad 2 - \chi(X). \end{aligned}$$

Proof. Let X and Y be two finite CW-complexes. Given $k \in \mathbb{N}_0$ we denote by c_k the number of k-cells of X and we denote by d_k the number of k-cells of Y. We equip $X \times Y$ with the product CW-structure from the CW-Category–Product Proposition 33.7 and the CW-Complex Product Proposition 33.9. By construction we have

$$\#n\text{-cells of } X \times Y \;=\; \sum_{k+l=n} \#k\text{-cells of } X \cdot \#l\text{-cells of } Y \;=\; \sum_{k+l=n} c_k \cdot d_l.$$

It follows that

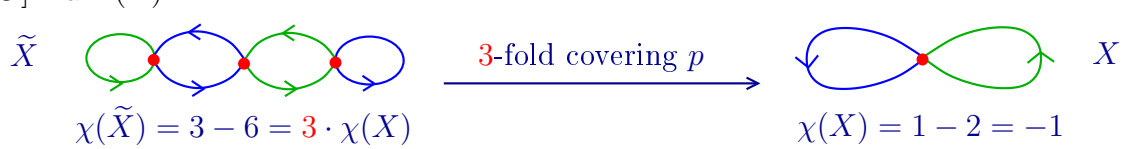
$$\begin{aligned} \chi(X) \cdot \chi(Y) \;=\; \Big(\sum_{k \in \mathbb{N}_0} (-1)^k \cdot c_k\Big) \cdot \Big(\sum_{l \in \mathbb{N}_0} (-1)^l \cdot d_l\Big) &\overset{\substack{\text{by multiplying out}\\ \downarrow}}{=} \sum_{n \in \mathbb{N}_0} \sum_{k+l=n} (-1)^{k+l} \cdot c_k \cdot d_l \\ &= \sum_{n \in \mathbb{N}_0} (-1)^n \cdot \sum_{k+l=n} c_k \cdot d_l \underset{\substack{\uparrow\\ \text{by the above discussion}}}{=} \chi(X \times Y). \end{aligned}$$

We have thus proved the desired equality $\chi(X) \cdot \chi(Y) = \chi(X \times Y)$. ■

Proposition 48.8. (Covering-χ-Proposition) Let $p\colon \widetilde{X} \to X$ be a finite covering of a finite CW-complex X. We equip $\widetilde{X}$ with the structure of a CW-complex given by the CW-Complex Covering Proposition 32.1. Then

$$\chi(\widetilde{X}) \;=\; [\widetilde{X} : X] \cdot \chi(X).$$

Example. In the Graph-π_1-Proposition 20.9 we showed that if X is a connected finite non-empty 1-dimensional CW-complex, then $\pi_1(X)$ is a free group of rank $1 - \chi(X)$. As we implicitly already argued in the Free Subgroup Proposition 32.5, it follows fairly easily from this observation and the Covering-χ-Proposition 48.8 that if F is a free group of finite rank and if G is a finite-index subgroup of F, then G is a free group with $\operatorname{rank}(G) - 1 = [F : G] \cdot \operatorname{rank}(F)$.



Proof. As we mentioned in the CW-Complex Covering Proposition 32.1 (1), it follows immediately from the construction of the CW-structure on $\widetilde{X}$ that for each $k \in \mathbb{N}_0$ we have

$$\#k\text{-cells of } \widetilde{X} \;=\; \big[\widetilde{X} : X\big] \cdot \#k\text{-cells of } X.$$

The proposition is an immediate consequence of this equality. ■

48.5. Applications of Euler characteristics.

Lemma 48.9. (Genus-of-Covering Lemma) Let Σ be the surface of genus g. For any connected finite covering $p\colon \widetilde{\Sigma} \to \Sigma$ we have

$$\text{genus}(\widetilde{\Sigma}) \;=\; \deg(p) \cdot (g-1) + 1.$$

Proof. We denote by $\widetilde{g}$ the genus of $\widetilde{\Sigma}$. The lemma follows immediately from the following two facts:

(1) By the discussion on page 412 we have $\chi(\Sigma) = 2 - 2g$ and $\chi(\widetilde{\Sigma}) = 2 - 2\widetilde{g}$.

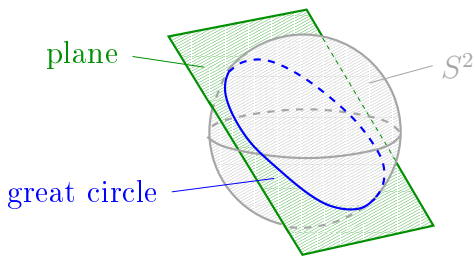
(2) By the Covering-χ-Proposition 48.8 we have $\chi(\widetilde{\Sigma}) = \deg(p) \cdot \chi(\Sigma)$. ■

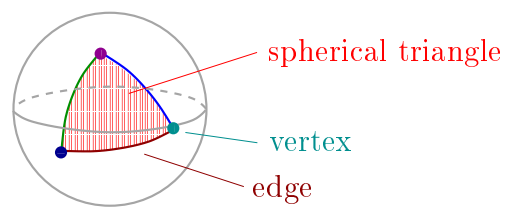
49. Applications of the Euler characteristic

49.1. Building a leather football.

Definition.

