ALGEBRAIC TORSION

1. Introduction

1.1. **Definition of $K_1(R)$.** By a *ring* we always mean an associative ring R with $1 \neq 0$. For such a ring R define $\mathrm{GL}(R)$ to be the group of all $\mathbb{N} \times \mathbb{N}$-matrices, that are the infinite identity matrix except for finitely many entries. More rigorously, $\mathrm{GL}(R) = \varinjlim \mathrm{GL}(R, d)$, where we have the following maps in the directed system:

$$\begin{array}{rcl} \mathrm{GL}(R,d) & \to & \mathrm{GL}(R,d+1) \\ A & \mapsto & \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}. \end{array}$$

Given $a \in R$ and $i \neq j \in \{1, \dots, d\}$ we denote by E_{ij} the $d \times d$-matrix which is zero except for the (i,j)-entry which equals 1. We refer to all matrices of form $\mathrm{id} + aE_{ij}$ with $a \in R$ as *elementary matrices*. We denote by $E(R)$ the subgroup of $\mathrm{GL}(R)$ generated by all elementary matrices.

In the following, given a group G and $g, h \in G$ we denote by $[g,h] = ghg^{-1}h^{-1}$ the commutator. Furthermore, we denote by

$$[G,G] := \Big\{ \prod_{i=1}^{n} [g_i, h_i] \,\Big|\, g_1, h_1, \dots, g_n, h_n \in G \Big\}$$

the *commutator subgroup of* G. The following proposition gives a convenient description of the commutator subgroup of $\mathrm{GL}(R)$.

Proposition 1.1.

$$E(R) = [\mathrm{GL}(R), \mathrm{GL}(R)].$$

Proof. For $i \neq j \neq k$ one can easily verify that

$$\begin{array}{rcl} \mathrm{id} + aE_{ik} & = & (\mathrm{id} + aE_{ij})(\mathrm{id} + E_{jk})(\mathrm{id} - aE_{ij})(\mathrm{id} - E_{jk}) \\ & = & (\mathrm{id} + aE_{ij})(\mathrm{id} + E_{jk})(\mathrm{id} + aE_{ij})^{-1}(\mathrm{id} + E_{jk})^{-1}. \end{array}$$

This shows that each elementary matrix in $\mathrm{GL}(d, A)$ with $d \geq 3$ is indeed a commutator.

Conversely, using the following three identities one can easily show that each commutator $XYX^{-1}Y^{-1} \in \mathrm{GL}(d,R)$ is a product of elementary matrices within the

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group $\mathrm{GL}(2d, R)$:

$$\begin{aligned}
&(1) & \begin{pmatrix} XYX^{-1}Y^{-1} & 0 \\ 0 & \mathrm{id} \end{pmatrix} &= \begin{pmatrix} X & 0 \\ 0 & X^{-1} \end{pmatrix} \begin{pmatrix} Y & 0 \\ 0 & Y^{-1} \end{pmatrix} \begin{pmatrix} (YX)^{-1} & 0 \\ 0 & YX \end{pmatrix}, \\
&(2) & \begin{pmatrix} X & 0 \\ 0 & X^{-1} \end{pmatrix} &= \begin{pmatrix} \mathrm{id} & X \\ 0 & \mathrm{id} \end{pmatrix} \begin{pmatrix} \mathrm{id} & 0 \\ \mathrm{id} - X^{-1} & \mathrm{id} \end{pmatrix} \begin{pmatrix} \mathrm{id} & -\mathrm{id} \\ 0 & \mathrm{id} \end{pmatrix} \begin{pmatrix} \mathrm{id} & 0 \\ \mathrm{id} - X & \mathrm{id} \end{pmatrix}, \\
&(3) & \begin{pmatrix} \mathrm{id} & X \\ 0 & \mathrm{id} \end{pmatrix} &= \prod_{i=1}^{d} \prod_{j=d+1}^{2d} (\mathrm{id} + x_{ij} E_{ij}).
\end{aligned}$$

□

The Whitehead group $K_1(R)$ is defined as the abelian group

$$K_1(R) := \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)] = \mathrm{GL}(R)/E(R).$$

For every d there exists a canonical homomorphism $\mathrm{GL}(R, d) \to K_1(R)$, in particular there exists a homomorphism from the units of R into $K_1(R)$. By abuse of notation we denote the image of $A \in \mathrm{GL}(R, d)$ in $K_1(R)$ by A as well.

Lemma 1.2. *For any two matrices $A \in \mathrm{GL}(R, d_1)$ and $B \in \mathrm{GL}(R, d_2)$ the product $AB \in K_1(R)$ is given by*

$$AB = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in K_1(R).$$

Proof. Without loss of generality we can assume that $d_1 = d_2$. Then

$$AB = \begin{pmatrix} AB & 0 \\ 0 & \mathrm{id} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} \in K_1(R).$$

In the proof of Proposition 1.1 we had seen that the second matrix on the right hand side is trivial in $K_1(R)$. □

The proof of the following lemma is left as an exercise.

Lemma 1.3. *Let a, b be units in R. Then*

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \text{ and } \begin{pmatrix} -a & 0 \\ 0 & b \end{pmatrix}$$

represent the same element in $K_1(R)$.

Corollary 1.4. *If $\mathbb{F}$ is a commutative field, then the map*

$$\begin{aligned} K_1(\mathbb{F}) &\to \mathbb{F}^* := \mathbb{F} \setminus \{0\} \\ [A] &\mapsto \det(A) \end{aligned}$$

is an isomorphism.

In the following, given a commutative field $\mathbb{F}$, we will use the isomorphism from Corollary 1.4 to identify $K_1(\mathbb{F})$ with $\mathbb{F}^*$.

Proof. It is straightforward to see that the map is a well-defined epimorphism. It remains to show that the map is injective. So let A be any matrix in $\mathrm{GL}(d,\mathbb{F})$. Using elementary linear algebra and Lemmas 1.2 and 1.3 one can show that A and the 1×1-matrix $(\det A)$ represent the same element in $K_1(\mathbb{F})$. It follows that the map under consideration is a monomorphism. $\square$

In the following we will mostly work with the *reduced Whitehead group*

$$\overline{K}_1(R) = K_1(R)/\text{subgroup of order 2 generated by } (-1).$$

On several occasions we will make use of the fact that a ring homomorphism $\varphi\colon R\to S$ induces group homomorphisms $\varphi_*\colon K_1(R)\to K_1(S)$ and $\varphi_*\colon \overline{K}_1(R)\to\overline{K}_1(S)$. Also, the maps

$$\begin{array}{ccc} \mathrm{GL}(d, M(n\times n,R)) & \to & \mathrm{GL}(dn,R) \\ \begin{array}{c} d\times d\text{-matrix } A \text{ with entries} \\ \text{that are } n\times n\text{-matrices over } R \end{array} & \mapsto & \begin{array}{c} A \text{ viewed as an} \\ nd\times nd\text{-matrix} \\ \text{with entries in } R \end{array} \end{array}$$

are homomorphisms that induce an isomorphism

$$K_1(M(n\times n,R)) \;\to\; K_1(R)$$

that is sometimes referred to as *Morita equivalence*.

1.2. **Definition of algebraic torsion.**

Henceforth we will assume that all rings in these lecture notes have the following property:

$$\text{if } r\neq s\in\mathbb{N}_0, \text{ then } R^r \text{ is not isomorphic to } R^s \text{ as an } R\text{–module.}$$

It is clear that any field, $\mathbb{Z}$, and more generally any domain have this property. The following lemma gives another important example.

Lemma 1.5. *If G is a group, then the group ring $\mathbb{Z}[G]$ has the above property.*

Proof. We view $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module via the trivial action of G on $\mathbb{Z}$. If $\mathbb{Z}[G]^r\cong\mathbb{Z}[G]^s$, then we also have

$$\mathbb{Z}^r \cong \mathbb{Z}\otimes_{\mathbb{Z}[G]}\mathbb{Z}[G]^r \cong \mathbb{Z}\otimes_{\mathbb{Z}[G]}\mathbb{Z}[G]^s\cong\mathbb{Z}^s,$$

hence $r=s$. $\square$

In the following, given a ring R we mean by an R-module a finitely generated left R-module, unless we say explicitly otherwise. We view elements in R^n as row vectors, and an $n\times m$-matrix over R gives rise to a homomorphism $R^n\to R^m$ of left R-modules via *right* multiplication on row vectors. A *based R-module* is a free R-module together with an (unordered) basis.

Given a free R-module V and two bases $d=\{d_1,\dots,d_n\}$ and $e=\{e_1,\dots,e_n\}$ we denote by $[d/e]\in\overline{K}_1(R)$ the element represented by the matrix of the basis change, i.e. $[d/e]=(a_{ij})$ where $d_i=\sum_j a_{ij}e_j$. We say that two bases are equivalent if $[d/e]$ is trivial in $\overline{K}_1(R)$.