(1) A great circle on the 2-dimensional sphere S^2 is the intersection of S^2 with a 2-dimensional vector subspace of $\mathbb{R}^3$.
(2) The angle between two great circles is defined as the smaller of the two angles given by the corresponding 2-dimensional vector subspaces.
(3) A spherical n-gon is a subset of S^2 such that the boundary is the union of n segments of n different great circles. These segments are called edges. The points where two edges intersect are called vertices.



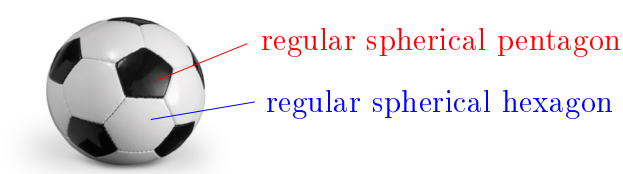


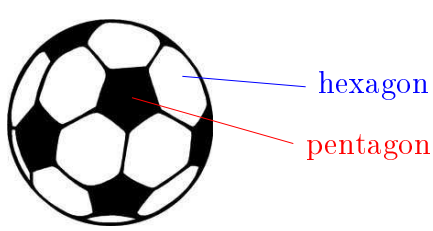
Definition. Let P be a subset of $\mathbb{R}^k$.

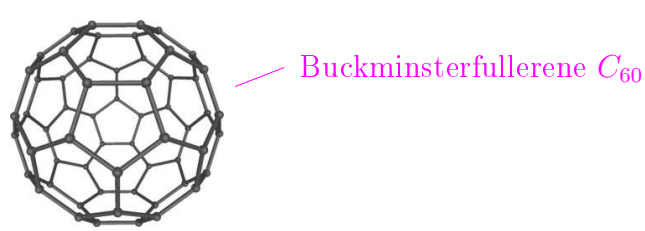
(1) A symmetry of P in $\mathbb{R}^k$ is a matrix $A \in \mathrm{O}(k)$ such that $A \cdot P = P$.
(2) The symmetry group $\mathrm{Sym}(P) \subset \mathrm{O}(k)$ is defined as the group of all symmetries of P.
(3) We set $\mathrm{Sym}^+(P) := \mathrm{Sym}(P) \cap \mathrm{SO}(k)$.

Definition. A spherical polygon P is called regular if the symmetry group $\mathrm{Sym}(P)$ acts transitively on the set of pairs (v, e) where v is a vertex of P and E is an edge of P with $v \in e$.

Example. The classical leather football is made out of 12 regular spherical pentagons and 20 regular spherical hexagons.







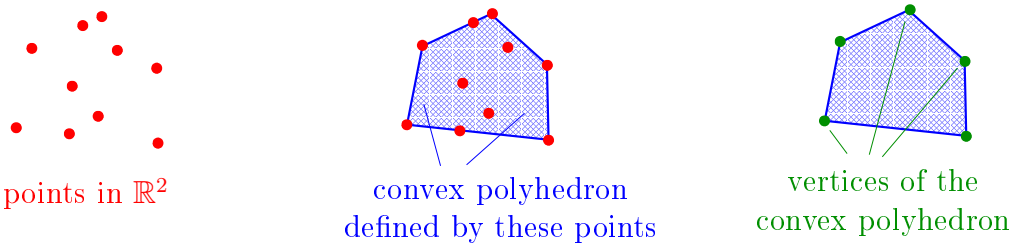
49.2. Euler's Formula.

Definition. Given a subset S of $\mathbb{R}^k$ the convex hull of S is defined as the intersection of all convex subsets of $\mathbb{R}^k$ that contain S.

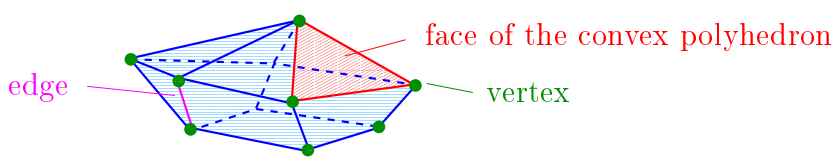
Definition.
(1) A convex polyhedron is the convex hull of finitely many points in $\mathbb{R}^k$.
(2) The dimension of a convex polyhedron is defined as the minimal dimension of an affine subspace of $\mathbb{R}^k$ that contains the convex polyhedron.