Given a short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ of free R-modules and bases $\mathcal{A} = \{a_i\}$ for A and $\mathcal{C} = \{c_j\}$ for C we pick lifts $\tilde{c}_j$ to B and we denote by $\mathcal{AC}$ the equivalence classes of bases represented by the a_i and the $\tilde{c}_j$. It is straightforward to verify that $\mathcal{AC}$ does not depend on the choice of the lifts. If B is also equipped with a basis $\mathcal{B}$, then we say that the bases $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are *compatible*, if $\mathcal{B}$ and $\mathcal{AC}$ are equivalent.

Let $(C_*, \mathcal{C}_*)$ be a finite chain complex of based R–modules. In the following we will associate to $(C_*, \mathcal{C}_*)$ its torsion

$$\tau(C_*, \mathcal{C}_*) \in \overline{K}_1(R) \cup \{*\}.$$

If C_* is not acyclic, then we define its torsion $\tau(C_*, \mathcal{C}_*)$ to be $*$. Now suppose that C_* is acyclic. In the following we make the technical assumption that each $B_i := \operatorname{Im}(C_{i+1}) \subset C_i$ is a free R-module. For each i we pick a basis $\mathcal{B}_i$ of B_i. Since C_* is acyclic the sequences

$$0 \to B_i \to C_i \to B_{i-1} \to 0$$

are exact. We define the *torsion* (sometimes also called *Reidemeister-torsion*) of the based acyclic complex $(C_*, \mathcal{C}_*)$ to be

$$\tau(C_*, \mathcal{C}_*) = \prod [\mathcal{B}_i \mathcal{B}_{i-1} / \mathcal{C}_i]^{(-1)^{i+1}} \in \overline{K}_1(R).$$

It is straightforward to see that $\tau(C_*, \mathcal{C}_*)$ is independent of the choice of the $\mathcal{B}_i$. If the R–modules B_i are not free, then one can show that they are stably free and a stable basis will then make the definition work again. We refer to [Mi66, p. 369] or [Tu01, p. 13] for the full details.[1]

For future reference we record the simplest example in the following lemma.

Lemma 1.6. *If* $0 \to C_k \xrightarrow{\partial_k} C_{k-1} \to 0$ *is an isomorphism between free based R-modules of rank n, then*

$$\tau(C_*) = [B_k]^{(-1)^{k-1}} \in \overline{K}_1(R),$$

where B_k denotes the $n \times n$-matrix over R which represents ∂_k with respect to the given bases.

The statement of the lemma is an immediate consequence of the definitions. It says in particular that the torsion of a chain complex of based R-modules can be viewed as the generalization of the determinant of a homomorphism between two based R-modules.

Example. Let A be an $n \times n$-matrix over $\mathbb{Z}$ with $\det(A) \neq 0$. Then

$$\tau\left(0 \to \mathbb{Q}^n \xrightarrow{A} \mathbb{Q}^n \to 0\right) = \det(A)$$

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[1] The torsion introduced in [Tu01] is the multiplicative inverse of the torsion introduced in [Mi66]. We follow the convention of [Mi66].

which up to sign equals the number of elements in the finite abelian group $\mathbb{Z}^n/A\mathbb{Z}^n$. This explains, perhaps, the name torsion for the invariant $\tau(C_*)$. The term 'torsion' was introduced by Franz [Fra35].

In the following we extend the multiplication of $\overline{K}_1(R)$ to

$$\tau(C_*, \mathcal{C}_*) \in \overline{K}_1(R) \cup \{*\}$$

by defining the product of 0 with any other element to be 0. The following elementary lemma now shows how the torsion depends on the choice of the bases.

Lemma 1.7. *Let C_* be a chain complex of free R-modules and let $\mathcal{C}_*$ and $\mathcal{C}'_*$ be two choices of bases. Then*

$$\tau(C_*, \mathcal{C}'_*) = \tau(C_*, \mathcal{C}_*) \cdot \prod_i [\mathcal{C}_i/\mathcal{C}'_i]^{(-1)^i}.$$

In many situations the above example together with the following theorem suffice to calculate the torsion invariants.

Theorem 1.8. *Let*

$$0 \to C'_* \xrightarrow{f} C_* \xrightarrow{p} C''_* \to 0$$

be a short exact sequence of based chain complexes of free R-modules, such that for each i the bases $\mathcal{C}'_i, \mathcal{C}_i$ and $\mathcal{C}''_i$ are compatible. If $\tau(C'_*, \mathcal{C}'_*) \neq *$ or $\tau(C''_*, \mathcal{C}''_*) \neq *$, then

$$\tau(C_*, \mathcal{C}_*) = \tau(C'_*, \mathcal{C}'_*) \cdot \tau(C''_*, \mathcal{C}''_*) \ \in \overline{K}_1(R) \cup \{*\}.$$

Proof. If one of the chain complexes is not acyclic, then it follows easily from the long exact sequence in homology, or hypothesis that $\tau(C'_*, \mathcal{C}'_*) \neq *$ or $\tau(C''_*, \mathcal{C}''_*) \neq *$, and the definition of the product structure on $\overline{K}_1(R) \cup \{*\}$ that both sides of the desired equality equal $*$.

Thus in the following we assume that all chain complexes are acyclic. In the proof we again assume that the following R-modules are free

$$\begin{aligned} B'_i &:= \operatorname{Im}(\partial'_{i+1} \colon C'_{i+1} \to C'_i), \\ B_i &:= \operatorname{Im}(\partial_{i+1} \colon C_{i+1} \to C_i), \\ B''_i &:= \operatorname{Im}(\partial''_{i+1} \colon C''_{i+1} \to C''_i). \end{aligned}$$

The general case is proved in Theorem 3.4 in [Tu01]. We start out with the following claim.

Claim. The sequence

$$0 \longrightarrow B'_i \xrightarrow{f_i} B_i \xrightarrow{p_i} B''_i \longrightarrow 0.$$

is exact.

We consider the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[r] & C'_{i-1} \arrow[rr, "f_i"] & & C_{i-1} \arrow[rr, "p_i"] & & C''_{i-1} \arrow[rr] & & 0 \\
 & 0 \arrow[r] & C'_i \arrow[ur] \arrow[rr, "f_i"] & & C_i \arrow[ur] \arrow[rr, "p_i"] & & C''_i \arrow[ur] \arrow[rr] & & 0 & \\
0 \arrow[r] & C'_{i+1} \arrow[ur] \arrow[dr] \arrow[rr] & & C_{i+1} \arrow[ur] \arrow[dr] \arrow[rr] & & C''_{i+1} \arrow[ur] \arrow[dr] \arrow[rr] & & 0 & & \\
 & 0 \arrow[r] & B'_i \arrow[uu] \arrow[rr, "f_i"] & & B_i \arrow[uu] \arrow[rr, "p_i"] & & B''_i \arrow[uu] \arrow[rr] & & 0. &
\end{tikzcd}$$

The diagonal sequences are by assumption exact and the first three horizontal sequences also also exact. A straightforward diagram chase shows that the bottom horizontal sequence is also exact. This concludes the proof of the claim.

For each i we pick bases $\mathcal{B}'_i$ of B'_i and $\mathcal{B}''_i$ of B''_i. By the claim we can and will equip each B_i with the basis $\mathcal{B}_i := \mathcal{B}'_i\mathcal{B}''_i$. We these choices we compute that

$$\begin{aligned}
\tau(C_*, \mathcal{C}_*) &= \prod [\mathcal{B}_i\mathcal{B}_{i-1}/\mathcal{C}_i]^{(-1)^i} \\
&= \prod [\mathcal{B}'_i\mathcal{B}'_{i-1}\mathcal{B}''_i\mathcal{B}''_{i-1}/\mathcal{C}'_i\mathcal{C}''_i]^{(-1)^i} \\
&= \prod [\mathcal{B}'_i\mathcal{B}'_{i-1}/\mathcal{C}'_i]^{(-1)^i} \cdot \prod [\mathcal{B}''_i\mathcal{B}''_{i-1}/\mathcal{C}''_i]^{(-1)^i} \quad \text{by Lemma 1.2} \\
&= \tau(C'_*, \mathcal{C}'_*) \cdot \tau(C''_*, \mathcal{C}''_*) \in \overline{K}_1(R).
\end{aligned}$$

□

The following theorem, which is proved on page 60 of [Co73], can be viewed as a generalization of the functoriality of the determinant.

Theorem 1.9. *Let $(C_*, \mathcal{C}_*)$ be an acyclic based chain complex over a ring R and let $\varphi\colon R \to S$ be a ring homomorphism. Then*

$$\tau(S \otimes_R C_*, 1 \otimes \mathcal{C}_*) = \varphi(\tau(C_*, \mathcal{C}_*)).$$

In fact the following somewhat more general theorem holds.

Theorem 1.10. *Let $(C_*, \mathcal{C}_*)$ be an acyclic based chain complex over a ring R. Let S be a ring and let $\varphi\colon R \to M(d \times d, S)$ be a homomorphism. Let $e_1, \dots, e_d$ be the standard basis of S^d. Then* [2]

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[2] On the right hand side, $\varphi_*(\tau(C_*, \mathcal{C}_*))$ is a priori an element in $\overline{K}_1(M(d \times d, S))$. The Morita equivalence $K_1(M(d \times d, S)) \xrightarrow{\cong} K_1(S)$ induces a map $\overline{K}_1(M(d \times d, S)) \xrightarrow{\cong} \overline{K}_1(S)$. Using this map we then view $\varphi_*(\tau(C_*, \mathcal{C}_*))$ as an element in $\overline{K}_1(S)$.

1.3. **Reidemeister torsion of a CW–complex.** Let X be a connected finite CW–complex and let $Y \subset X$ be a subcomplex. We write $\pi = \pi_1(X)$. We denote the universal covering of X by $p\colon \widetilde{X} \to X$. We write $\widetilde{Y} := p^{-1}(Y)$. A careful inspection of the definitions shows that the action of π via deck transformations on the universal cover is a left action. In particular $C_*(\widetilde{X}, \widetilde{Y})$ is a chain complex of left $\mathbb{Z}[\pi]$-modules. A lift of each cell in $X \setminus Y$ gives a basis for $C_*(\widetilde{X}, \widetilde{Y})$ as a free $\mathbb{Z}[\pi]$-module.

Now let R be a ring[3] and let V be a free d-dimensional (left) R-module. Let $\alpha\colon \pi_1(X) \to \operatorname{Aut}_R(V)$ be a representation. This equips V with a right $\mathbb{Z}[\pi]$–module structure. We can therefore consider the chain complex

$$V \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\widetilde{X}, \widetilde{Y})$$

with boundary maps $\mathrm{id} \otimes \partial_*$. This is a chain complex of free (left) R–modules.

We denote the i–cells of $X \setminus Y$ by $\sigma_i^1, \dots, \sigma_i^{r_i}$. We pick a basis $e_1, \dots, e_d$ for V. Furthermore we pick an orientation for each cell σ_i^j, and we also pick a lift $\widetilde{\sigma}_i^j$ for each cell σ_i^j to the universal cover $\widetilde{X}$. We get a basis

$$\mathcal{C}_i = \{e_1 \otimes \widetilde{\sigma}_i^1, \dots, e_d \otimes \widetilde{\sigma}_i^1, \dots, e_1 \otimes \widetilde{\sigma}_i^{r_i}, \dots, e_d \otimes \widetilde{\sigma}_i^{r_i}\}$$

for $V \otimes_{\mathbb{Z}[\pi_1(X)]} C_i(\widetilde{X}, \widetilde{Y})$.

Proposition 1.11. *The torsion*

$$\tau(X, Y, \alpha) := \tau\Big(V \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\widetilde{X}, \widetilde{Y}), \{\mathcal{C}_i\}\Big),$$

viewed as an element in $\overline{K}_1(R)/\alpha(\pi_1(X)) \cup \{*\}$*, is an invariant of the CW-complex* X *together with the representation* α*.*

Not surprisingly, for $Y = \emptyset$ we write $\tau(X, \alpha) := \tau(X, \emptyset, \alpha)$.

Proof. Using Lemma 1.7 it is a straightforward that a different choice of basis for V does not affect the torsion at all. A different choice of orientation for any one cell changes the sign in $K_1(R)$, but does the element it represents in $\overline{K}_1(R)/\alpha(\pi_1(X)) \cup \{*\}$. Finally, once again using Lemma 1.7 it is straightforward to see that different lifts of cells change the torsion in $\overline{K}_1(R) \cup \{*\}$ by multiplication by an element of the form $\alpha(g)$ with $g \in \pi_1(X)$. □

The following lemma says in particular, that we do not need to worry about base point issues. The proof is left as an exercise.

Lemma 1.12. *Let* X *be a finite CW-complex and let* $\alpha, \beta\colon \pi_1(X) \to \mathrm{GL}(d, R)$ *be two representations. If* α *and* β *are conjugate, i.e. if there exists an* $A \in \mathrm{GL}(d, R)$ *such that* $\alpha(g) = A\beta(g)A^{-1}$ *for all* $g \in \pi_1(X)$*, then*

$$\tau(X, \alpha) = \tau(Y, \alpha) \in \overline{K}_1(R)/\alpha(\pi_1(X)) \cup \{*\}.$$

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[3] As we mentioned before, we assume throughout these notes that all rings have the Property $(*)$ from the previous section.

Summarizing we just defined an invariant for a CW-complex together with a representation. The question that now arises is, to what degree does the invariant depend on the precise structure of the CW-complex? Does it, as most invariants in algebraic topology, depend only on the homotopy type? To answer this question we need to study the torsion in more detail.

1.4. **Whitehead torsion of a deformation retract.** Let X be a connected CW-complex and let Y be a subcomplex of X which is a deformation retract of X. Here, recall that this means that there exists a deformation retraction, i.e. a map

$$h\colon X \times [0,1] \to X$$

such that the following conditions hold:

1. $h(-,0)$ is the identity map,
2. for each $t \in [0,1]$ the restriction of $h(-,t)$ to Y is the identity map,
3. $h(X,1) \subset Y$.

Since Y is a deformation retract of X it follows that the inclusion $Y \to X$ is a homotopy equivalence, in particular, given $y \in Y$, the inclusion induces an isomorphism $\pi_1(Y,y) \to \pi := \pi_1(X,y)$. We denote by $p\colon \widetilde{X} \to X$ the universal covering and we write $\widetilde{Y} := p^{-1}(Y)$. Elementary covering theory shows that the deformation retraction h lifts to a deformation retraction $\widetilde{h}\colon \widetilde{X} \times [0,1] \to \widetilde{X}$. In particular the chain complex

$$C_*(\widetilde{X},\widetilde{Y}) = \mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X},\widetilde{Y})$$

is acyclic as a complex of $\mathbb{Z}[\pi]$-modules.

If X is finite we can thus define

$$\tau(X,Y) := \tau(X,Y,\pi \xrightarrow{\text{id}} \mathbb{Z}[\pi]) = \tau\Big(C_*(\widetilde{X},\widetilde{Y})\Big) \in \overline{K}_1(\mathbb{Z}[\pi])/\pi.$$

In the following, given a group G we write $\mathrm{Wh}(G) := \overline{K}_1(\mathbb{Z}[G])/G$ and refer to it as the *Whitehead group of* G.

1.5. **Whitehead torsion of a homotopy equivalence.** Given a map $f\colon X \to Y$ between topological spaces we refer to

$$M_f := \Big((X \times [0,1]) \cup Y\Big)/\sim \ \text{ with } (x,1) \sim f(x) \text{ for all } x \in X$$

as the *mapping cylinder of* f. We identify X with $X \times 0$ and we also identify Y with the image of Y in M_f. It is (supposedly) straightforward to see that the following are equivalent:

1. $f\colon X \to Y$ is a homotopy equivalence.
2. the inclusion $X \to M_f$ is a homotopy equivalence,
3. X is a deformation retract of M_f.

Now let $f\colon X \to Y$ be a cellular map between CW–complexes. We equip M_f with the structure of a CW-complex where the cells are given by all the cells of X and Y, together with the product cells $e \times (0,1)$, where e is a cell in X.

If X and Y are finite connected CW-complexes and $f\colon X \to Y$ is a cellular homotopy equivalence, then we define the *Whitehead torsion of f* as

$$\tau(f) := \tau(M_f, X) \in \mathrm{Wh}(\pi_1(X)) = \mathrm{Wh}(\pi_1(Y)).$$

As an example we will prove the following lemma.