Lemma 49.1. (Vertices-of-Polyhedron Lemma) If P is a convex polyhedron in $\mathbb{R}^k$, then there exists a unique finite set $V \subset \mathbb{R}^k$ such that P is the convex hull of V, but such that for any proper subset W of V, P is not the convex hull of W.

Proof. We leave the proof of this lemma to the reader. ■



Definition. Let P be a convex polyhedron in $\mathbb{R}^k$.
(1) We refer to the finite set of the Vertices-of-Polyhedron Lemma 49.1 as the vertices of the convex polyhedron P.
(2) (a) A closed affine half-space in $\mathbb{R}^k$ is a set of the form $\{v \in \mathbb{R}^k \mid \langle v, w\rangle \geq a\}$ for some fixed non-zero vector $w \in \mathbb{R}^k$ and some $a \in \mathbb{R}$.
(b) A subset $F \subset P$ is called a face of P if there exists a closed affine half-space H of $\mathbb{R}^k$ with $P \subset H$ such that $F = P \cap \partial H$.
Note that a face is a convex polyhedron in its own right, which implies in particular that it makes sense to talk of the dimension of a face.
(3) An edge is a 1-dimensional face. Sometimes we refer to a 2-dimensional face just as a face, without specifying the dimension.



Proposition 49.2. (Euler's Formula) Let P be a convex k-dimensional polyhedron in $\mathbb{R}^k$. The following statements hold:
(1) The interior $\mathring{P}$ of P is non-empty and all faces of P are of dimension $\leq k-1$.
(2) The boundary ∂P admits a CW-structure such that for each $j \in \mathbb{N}_0$ the j-cells are precisely the j-dimensional faces of P.
(3) The following equality hold:

$$\sum_{j=0}^{k-1} (-1)^j \cdot \#j\text{-dimensional faces} \;=\; 1 + (-1)^{k-1}.$$

Sketch of proof. We leave the pleasure of proving the first two statements to the reader. Now we calculate that

$$\sum_{j=0}^{k-1}(-1)^j \cdot \#j\text{-dimensional faces} \underset{\substack{\uparrow \\ \text{by (1) and (2)}}}{=} \chi(\partial P) \overset{\substack{\text{by (1) together with the} \\ \text{the Convex-to-Ball Proposition 2.12(2)} \\ \downarrow}}{=} \chi(S^{k-1}) \underset{\substack{\uparrow \\ \text{see page 411}}}{=} 1+(-1)^{k-1}.$$

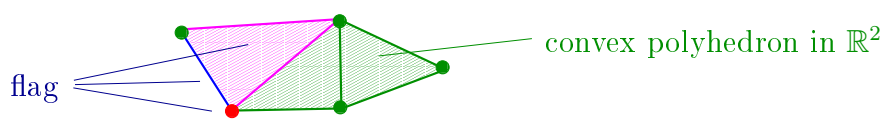
■

49.3. Platonic solids.

Definition.

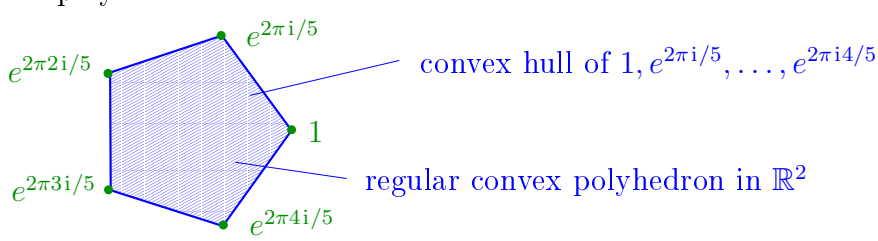
(1) Let P be a convex polyhedron in $\mathbb{R}^k$.
   (a) A flag of P is a sequence $F_0 \subset F_1 \subset \cdots \subset F_{m-1}$ of faces of P such that each F_i is i-dimensional.
   (b) We say P is regular if the symmetry group $\mathrm{Sym}(P)$ acts transitively on the set of flags of P.

(2) Two convex polyhedra P and Q in $\mathbb{R}^k$ are called equivalent if there exists an $r > 0$, a $v \in \mathbb{R}^k$ and an orthogonal matrix $A \in \mathrm{O}(k)$ with $P = A \cdot (r \cdot Q) + v$.



Examples.