Lemma 1.13. *The Whitehead torsion of the identity* $\mathrm{id}\colon X \to X$ *is trivial.*

Proof. The mapping cylinder is evidently just $X \times [0,1]$, equipped with the usual product CW-structure. For each cell e of X we pick a lift $\widetilde{e}$ to $\widetilde{X}$. We lift the cells $e \times \{1\}$ and $e \times (0,1)$ in the obvious way. We write

$$\begin{array}{rcl} C_i & := & C_i(\widetilde{X} \times [0,1], \widetilde{X} \times \{0\}) \\ D_i & := & \text{span over } \mathbb{Z}[\pi] \text{ of } \{\widetilde{e} \times \{1\} \,|\, e \text{ is an } i\text{-cell of } X\}, \\ E_i & := & \text{span over } \mathbb{Z}[\pi] \text{ of } \{\widetilde{e} \times (0,1) \,|\, e \text{ is an } (i-1)\text{-cell of } X\}. \end{array}$$

We view these modules as equipped with the obvious bases. The chain complex $C_i = D_i \oplus E_i = D_i \oplus D_{i-1}$ then looks as follows

$$\begin{tikzcd} D_{i+1} \oplus D_i \arrow[r, "\begin{pmatrix} \partial & 0 \\ \mathrm{id} & \partial \end{pmatrix}"] & D_i \oplus D_{i-1} \arrow[r, "\begin{pmatrix} \partial & 0 \\ -\mathrm{id} & \partial \end{pmatrix}"] & D_{i-1} \oplus D_{i-2} \arrow[r, "\begin{pmatrix} \partial & 0 \\ \mathrm{id} & \partial \end{pmatrix}"] & D_{i-2} \oplus D_{i-3} \arrow[r, "\begin{pmatrix} \partial & 0 \\ -\mathrm{id} & \partial \end{pmatrix}"] & {} \end{tikzcd} .$$

We now consider the following short exact sequence of chain complexes

$$\begin{tikzcd} {} \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\ {} \arrow[r] & D_{i+1} \arrow[r, "0"] \arrow[d, "\begin{pmatrix} \mathrm{id} & -\partial_{i+1} \end{pmatrix}"] & D_{i-1} \arrow[r, "\mathrm{id}"] \arrow[d, "\begin{pmatrix} 0 & -\mathrm{id} \end{pmatrix}"] & D_{i-1} \arrow[r, "0"] \arrow[d, "\begin{pmatrix} \mathrm{id} & -\partial_{i-1} \end{pmatrix}"] & D_{i-3} \arrow[r] \arrow[d, "\begin{pmatrix} 0 & -\mathrm{id} \end{pmatrix}"] & {} \\ {} \arrow[r] & D_{i+1} \oplus D_i \arrow[r, "\begin{pmatrix} \partial_{i+1} & 0 \\ \mathrm{id} & \partial_i \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \partial_{i+1} \\ \mathrm{id} \end{pmatrix}"] & D_i \oplus D_{i-1} \arrow[r, "\begin{pmatrix} \partial_i & 0 \\ -\mathrm{id} & \partial_{i-1} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \mathrm{id} \\ 0 \end{pmatrix}"] & D_{i-1} \oplus D_{i-2} \arrow[r, "\begin{pmatrix} \partial_{i-1} & 0 \\ \mathrm{id} & \partial_{i-2} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \partial_{i-1} \\ \mathrm{id} \end{pmatrix}"] & D_{i-2} \oplus D_{i-3} \arrow[r] \arrow[d, "\begin{pmatrix} \mathrm{id} \\ 0 \end{pmatrix}"] & {} \\ {} \arrow[r] & D_i \arrow[r, "\mathrm{id}"] \arrow[d] & D_i \arrow[r, "0"] \arrow[d] & D_{i-2} \arrow[r, "\mathrm{id}"] \arrow[d] & D_{i-2} \arrow[r] \arrow[d] & {} \\ {} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & {} \end{tikzcd} .$$

It follows from Theorem 1.8 that the torsion of $C_i = D_i \oplus E_i = D_i \oplus D_{i-1}$ equals the products of the torsions of the chain complexes on the top and at the bottom. But clearly these torsions are trivial. $\square$

The following is [Mi66, Lemma 7.8].

Lemma 1.14. *Let $f, g\colon X \to Y$ be two cellular homotopy equivalences. If f and g are homotopic, then $\tau(f) = \tau(g)$.*

Let $f\colon X \to Y$ be a homotopy equivalence between finite CW-complexes. Then f is homotopic to a cellular map f' and we define $\tau(f) = \tau(f')$. It follows from Lemma 1.14 that this definition of $\tau(f)$ does not depend on the choice of f'.

Before we continue we note that the assignment $G \mapsto \operatorname{Wh} G)$ obviously defines a function from the category of groups to the category of abelian group. We refer to [Mi66, Section 7] for a proof of the following lemma.

Lemma 1.15. *Let X, Y, Z be finite connected CW-complexes and let $f\colon X \to Y$ and $g\colon Y \to Z$ be homotopy equivalences. Then*

$$\tau(g \circ f) = \tau(g) \cdot g_*(\tau(f)) \in \operatorname{Wh}(\pi_1(Z)) = g_*(\operatorname{Wh}(\pi_1(Y))).$$

1.6. Simple homotopy equivalences.

A homotopy equivalence $f\colon X \to Y$ of finite connected CW–complexes is called *simple* if $\tau(f) = 1 \in \operatorname{Wh}(\pi_1(Y))$.

The goal of this section is to give a more topological interpretation of a simple homotopy equivalence.

Definition. We write $S^n_- = \{(x_1, \dots, x_{n+1}) \,|\, x_{n+1} < 0\}$ and similarly we define S^n_+. Now let X be a CW-complex. If Y is a CW-complex that is obtained from X by adding first an n-cell e_n and then an $(n+1)$-cell along an attaching map $f\colon S^n \to X \cup e_n$ such that $f(S^n_-) \subset X$ and such that $f(S^n_+) \to e_n$ is a homeomorphism, then we refer to Y as an *elementary expansion* of X and we refer to X as an *elementary collapse* of Y.

It is straightforward to see that two CW-complexes that are related by a sequence of elementary expansions and elementary collapses are homotopy equivalent. An elementary argument, similar to the proof of Lemma 1.13, shows that that these homotopy equivalences are in fact simple homotopy equivalences.

More interestingly, the converse to the above observation also holds. More precisely, the following theorem is proved in [Co73, (22.2)].

Theorem 1.16. *Two finite CW-complexes are simple homotopy equivalent if and only if they are related by a finite sequence of elementary expansions and elementary collapses.*

1.7. Whitehead torsion and torsions of CW-complexes.

Theorem 1.17. *Let* $f\colon X \to Y$ *be a homotopy equivalence between finite CW-complexes and let* $\alpha\colon \pi_1(Y) \to \mathrm{GL}(d, R)$ *be a representation. Then*

$$\tau(Y,\alpha) = \tau(X,\alpha\circ f_*)\cdot \alpha_*(\tau(f)) \in \overline{K}_1(R)/\alpha(\pi_1(Y)) \cup \{*\}.$$

Proof. Let $f\colon X \to Y$ be a homotopy equivalence between finite CW-complexes and let $\alpha\colon \pi_1(Y) \to \mathrm{GL}(d, R)$ be a representation. Without loss of generality we can assume that f is cellular. We identify $\pi := \pi_1(Y)$ with $\pi_1(X)$ and $\pi_1(M_f)$. Since X, Y and M_f are homotopy equivalent we see that if one of the twisted chain complexes is acyclic, then are all the others. In particular, $\tau(X,\alpha\circ f_*) = *$ if and only if $\tau(Y,\alpha) = *$.

In the following we assume that $\tau(X,\alpha\circ f_*) \neq *$ and thus also $\tau(Y,\alpha) \neq *$. The inclusion $X \subset M_f$ gives rise to a short exact sequence of chain complexes

$$0 \to C_*(\widetilde{X}) \to C_*(\widetilde{M_f}) \to C_*(\widetilde{M_f},\widetilde{X}) \to 0.$$

We pick lifts of the cells of M_f to $\widetilde{M_f}$, and we use the same lifts for $\widetilde{X}$ and $\widetilde{M_f}\setminus\widetilde{X}$. Since the modules are free $\mathbb{Z}[\pi]$-modules it follows that this short exact sequence splits. Hence the sequence

$$0 \to R^d \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}) \to R^d \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{M_f}) \to R^d \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{M_f},\widetilde{X}) \to 0$$

is also exact.

It follows from Theorems 1.10 and 1.8 that

$$\begin{aligned}\tau\Big(R^d \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{M_f})\Big) &= \tau\Big(R^d \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X})\Big)\cdot\tau\Big(R^d \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{M_f},\widetilde{X})\Big)\\ &= \tau(X,\alpha\circ f_*)\cdot\alpha_*(\tau(f)).\end{aligned}$$

Similarly one can show that

$$\tau\Big(R^d \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{M_f})\Big) = \tau\Big(R^d \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{Y})\Big)\cdot\tau\Big(R^d \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{M_f},\widetilde{Y})\Big).$$

Note that we have a canonical isomorphism $C_*(\widetilde{M_f},\widetilde{Y}) = C_*(\widetilde{X}\times[0,1],\widetilde{X}\times\{1\})$ of $\mathbb{Z}[\pi]$-chain complexes. By Lemma 1.13 we known that $\tau(C_*(\widetilde{X}\times[0,1],\widetilde{X}\times\{1\}))$ is trivial. It thus follows from Theorem 1.10 that

$$\tau(R^d \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{M_f},\widetilde{Y})) = \alpha_*(\tau(C_*(\widetilde{X}\times[0,1],\widetilde{X}\times\{1\}))) = 1 \in \overline{K}_1(R).$$

Combining the above three equalities gives us the desired equality

$$\tau(X,\alpha\circ f_*)\cdot\alpha_*(\tau(f)) = \tau(Y,\alpha) \in \overline{K}_1(R).$$

□

We have the following immediate corollary.

Corollary 1.18. *Let* $f\colon X \to Y$ *be a simple homotopy equivalence between finite CW-complexes and let* $\alpha\colon \pi_1(Y) \to \mathrm{GL}(d, R)$ *be a representation. Then*

$$\tau(X,\alpha\circ f_*) = \tau(Y,\alpha) \in \overline{K}_1(R)/\alpha(\pi_1(Y)) \cup \{*\}.$$

1.8. **Computations of the Whitehead group.** From the discussion of the previous section the following question naturally arises: how big is the Whitehead group in general? We start out with the trivial group.

Lemma 1.19. *The Whitehead group of the trivial group is trivial.*

Proof. Let A be an invertible matrix over $\mathbb{Z}[\{e\}]=\mathbb{Z}$. Since $\mathbb{Z}$ is a Euclidean ring one can use row and column operations over $\mathbb{Z}$ to turn A into a diagonal matrix D. By Lemma 1.2 the matrix is equivalent in $\operatorname{Wh}(\{e\})=\overline{K_1}(\mathbb{Z})$ to the element represented by the 1×1-matrix $(\pm\det(A))$, which is trivial in $\operatorname{Wh}(\{e\})=\overline{K_1}(\mathbb{Z})$. □

The following theorem is much harder to prove.

Theorem 1.20.

1. *The Whitehead group of any free abelian group is trivial.*
2. *If G and G' are two groups, then*
$$\operatorname{Wh}(G*G')\cong\operatorname{Wh}(G)\times\operatorname{Wh}(G').$$
3. *The Whitehead group of any free group is trivial.*

Higman [Hi40] (see also [Co73, p. 42]) showed that the Whitehead group of the infinite cyclic group is trivial. This was extended by Bass–Heller–Swan [BHS64] to all free abelian groups. The second statement was proved by Stallings [Sta65]. The last statement is a consequence of the first two statements.

The next lemma shows that the Whitehead group of a finite cyclic group can be non-trivial, in fact infinite.

Lemma 1.21. *Let $G=\langle x|x^5=1\rangle$ be the cyclic group of order 5. Then*
$$(x+x^{-1}-1)$$
represents an element of infinite order in $\operatorname{Wh}(G)$.

More information on Whitehead groups of finite groups can be found in the survey article [Ste78].

Proof. The identity
$$\begin{aligned}(x+x^{-1}-1)(x^2+x^{-2}-1) &= (x^3+x^{-1}-x)+(x+x^{-3}-x^{-1})+(-x^2-x^{-2}+1)\\ &= x^3+x^{-3}-x^2-x^{-2}+1=1\end{aligned}$$
shows that $(x+x^{-1}-1)$ defines an element in $\operatorname{Wh}(G)$. We write $\xi=e^{2\pi i/5}$. The ring homomorphism
$$\begin{aligned}\phi\colon \mathbb{Z}[\langle x|x^5=1\rangle] &\to \mathbb{C}\\ \sum_{i=0}^{4}a_ix^i &\to \sum_{i=0}^{4}a_i\xi^i\end{aligned}$$
induces a homomorphism
$$\phi_*\colon\ \operatorname{Wh}(\langle x|x^5=1\rangle)\to K_1(\mathbb{C})/\{\pm\xi^i\,|\,i\in\mathbb{Z}\}=\mathbb{C}^*/\{\pm\xi^i\,|\,i\in\mathbb{Z}\}.$$

It follows from $|\xi+\xi^{-1}-1|\neq 1$ that $\phi_*(x+x^{-1}-1)=\xi+\xi^{-1}-1$ is an element of infinite order in $\mathbb{C}^*/\{\pm\xi^i \,|\, i\in\mathbb{Z}\}$. □

The two results suggests that perhaps Whitehead groups behave very differently for torsion-free groups and for groups with torsion. In fact the following conjecture was formulated many years ago. It can be viewed as a special case of the much more general Farrell–Jones Conjecture.

Conjecture 1.22. *The Whitehead group is trivial for any torsion-free group.*

This conjecture has been established in many cases. For example, Bartels–Lück–Reich [BLR08] showed that it holds for all word-hyperbolic groups. For future reference we also record the following theorem which follows from the Geometrization Theorem together with the work of Farrell–Jones [FJ86, Corollary 1], Waldhausen [Wa78, Theorem 17.5], Farrell–Hsiang [FH81] and Plotnick [Pl80].

Theorem 1.23. *The Whitehead group of the fundamental group of any compact, orientable, non-spherical prime* 3*-manifold is trivial.*

In particular the Whitehead group is trivial for the fundamental group of any knot complement. That particular result was already obtained by Waldhausen [Wa78, Theorem 17.5] without appealing to the Geometrization Theorem.

1.9. **Chapman's Theorem.** We have seen that torsion is an invariant of a finite CW-complex together with a representation which stays invariant under simple homotopy equivalences. The following deep theorem of Chapman [Ch74] shows that homeomorphism are examples of simple homotopy equivalences.

Theorem 1.24. *Any homeomorphism between two finite CW-complexes is a simple homotopy equivalence.*

Together with Corollary 1.18 we obtain the following theorem.

Theorem 1.25. *Let* $f\colon X\to Y$ *be a homeomorphism of finite CW–complexes. Let* $\alpha\colon \pi_1(Y)\to \mathrm{GL}(d,R)$ *be a representation. Then*

$$\tau(X,\alpha\circ f_*)=\tau(Y,\alpha)\in \overline{K}_1(R)/\alpha(\pi_1(Y))\cup\{*\}.$$

1.10. **Torsions of manifolds.** Whitehead [Wh41] showed that any smooth manifold M admits a triangulation, i.e. there exists a simplicial complex that is homeomorphic to M. Furthermore any two triangulations are equivalent, in the sense that they have a common refinement, i.e. a single triangulation that is a subdivision of both of them.

In the following, given a compact smooth manifold N, together with a representation $\alpha\colon \pi_1(N)\to \mathrm{GL}(d,R)$ we define

$$\tau(N,\alpha)=\tau(X,\alpha)\in \overline{K}_1(R)\cup\{*\},$$

where X is any CW-structure for N, e.g. given by a triangulation. By Theorem 1.25 this definition does not depend on the choice of X. More precisely it only depends on the pair (N,α) up to homeomorphism.

In dimensions two and three a compact topological manifold also admits a triangulation, see [Ra25] and [Mo52], we can thus define the Reidemeister torsion of any compact topological 2-dimensional and 3-dimensional manifold. On the other hand, Freedman and Kirby–Siebenmann showed that there exist closed topological 4-manifolds that do not admit a triangulation.

Nonetheless, it seems like the following question is still open.

Question 1.26. *Is every compact topological 4-manifold homeomorphic to a CW-complex?*

2. Calculations of torsion

We start out with the calculation of torsion for the circle, the solid torus and the standard torus. First we consider S^1. Throughout this course we denote by x the standard generator of $H_1(S^1;\mathbb{Z})$. We then have the following lemma.

Lemma 2.1. *Let* $\alpha\colon S^1 \to \mathrm{GL}(1,\mathbb{F})$ *be a one-dimensional representation over a field* $\mathbb{F}$. *Then*

$$\tau(S^1,\alpha) = \begin{cases} 1-\alpha(x), & \textit{if } \alpha \textit{ is non-trivial,} \\ *, & \textit{otherwise.} \end{cases}$$

Proof. We equip S^1 with the usual CW-structure with one 0-cell $q=\{1\}$ and one 1-cell $c=s^1$. We denote by $p\colon \mathbb{R}\to S^1$ the universal covering. We pick THE lift $\widetilde{q}=\{0\}$ of q and we pick the lift $\widetilde{c}=[0,1]$ of c to $\mathbb{R}$. Then $\partial\widetilde{c} = x\cdot\widetilde{q}-\widetilde{q}$. With respect to the bases $\widetilde{c}$ and $\widetilde{q}$ the cellular chain complex $C_*(\mathbb{R})$ is thus given by

$$0\to \mathbb{Z}[x^{\pm 1}] \xrightarrow{(x-1)} \mathbb{Z}[x^{\pm 1}] \to 0.$$

It follows that

$$\begin{array}{rcl} \tau(S^1,\alpha) & = & \tau\left(0\to \mathbb{F}\otimes_{\mathbb{Z}[x^{\pm1}]}\mathbb{Z}[x^{\pm1}] \xrightarrow{\mathrm{id}\otimes(1-x)} \mathbb{F}\otimes_{\mathbb{Z}[x^{\pm1}]}\mathbb{Z}[x^{\pm1}]\to 0\right) \\ & = & \tau\left(0\to\mathbb{F}\xrightarrow{(1-\alpha(x))}\mathbb{F}\to 0\right) \\ & = & \begin{cases} 1-\alpha(x), & \text{if } \alpha \text{ is non-trivial,} \\ *, & \text{otherwise.}\end{cases} \end{array}$$

□

By a slight abuse of notation we denote by x also the standard generator of $H_1(S^1\times D^2;\mathbb{Z})$. Since the solid torus is simple homotopy equivalent to S^1 we have the following corollary to Lemma 2.1.