(1) Let $n \in \mathbb{N}_{\geq 3}$ and $r > 0$. The convex hull of the n points $r, r \cdot e^{\pi \mathrm{i}/n}, \ldots, r \cdot e^{2\pi \mathrm{i}(n-1)/n}$ in $\mathbb{C} = \mathbb{R}^2$ is a regular convex polyhedron with vertices $r, r \cdot e^{2\pi \mathrm{i}/n}, \ldots, r \cdot e^{2\pi \mathrm{i}(n-1)/n}$. We leave it to the reader to show that, up to equivalence, these are all regular convex 2-dimensional polyhedra in $\mathbb{R}^2$.



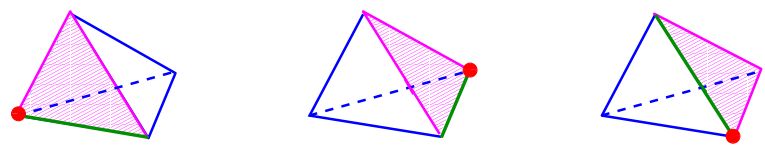
(2) The standard n-simplex Δ^n is a convex n-dimensional polyhedron in $\mathbb{R}^{n+1}$ with vertex set $\{(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)\}$. It is straightforward to see that Δ^n is regular.

Definition. A Platonic solid is a regular convex 3-dimensional polyhedron in $\mathbb{R}^3$.

Definition. The valence of a vertex of a convex polyhedron is defined as the number of edges that contain the vertex.

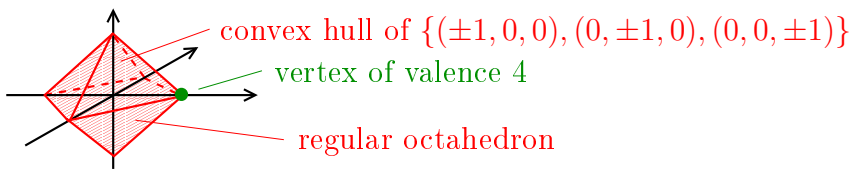
Examples.

(1) Let T be the 3-dimensional polyhedron in $\mathbb{R}^3$ that has four 2-dimensional faces, each of which is an equilateral triangle. In the figure below we show T together with three flags. Some classical geometry shows that T is, up to equivalence, uniquely defined and it is regular. We refer to T as *the regular tetrahedron*. Finally note that each vertex has valence 3.

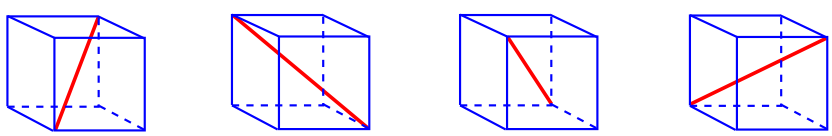


three flags in the regular tetrahedron T

(2) The convex polyhedron in $\mathbb{R}^3$ spanned by $\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$ is a regular convex 3-dimensional polyhedron where each vertex has valence 4. This convex polyhedron is usually called *the regular octahedron*.



(3) We consider the cube $[-1, 1]^3 \subset \mathbb{R}^3$. In [**Arm1988**] it is shown that the group $\mathrm{Sym}^+([-1, 1]^3)$ is isomorphic to S_4. More precisely, we denote by D the set of the diagonals in the cube, i.e. the segments in $\mathbb{R}^3$ spanned by opposite vertices of the cube. Note that D has precisely four elements.



In [**Arm1988**] it is shown that the map

$$\begin{array}{rcl} \mathrm{Sym}^+([-1, 1]^3) & \to & S_4 \cong \text{permutation group of } D \\ A & \mapsto & \begin{pmatrix} D & \to & D \\ d & \mapsto & A \cdot d \end{pmatrix} \end{array}$$

is an isomorphism. It is clear that the cube is a regular convex polyhedron. Since it has six sides it is in our context also referred to as *the regular hexahedron*.

(4) Let $z := 1 + \frac{\sqrt{5}}{2}$ be the golden ratio. We denote by D the polyhedron that is spanned by the 20 vertices

$$(\pm 1, \pm 1, \pm 1), \quad (0, \pm z, \pm \tfrac{1}{z}), \quad (\pm \tfrac{1}{z}, 0, \pm z), \quad \text{and} \quad (\pm z, \pm \tfrac{1}{z}, 0).$$

By [**Cox1948**] this is a regular polyhedron. We refer to is as *the regular dodecahedron*. Note that it has 12 faces and 20 vertices. We can inscribe five cubes $C_1, \ldots, C_5$ into the regular dodecahedron. In [**Arm1988**] it is shown that the map

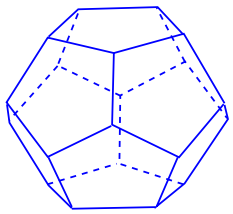
$$\begin{array}{rcl} \mathrm{Sym}^+(D) & \to & A_5 \cong \text{positive permutations of } \{C_1, \ldots, C_5\} \\ A & \mapsto & (C_i \mapsto A \cdot C_i) \end{array}$$

is an isomorphism.

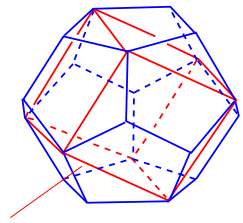
(5) Let $z := 1 + \frac{\sqrt{5}}{2}$ be the golden ratio. We denote by I the polyhedron that is spanned by the 12 vertices

$$(0, \pm 1, \pm z), \quad (\pm z, 0, \pm 1) \quad \text{and} \quad (\pm 1, \pm z, 0).$$

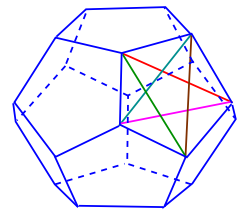
By [**Cox1948**] this is a regular polyhedron. We refer to is as the *regular icosahedron*.



regular dodecahedron

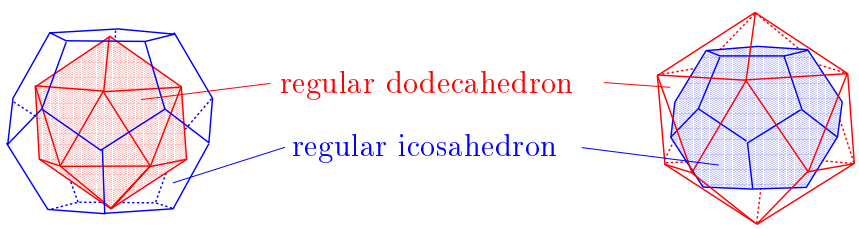


cube inscribed into the regular dodecahedron



the five diagonals of a face correspond precisely to one inscribed cube

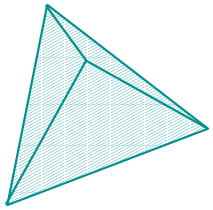
Note that the regular dodecahedron has 20 vertices and 12 faces whereas the regular icosahedron has 12 vertices and 20 faces. In fact the regular icosahedron is dual to the regular dodecahedron in the following sense: up to a scaling factor and a rotation, the regular icosahedron is spanned by the centers of the faces of the regular dodecahedron and vice-versa. We refer to the figure below for an illustration of this fact. It follows that the symmetry group of the regular icosahedron I is in a suitable sense equal to the symmetry group of the regular dodecahedron D. In particular the group $\mathrm{Sym}^+(I)$ is isomorphic to A_5.



Theorem 49.3. (Platonic Solid Theorem) Up to equivalence the only regular convex 3-dimensional polyhedra in $\mathbb{R}^3$ are the platonic solids discussed above and illustrated in the figure below. They have the following types of faces and valences and they have the following symmetry groups:

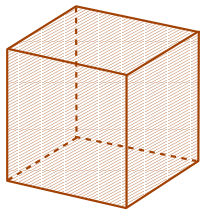
	tetrahedron	hexahedron	octahedron	dodecahedron	icosahedron
number of faces	4	6	8	12	20
shape of face	3-gon	4-gon	3-gon	5-gon	3-gon
valence of vertex	4	3	4	3	5
$\mathrm{Sym}^+(P)$	A_4	S_4	S_4	A_5	A_5

tetrahedron



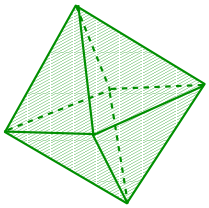
four 3-gons
valence 4

hexahedron



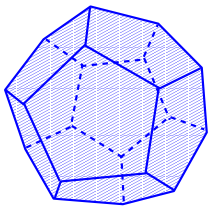
six 4-gons
valence 3

octahedron



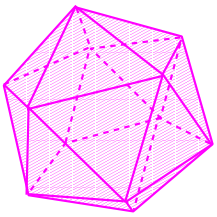
eight 3-gons
valence 4

dodecahedron



twelve 5-gons
valence 3

icosahedron



twenty 3-gons
valence 5

Sketch of proof. We will not provide a full proof for the theorem. Instead we refer to [**Cox1948**] for a full proof that the list of five platonic solids is indeed complete. Furthermore we refer to [**Arm1988**] or alternatively to [**Ae2008**] for a proof of the statement regarding the symmetry groups.

In the following we will show the weaker statement that any platonic solid fits into the table of the theorem. Put differently, we will show that given any platonic solid the number of faces, the shape of faces and the valence of a vertex corresponds to one of the five columns in the above table.