Corollary 2.2. *Let* $\alpha\colon S^1\times D^2 \to \mathrm{GL}(1,\mathbb{F})$ *be a one-dimensional representation over a field* $\mathbb{F}$. *Then*

$$\tau(S^1\times D^2,\alpha) = \begin{cases} 1-\alpha(x), & \textit{if } \alpha \textit{ is non-trivial,} \\ *, & \textit{otherwise.} \end{cases}$$

Lemma 2.3. *Let* $\alpha\colon S^1 \times S^1 \to \mathrm{GL}(1,\mathbb{F})$ *be a one-dimensional representation over a field. Then*

$$\tau(S^1 \times S^1, \alpha) = \begin{cases} 1, & \text{if } \alpha \text{ is non-trivial,} \\ *, & \text{otherwise.} \end{cases}$$

Proof. We equip $T := S^1 \times S^1$ with the usual CW-structure with one 0-cell q, two 1-cells x and y and one 2-cell c. We denote by $\widetilde{T}$ the universal cover of T. For an appropriate choice of lifts of q, x, y and c to the universal cover the based chain complex $C_*(\widetilde{T})$ is isomorphic to

$$0 \to \mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \xrightarrow{\begin{pmatrix} y-1 & 1-x \end{pmatrix}} \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]^2 \xrightarrow{\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}} \mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \to 0.$$

Thus the based chain complex $\mathbb{F} \otimes_{\mathbb{Z}[\pi_1(T)]} C_*(\widetilde{T})$ is isomorphic to

$$0 \to \mathbb{F} \xrightarrow{\begin{pmatrix} \alpha(y)-1 & 1-\alpha(x) \end{pmatrix}} \mathbb{F}^2 \xrightarrow{\begin{pmatrix} 1-\alpha(x) \\ 1-\alpha(y) \end{pmatrix}} \mathbb{F} \to 0.$$

If α is trivial, then this chain complex is not acyclic, hence $\tau(T, \alpha) = *$. Now suppose that α is non-trivial. Without loss of generality we can assume that $\alpha(x) \neq 1$. We consider the following vertical short exact sequence of horizontal chain complexes

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{F} \oplus 0 \arrow[r, "\begin{pmatrix} 1-\alpha(x) \end{pmatrix}"] \arrow[d] & \mathbb{F} \arrow[r] \arrow[d] & 0 \,. \\
0 \arrow[r] & \mathbb{F} \arrow[r, "(\alpha(y)-1\, 1-\alpha(x))"] \arrow[d] & \mathbb{F} \oplus \mathbb{F} \arrow[r, "\begin{pmatrix} 1-\alpha(x) \\ 1-\alpha(y) \end{pmatrix}"] \arrow[d] & \mathbb{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{F} \arrow[r, "\begin{pmatrix} 1-\alpha(x) \end{pmatrix}"] & 0 \oplus \mathbb{F} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

It follows from Theorem 1.8 that the torsion of the middle chain complex equals the product of the torsions of the chain complexes on top and on bottom, but this product equals $(1-\alpha(x))(1-\alpha(x))^{-1} = 1 \in \overline{K}_1(\mathbb{F})$. $\square$

In the following, given a connected space X, a representation $\alpha\colon \pi_1(X) \to \mathrm{GL}(d, R)$ and a connected subspace Y we denote by a slight abuse of notation the induced representation $\pi_1(Y) \to \pi_1(X) \to \mathrm{GL}(d, R)$ by α as well. (By Lemma 1.12 the torsion $\tau(Y, \alpha)$ is independent of the choice of base points and path connecting the base points of X and Y.)

The following theorem plays the role of the Mayer–Vietoris sequence in homology.

Theorem 2.4. *Let $X = Y \cup_F Z$ be a decomposition of a finite CW-complex along a connected subcomplex F into two connected subcomplexes Y and Z. Furthermore let $\alpha\colon \pi_1(X) \to \mathrm{GL}(d, R)$ be a representation such that $\tau(F, \alpha) \neq *$. Then*

$$\tau(X, \alpha) = \tau(Y, \alpha) \cdot \tau(Z, \alpha) \cdot \tau(F, \alpha)^{-1} \in \overline{K}_1(R)/\alpha(\pi_1(X)) \cup \{*\}.$$

In the proof of Theorem 2.4 we will need the following lemma.

Lemma 2.5. *Let X be a connected finite CW-complex and let $Y \subset X$ be a connected subcomplex. We denote by $p\colon \widetilde{X} \to X$ the universal covering of X and we write $\widetilde{Y} := p^{-1}(Y)$. Let $\alpha\colon \pi_1(Y) \to \mathrm{GL}(d, R)$ be a representation. Then*

$$\tau(Y, \alpha) = \tau\left(R^d \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\widetilde{Y})\right).$$

Proof. We can pick the same base point for X and Y. We write $\pi = \pi_1(X)$, $\Gamma = \pi_1(Y)$ and $\pi_0 := \mathrm{Im}\{\Gamma \to \pi\}$. Furthermore we denote by $\widetilde{Y}_0$ a component of $\widetilde{Y}$. Finally we denote by $\widehat{Y}$ the universal cover of Y. It is straightforward to verify that the maps

$$\begin{array}{rcl} R^d \otimes_{\mathbb{Z}[\pi_0]} C_*(\widetilde{Y}_0) & \to & R^d \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{Y}) \\ v \otimes \sigma & \to & v \otimes \sigma \end{array}$$

and

$$\begin{array}{rcl} R^d \otimes_{\mathbb{Z}[\Gamma]} C_*(\widehat{Y}) & \to & R^d \otimes_{\mathbb{Z}[\pi_0]} C_*(\widetilde{Y}_0) \\ v \otimes \sigma & \to & v \otimes \text{projection of } \sigma \text{ to } \widetilde{Y}_0 \end{array}$$

are isomorphisms of chain complexes of based R-modules. The lemma is now an immediate consequence of the definitions. □

Now we turn to the proof of Theorem 2.4.

Proof of Theorem 2.4. We write $\pi = \pi_1(X)$ and we denote by $p\colon \widetilde{X} \to X$ the universal covering of X. We write $\widetilde{Y} = p^{-1}(Y)$, $\widetilde{Z} = p^{-1}(Z)$ and $\widetilde{F} = p^{-1}(F)$. Then we have the following short exact sequence of free $\mathbb{Z}[\pi]$-modules

$$0 \to C_*(\widetilde{F}) \to C_*(\widetilde{Y}) \oplus C_*(\widetilde{Z}) \to C_*(\widetilde{X}) \to 0.$$

We can and will pick lifts of the cells such that we obtain compatible bases for the free $\mathbb{Z}[\pi]$-modules. This short exact sequence splits since the modules are free. Since this is a split exact sequence the sequence stays exact after tensoring, i.e. the following sequence of chain complexes is also exact

$$0 \to R^d \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{F}) \to R^d \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{Y}) \oplus R^d \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{Z}) \to R^d \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}) \to 0.$$

The theorem follows from Theorem 1.8 and Lemma 2.5. □

The proof of the following proposition is similar to the proof of Theorem 2.4 and is left as an exercise.

Proposition 2.6. *Let X be a connected CW–complex and let $Y \subset X$ be a connected subcomplex. Let $\alpha\colon \pi_1(X) \to \mathrm{GL}(d,R)$ be a representation. If $\tau(X,Y,\alpha) \neq *$ or $\tau(Y,\alpha) \neq *$, then*

$$\tau(X,\alpha) = \tau(X,Y,\alpha)\cdot\tau(Y,\alpha) \in \overline{K}_1(R)/\alpha(\pi_1(X)) \cup \{*\}.$$

We conclude this section with the following theorem. The theorem is a consequence of Poincaré duality and a theorem relating the torsion of a complex to the torsion of the corresponding dual chain complex. We refer to [Tu01, Theorem 1.9 and Section 14] for details.