Thus let P be a regular convex 3-dimensional polyhedron in $\mathbb{R}^3$. We make the following observations:

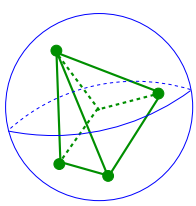
(1) It follows easily from the definition of a regular polyhedron that there exists an $m \in \mathbb{N}$ such that each face of P is a regular m-gon.
(2) The number of faces that meet at a vertex equals the valence of a vertex.
(3) The valences of the vertices are the same and it is at least three.

The following claim tells us the possible values for m and the number f of faces that meet at a vertex.

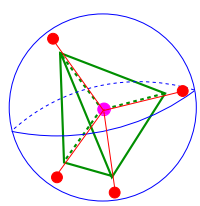
Claim.

possible values for m	3	4	5
f = number of faces that meet at a vertex	$3, 4, 5$	4	3

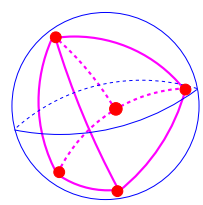
Proof. After a translation we can assume that the sum of all vertices is in the origin. Next we shrink the convex polyhedron so that it fits into the sphere S^2. Then we consider the projection of the vertices and edges of the convex polyhedron onto the sphere S^2.



embed polyhedron in the 3-ball



projection of vertices onto the sphere



projection of edges onto the sphere

Recall that on page **??** we noted that the interior angle of a regular spherical m-gon is greater than the angle of a regular Euclidean m-gon. So it follows that in our case the interior angle is greater than the angle of a regular m-gon in $\mathbb{R}^3$, i.e. it is greater than $\pi - \frac{2\pi}{m}$. Since at any vertex at least three faces meet we obtain the restriction that $f \geq 3$ and that $f \cdot (\pi - \frac{2\pi}{m}) < 2\pi$. It is now straightforward to see that we obtain the stated table. $\boxplus$

Now we treat the cases $m = 3, 4, 5$ separately. We start out with the case $m = 3$, i.e. we assume that all the faces of the convex polyhedron are equilateral triangles. Let us first consider the case that three triangles meet at a vertex. Then we have

$$\#\text{faces} =: k, \qquad \#\text{edges} \underset{\substack{\uparrow \\ \text{since } m=3 \text{ and since each} \\ \text{edge cobounds two faces}}}{=} \frac{1}{2} \cdot 3k \qquad \text{and} \qquad \#\text{vertices} \underset{\substack{\uparrow \\ \text{since } m=3 \text{ and since each} \\ \text{vertex cobounds three faces}}}{=} \frac{1}{3} \cdot 3k.$$

Now we see that

$$2 \underset{\substack{\uparrow \\ \text{Euler's Formula 49.2}}}{=} \#\text{faces} - \#\text{edges} + \#\text{vertices} \underset{\substack{\uparrow \\ \text{by the above}}}{=} k - \frac{3}{2}k + \frac{3}{3}k = \frac{1}{2}k.$$

It follows that $k = 4$. Note that this case is realized by the regular tetrahedron. Almost the same calculation shows that if four triangles meet at a vertex, then $2 = k - \frac{3}{2}k + \frac{3}{4}k$, i.e. $k = 8$. Note that this case is realized by the regular octahedron. Finally by the above claim it remains to consider the case that five triangles meet at a vertex. In this case we see that $2 = k - \frac{3}{2}k + \frac{3}{5}k$, i.e. $k = 20$. Note that this case is realized by the regular icosahedron.

Next we consider the case $m = 4$, i.e. each face is a square. The same argument as above shows that the valence of a vertex can only be three and the same calculation shows that there have to be precisely six faces. Note that this case is realized by the regular hexahedron (aka cube).

Finally if $m = 5$, i.e. if each face is a regular pentagon, then as above we see that the valence of a vertex can only be three and that there have to be precisely twelve faces. Note that this case is realized by the regular dodecahedron. ■

49.4. Planar graphs.

Definition.

(1) An (undirected abstract) graph G is a triple (V, E, φ) where V is a set, E is a set and φ is a map

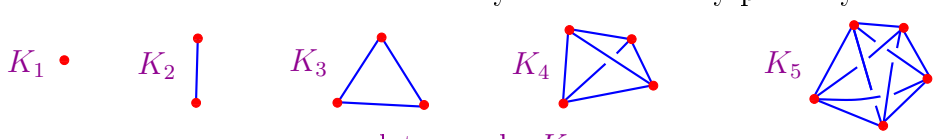
$$\varphi\colon E \to \{\text{subsets of } V \text{ with one or two elements}\}.$$

The elements of V are called vertices of G and the elements of E are called the edges of G.

(2) The topological realization $|\mathrm{G}|$ of G is defined as the topological space that is obtained from the vertex set V (equipped with the discrete topology) by attaching an interval for each edge $e \in E$ to the vertices given by $\varphi(e) \subset V$.

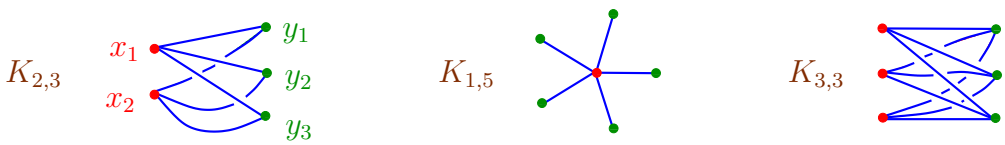
Examples.

(1) For any $n \in \mathbb{N}$ we define the complete graph K_n as the abstract graph with vertex set $V = \{v_1, \dots, v_n\}$, edge set $E = \{\{v_i, v_j\} \mid i, j \in \{1, \dots, n\} \text{ with } i \neq j\}$ and where we define $\varphi\colon E \to \mathcal{P}(V)$ via $\varphi(e) = e$. Put differently, the complete graph K_n has n vertices and each vertex is connected to any other vertex by precisely one edge.



complete graphs K_n

(2) For every choice of $m, n \in \mathbb{N}$ we define the complete bipartite graph $K_{m,n}$ as the abstract graph which is defined by the vertex set $V = \{x_1, \dots, x_m\} \cup \{y_1, \dots, y_n\}$, the edge set $E = \{\{x_i, y_j\} \mid i \in \{1, \dots, m\} \text{ and } j \in \{1, \dots, n\}\}$ and the map $\varphi\colon E \to \mathcal{P}(V)$ that is given by $\varphi(e) = e$. Put differently, the complete bipartite graph $K_{m,n}$ consists of m "red-colored" vertices, n "green-colored" vertices and each "red-colored" vertex is connected to any "green-colored" vertex by precisely one edge.

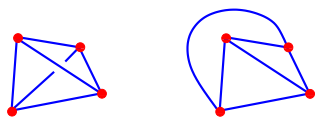


complete bipartite graphs $K_{m,n}$

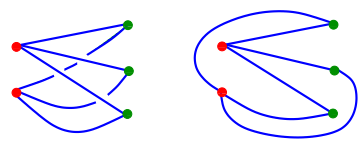
Definition. We say a finite graph G is planar if its topological realization $|G|$ admits an embedding $|G| \to \mathbb{R}^2$.

Examples.

(1) The above figures show that the graphs K_1, K_2, K_3 and $K_{1,5}$ are planar. On the other hand the above topological realizations of K_4 and $K_{2,3}$ are not subspaces of $\mathbb{R}^2$. But as we see in the figure below both graphs admit topological realizations that are subspaces of $\mathbb{R}^2$.



topological realizations of K_4

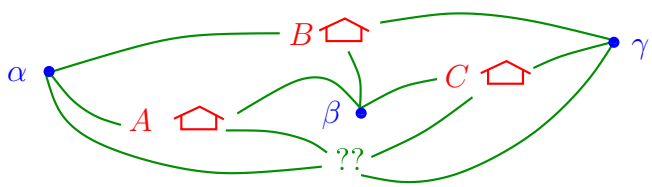


topological realizations of $K_{2,3}$

(2) One can easily verify that given an $m \in \mathbb{N}$ the bipartite graph $K_{2,m}$ is planar.

Question 49.4. Are the graphs K_5 and $K_{3,3}$ also planar?

Problem 49.5. (Utility Problem) Suppose that three new houses A, B and C are built and they need to be connected to three utilities, e.g. a water source α, an electricity provider β and an internet provider γ. Can we perform the connections so that no two lines/pipes intersect, except at the endpoints?



Definition. Let G be a graph.

(1) Given $n \in \mathbb{N}$ we denote by X_n the "circular graph with n vertices and n edges", more precisely, X_n is the graph with

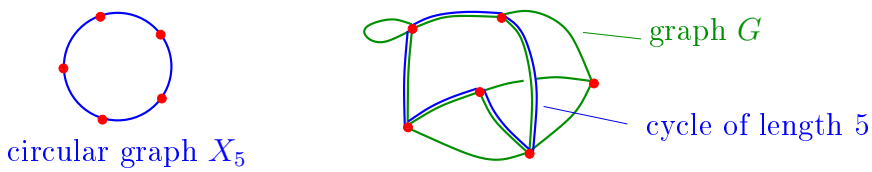
$$V(X_n) = \{1, \dots, n\} \quad \text{and} \quad E(X_n) = \{\{1,2\}, \{2,3\}, \dots, \{n-1,n\}, \{n,1\}\}$$

and where $\varphi\colon E(X_n) \to \mathcal{P}(V(X_n))$ is the obvious map.