Theorem 2.7. *Let M be an orientable compact n-dimensional manifold and let $\varphi\colon \pi_1(M) \to \mathrm{GL}(d,R)$ be a representation. We denote by*

$$\begin{array}{rcl} \varphi^\dagger\colon \pi_1(M) & \to & \mathrm{GL}(d,R) \\ g & \mapsto & \alpha(g^{-1})^t \end{array}$$

the dual representation. Then

$$\tau(M,\partial M,\varphi^\dagger) = \tau(M,\varphi)^{(-1)^{n+1}} \in \overline{K}_1(R)/\alpha(\pi_1(M)) \cup \{*\}.$$

Hereby we use the convention that $*^{-1} = *$.

3. The classification of lens spaces

Given coprime natural numbers p and q we denote by $L(p,q)$ the corresponding *lens space*, defined as

$$L(p,q) := S^3/\mathbb{Z}_p = \{(z_1,z_2)\in\mathbb{C}^2 \,|\, |z_1|^2+|z_2|^2=1\}/\mathbb{Z}_p,$$

where $k \in \mathbb{Z}_p$ acts freely on S^3 by $(z_1,z_2)\mapsto(z_1e^{2\pi ik/p}, z_2e^{2\pi ikq/p})$. Note that $L(1,1) = S^3$ and $L(2,10 = \mathbb{R}P^3$.

For calculations it is often more useful to view the lens space $L(p,q)$ as a union of two solid tori. Before we give that description we introduce one more convention. Throughout this course, given the 2-torus $S^1\times S^1$ we denote by x and y the two standard generators of $H_1(S^1\times S^1;\mathbb{Z})$. Then there exists a homeomorphism

$$L(p,q) \cong (S^1\times D^2) \cup_{\varphi_{p,q}} (S^1\times D^2)$$

where $\varphi_{p,q}\colon S^1\times S^1 \to S^1\times S^1$ is an orientation reversing homeomorphism with $(\varphi_{p,q})_*(y) = px+qy$. The homeomorphism $\varphi_{p,q}$ is not unique, not even up to isotopy, but any two such homeomorphisms extend to a self-homeomorphism of $S^1\times D^2$, and hence give rise to the same gluing of solid tori. [4]

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[4] The fact that the two descriptions of $L(p,q)$ give homeomorphic spaces can be seen by decomposing S^3 into the two solid tori $V_1 = \{(z_1,z_2)\in S^3 \,|\, |z_1|^2 \le \frac{1}{2}\}$ and $V_2 = \{(z_1,z_2)\in S^3 \,|\, |z_2|^2 \le \frac{1}{2}\}$, and noting that the action by $\mathbb{Z}_p$ on S^3 descends to an action on each of the two solid tori.

3.1. **The classification of lens spaces up to homotopy.** It is clear from the first definition of lens spaces that $\pi_1(L(p,q)) = \mathbb{Z}_p$. It is also straightforward to verify that

$$H_i(L(p,q)) = \begin{cases} \mathbb{Z}, & \text{for } i=0 \text{ and } i=3, \\ \mathbb{Z}_p, & \text{for } i=1, \\ 0, & \text{for all other } i\text{'s.} \end{cases}$$

Furthermore it is clear that the homotopy groups of lens spaces $L(p,q)$ and $L(p,q')$ agree. Even though all these invariants agree, we will see that lens spaces $L(p,q)$ and $L(p,q')$ are not necessarily homotopy equivalent.

In order to distinguish lens spaces with the same fundamental group we introduce the linking pairing on oriented 3-dimensional rational homology spheres. In the following let N be an oriented 3-dimensional rational homology sphere, i.e. a 3-manifold with $H_*(N;\mathbb{Q}) = H_*(S^3;\mathbb{Q})$. Put differently, a 3-dimensional rational homology sphere is an orientable closed 3-manifold with finite first homology.

We then consider the following sequence of homomorphisms

$$\begin{tikzcd}
 & H^2(N;\mathbb{Q}) = 0 & \\
H_1(N;\mathbb{Z}) \arrow[r, "PD"] & H^2(N;\mathbb{Z}) \arrow[u] & \\
 & H^1(N;\mathbb{Q}/\mathbb{Z}) \arrow[u, "\cong"] \arrow[r, "\cong"] & \mathrm{Hom}(H_1(N;\mathbb{Z}),\mathbb{Q}/\mathbb{Z}) \\
 & H^1(N;\mathbb{Q}) = 0. \arrow[u] &
\end{tikzcd}$$

Here the vertical sequence of maps is the long exact sequence in cohomology corresponding to the short exact sequence of coefficients $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. The horizontal map on the left is the Poincaré duality isomorphism and the horizontal map on the right is the evaluation map. We denote the resulting isomorphism $H_1(N;\mathbb{Z}) \to \mathrm{Hom}(H_1(N;\mathbb{Z}),\mathbb{Q}/\mathbb{Z})$ by Φ. This isomorphism gives rise to a bilinear non-singular pairing

$$\begin{array}{rcl} \lambda(N)\colon H_1(N;\mathbb{Z}) \times H_1(N;\mathbb{Z}) & \to & \mathbb{Q}/\mathbb{Z} \\ (a,b) & \mapsto & \Phi(a)(b). \end{array}$$

Even though it is not entirely obvious from the definitions, this pairing is in fact symmetric. The isometry type of the linking form is an invariant of the homotopy type of the underlying manifold.

For a lens space $L(p,q)$ its linking form is isometric to

$$\begin{array}{rcl} \mathbb{Z}_p \times \mathbb{Z}_p & \to & \mathbb{Q}/\mathbb{Z} \\ (a,b) & \mapsto & \pm\frac{1}{p}abq \end{array}$$

where the sign depends on the orientation of $L(p,q)$, see [Li85, Lemma 2] for a proof. It is now straightforward to see that if $L(p,q)$ and $L(p,q')$ are isotopic, then $q \equiv \pm q' a^2 \mod p$ for some $a \in \mathbb{Z}$. In fact the converse also holds, we thus see that the linking form classifies the homotopy type of lens spaces. We refer to [Wh41] for details on the classification of lens spaces up to homotopy.

3.2. **The classification of lens spaces up to homeomorphism.**

Theorem 3.1. *Let p, q be coprime natural numbers. We write $\xi = e^{2\pi i/p}$. Let $\alpha\colon \pi_1(L(p,q)) \to \mathrm{GL}(1,\mathbb{C})$ be a faithful (i.e. injective) representation. Then there exists a k coprime to p such that*

$$\tau(L(p,q),\alpha) = (1-\xi^k)(1-\xi^{-qk}).$$

Proof. We pick two copies V_1 and V_2 of the solid torus $S^1 \times D^2$. For $i = 1,2$ we denote the curves $S^1 \times \{0\}$ and $\{*\} \times S^1$ in the solid torus T_i by x_i and y_i. We pick $a, b \in \mathbb{N}$ with $aq - bp = -1$. Recall that

$$L(p,q) = V_1 \cup_{\varphi_{p,q}} V_2$$

where $\varphi_{p,q}\colon \partial V_1 \to \partial V_2$ is a homeomorphism such that the map

$$\varphi_*\colon H_1(V_1;\mathbb{Z}) = \mathbb{Z}x_1 \oplus \mathbb{Z}y_1 \to H_1(V_1;\mathbb{Z}) = \mathbb{Z}x_1 \oplus \mathbb{Z}y_1$$

is represented by the matrix

$$A = \begin{pmatrix} a & b \\ p & q \end{pmatrix} \text{ with } A^{-1} = \begin{pmatrix} -q & b \\ p & -a \end{pmatrix}.$$

(Here we also use the convention that matrices act by right multiplication.) Note that x_1 generates $H_1(L(p,q);\mathbb{Z})$ and that y_1 is null-homologous. It follows that $\alpha(x_1) = \xi^k$ for some k coprime to p. By the choice of the gluing we have

$$\alpha(x_2) = \alpha(x_1)^{-q}\alpha(y_1)^b = \alpha(x_1)^{-q} = \xi^{-qk}.$$

Now it follows from Theorem 2.4 together with Corollary 2.2 and Lemma 2.3 that

$$\tau(L(p,q),\alpha) = \tau(V_1,\alpha)\cdot\tau(V_2,\alpha) = (1-\xi^k)(1-\xi^{-qk}).$$

□

Now we can prove the following theorem, which was first proved, modulo Chapman's theorem, by Reidemeister [Re35].

Theorem 3.2. *Two lens spaces $L(p,q)$ and $L(p,q')$ are homeomorphic if and only if $q' \equiv q^{\pm 1} \mod p$.*

Proof. If $q' \equiv q \mod p$, then evidently $L(p,q') = L(p,q)$. If $q' \equiv q^{-1} \mod p$, then it is straightforward to see that there exists an orientation reversing homeomorphism $L(p,q') \cong L(p,q)$.