(2) A cycle of length n in G is an injective morphism $\varphi\colon X_n \to G$ of graphs.

(3) The girth of G is the shortest length of a cycle in G. If G has no cycles, then we define its girth as ∞.

Examples.



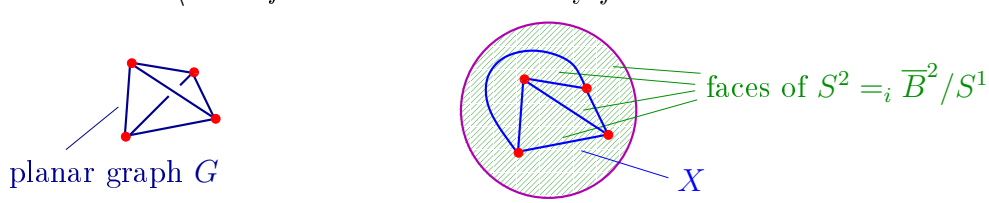
(1) The girth of the graph shown in the figure above to the right is one, since it has an edge where the two endpoints agree.
(2) One can easily show that for any $n \geq 3$ the girth of the complete graphs K_n equals three.
(3) Similarly one can easily show that for any $n, m \geq 2$ the girth of the complete bipartite graphs $K_{n,m}$ equals four.

Proposition 49.6. (Planar Graph–Girth Proposition) Let G be a graph with v vertices and e edges. We denote by g the girth of G. If G is planar and if $g < \infty$, then

$$e \leq \frac{g}{g-2} \cdot (v-2).$$

Sketch of proof. Let G be a graph with v vertices and e edges. We suppose that the girth g of G is finite. Furthermore we suppose that G is planar, i.e. we suppose that there exists an embedding $\varphi\colon |G| \to \mathbb{R}^2$. We write $X = \varphi(|G|)$. Without loss of generality we can assume that X lies in the open disk $B^2 \subset \mathbb{R}^2$.

Now we view X as a subspace of the sphere $S^2 =_i \overline{B}^2/S^1$. We refer to the closures of the components of $S^2 \setminus X$ as *faces* and we denote by f the number of faces.



We make the following observation:

(1) The vertices, edges and faces define a CW-structure for $S^2 =_i \overline{B}^2/S^1$.
(2) We make the following two observations:
  (a) The boundary of each face corresponds to a cycle in the graph. By the definition of the girth each face is bounded by at least g edges.
  (b) Each edge lies in the boundary of at most two faces.

Now we deduce that

$$\begin{aligned} g \cdot f &\overset{\substack{\text{follows from (a)}\\ \downarrow}}{\leq} \sum_{\text{faces } f} \text{number of edges in the boundary of } f \\ &= \text{number of pairs } (f, e) \text{ where } f \text{ is a face and } e \text{ an edge in the boundary of } f \\ &= \sum_{\text{edges } e} \text{number of faces that contain } e \text{ in the boundary } \underset{\substack{\uparrow\\ \text{follows from (b)}}}{\leq} 2e. \end{aligned}$$

Summarizing we see that $g \cdot f \leq 2e$.

Now we see that

$$2 \underset{\substack{\uparrow\\ \text{see page 411}}}{=} \chi(S^2) \underset{\substack{\uparrow\\ \text{by (1)}}}{=} v - e + f \underset{\substack{\uparrow\\ \text{since by (2) we have } g \cdot f \leq 2e}}{\leq} v - e + \frac{2}{g} \cdot e = v - \frac{g-2}{g} \cdot e.$$

Solving for e we get the desired inequality $e \leq \frac{g}{g-2} \cdot (v - 2)$. ■

Proposition 49.7. (Non-Planar Graph Proposition) If a graph G contains the complete graph K_5 or the complete bipartite graph $K_{3,3}$ as a subgraph, then G itself cannot be planar.

Proof.

(1) Let $m \in \mathbb{N}_{\geq 3}$. It is straightforward to see that the complete graph K_m has m vertices and $\frac{1}{2}m(m-1)$ edges. Furthermore, as we pointed out on page 425, the girth is three. It is straightforward to verify that the inequality of the Planar Graph–Girth Proposition 49.6 is only satisfied for $m = 3$ and $m = 4$.

(2) Let $m, n \in \mathbb{N}_{\geq 2}$. It is easy to verify that the complete bipartite graph $K_{m,n}$ has $m + n$ vertices and $m \cdot n$ edges. Furthermore, as we pointed out above, the girth is four. It is straightforward to verify that the inequality of the Planar Graph–Girth Proposition 49.6 is only satisfied if $m = 2$ or $n = 2$.

(3) The proposition now follows from (1) and (2) and the observation that if a graph is planar, then evidently any subgraph is also planar. ■