Conversely, suppose that there exists a homeomorphism $f\colon L(p,q)\to L(p,q')$. We write $\xi = e^{2\pi i/p}$. Let $\alpha\colon \pi_1(L(p,q'))\to \mathrm{GL}(1,\mathbb{C})$ be a faithful representation. It follows from Theorem 1.25 that

$$\tau(L(p,q),\alpha\circ f_*) = \tau(L(p,q'),\alpha)\in \mathbb{C}^*/\{\pm\xi^i\,|\,i=0,\dots,p-1\}.$$

We can compute both torsions using Theorem 3.1. We conclude that there there exist k and k' coprime to p, a $l\in\mathbb{Z}$ and an $\epsilon\in\{-1,1\}$ such that

$$\begin{array}{rcl} (1-\xi^k)(1-\xi^{-qk}) &=& \epsilon(1-\xi^{k'})(1-\xi^{-qk'})\xi^l, \\ \text{i.e. } 1-\xi^k-\xi^{-qk}+\xi^{k-qk} &=& \epsilon\xi^l-\epsilon\xi^{k+l}-\epsilon\xi^{-q'k+l}+\epsilon\xi^{k-q'k+l}. \end{array}$$

It is now a slightly painful, but elementary exercise to verify that this is only possible if $q'\equiv \pm q^{\pm 1} \mod p$. □

4. Applications to knot theory

Let $K\subset S^3$ be an oriented knot. We denote by νK an open tubular neighborhood of K and we refer to $X_K = S^3\setminus \nu K$ as the *exterior* of K. Furthermore we denote the oriented meridian by μ_K.

It follows from Alexander duality that the inclusion $\mu_K\to X_K$ induces an isomorphism $H_*(\mu_K;\mathbb{Z})\to H_*(X_K;\mathbb{Z})$, in particular $H_1(X_K;\mathbb{Z})\cong\mathbb{Z}$, where this group is generated by the meridian.

In the following we also write $\pi_K=\pi_1(X_K)$. We denote by

$$\phi_K\colon \pi_K\to\langle t\rangle$$

the unique epimorphism with $\phi_K(\mu_K)=t$. Throughout these notes we will view $\langle t\rangle$ as a subset of $\mathbb{Z}[t^{\pm 1}]$ and also of $\mathbb{Q}(t)$. When the knot K is understood from the context, then we will sometimes drop the subscript K in the notation.

The *Milnor torsion* of K is defined as

$$\tau_K(t) := \tau\left(X_K,\pi_K\xrightarrow{\phi_K}\langle t\rangle\to \mathrm{GL}(1,\mathbb{Q}(t))\right)\in \mathbb{Q}(t)/\{\pm t^i|i\in\mathbb{Z}\}.$$

In the following sections we will show that $\tau_K(t)$ is always non-zero, we will show how to calculate it efficiently and we will relate $\tau_K(t)$ to various topological properties of K.

4.1. The non-vanishing and symmetry of Milnor torsion.

We start out with the following proposition.

Proposition 4.1. *Let C_* be a finite chain complex of free R-modules and let $\varphi\colon R\to S$ be a ring homomorphism, such that the following condition holds:*

(∗) *if A is a matrix over R such that $\varphi(A)$ is invertible over S, then A is already invertible over A.*

Then, if $S\otimes_R C_*$ is acyclic over S, then C_* is already acyclic over R.

Proof. We first recall that a chain complex D_* over a ring T is acyclic if and only if there exists a chain homotopy from an automorphism of D_* to 0. More precisely, the chain complex is acyclic if there exists a chain automorphism h_* of D_* and maps $s_i\colon D_i \to D_{i+1}$, such that

$$\partial_{i+1} \circ s_i + s_{i+1} \circ \partial_i = h_i \text{ for all } i.$$

If such maps s_i exist for $S \otimes_R C_*$ then, using that the C_i are free R-modules, we can inductively lift the chain maps h_* and the maps s_* to C_*. Since the maps $\partial_{i+1} \circ s_i + s_{i+1} \circ \partial_i$ are isomorphisms it follows from our assumptions that the lifts to C_* are also isomorphisms. □

Now we can show that the Milnor torsion for any knot is non-zero. In fact we have the following more precise statement.

Theorem 4.2. *Given any oriented knot $K \subset S^3$ there exists a polynomial $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$ with $\Delta_K(1) = \pm 1$ such that*

$$\tau_K(t) = (1-t) \cdot \Delta_K(t)^{-1} \in \mathbb{Q}(t)/\{\pm t^i | i \in \mathbb{Z}\}.$$

The polynomial $\Delta_K(t)$ has the same indeterminacy as $\tau_K(t)$, i.e. it is well-defined up to multiplication by $\pm t^i$. The polynomial $\Delta_K(t)$ is called the *Alexander polynomial of K*.

Proof. We write $X = X_K$, $\pi = \pi_K$ and $\mu = \mu_K$. We view any $\mathbb{Z}[t^{\pm 1}]$-module M as a module over $\mathbb{Z}[\pi]$ via the homomorphism ϕ_K. We write

$$\tau(X, M) := \tau(M \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}).$$

Similarly we define $\tau(\mu, M)$ and $\tau(X, \mu, M)$.

It follows from Proposition 2.6 that

$$\tau(X, \mathbb{Q}(t)) = \tau(X, \mu, \mathbb{Q}(t)) \cdot \tau(\mu, \mathbb{Q}(t)).$$

By Lemma 2.1 we have $\tau(\mu, \mathbb{Q}(t)) = 1 - t$. Thus it suffices to prove the following claim.

Claim. There exists a polynomial $p(t) \in \mathbb{Z}[t^{\pm 1}]$ with $\tau(X, \mu, \mathbb{Q}(t))$ such that $p(1) = \pm 1$.

We write

$$S := \{f(t) \in \mathbb{Z}[t^{\pm 1}] \,|\, f(1) = \pm 1\}.$$

We denote by $\varphi\colon S^{-1}\mathbb{Z}[t^{\pm 1}] \to \mathbb{Z}$ the ring homomorphism induced by $\varphi(t) = 1$. Then

$$\mathbb{Z} \otimes_{S^{-1}\mathbb{Z}[t^{\pm 1}]} \left(S^{-1}\mathbb{Z}[t^{\pm 1}] \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}, \widetilde{\mu})\right) = \mathbb{Z} \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}, \widetilde{\mu}) = C_*(X, \mu)$$

is acyclic.

It follows from Proposition 4.1 that $\tau(X, \mu, S^{-1}\mathbb{Z}[t^{\pm 1}])$ is non-zero, i.e. it lies in $S^{-1}\mathbb{Z}[t^{\pm 1}]$. Since X is simple homotopy equivalent to a 2-dimensional CW-complex with one 0-cell we can calculate $\tau(X, \mu, S^{-1}\mathbb{Z}[t^{\pm 1}])$ using a chain complex which is zero except in dimensions 1 and 2. Since the corresponding matrix is defined over $\mathbb{Z}[t^{\pm 1}]$ we see that $\tau(X, \mu, S^{-1}\mathbb{Z}[t^{\pm 1}]) = p(t)^{-1}$ for some $p(t) \in \mathbb{Z}[t^{\pm 1}]$. Since $p(t)^{-1} \in S^{-1}\mathbb{Z}[t^{\pm 1}]$ we see that $p(t) \in S$, i.e. $p(1) = \pm 1$. This concludes the proof of the claim. □

Theorem 4.3. *Let $K \subset S^3$ be an oriented knot. Then*

$$\tau_K(t) = \tau_K(t^{-1}) \in \mathbb{Q}(t)/\{\pm t^i | i \in \mathbb{Z}\}.$$

Proof. We denote by $p(t) \mapsto \overline{p(t)}$ the involution on $\mathbb{Q}(t)$ that is induced by $\bar{t} := t^{-1}$. Recall that the orientation of K gives rise to a representation

$$\alpha\colon \pi_K \xrightarrow{\phi_K} \langle t \rangle \to \mathrm{GL}(1, \mathbb{Q}(t)).$$

It follows immediately from the definitions that for any $g \in \pi_K$ we have $\alpha^\dagger(g) = \overline{\alpha(g)} \in \mathbb{Q}(t)$. Since $p(t) \mapsto \overline{p(t)}$ is a ring homomorphism it follows immediately from Theorem 2.7, Proposition 2.6 and Lemma 2.3 that the following equalities hold in $\mathbb{Q}(t)/\{\pm t^i | i \in \mathbb{Z}\}$

$$\begin{aligned} \tau_K(t) = \tau(X_K, \alpha) &= \tau(X_K, \partial X_K, \alpha^\dagger) \\ &= \overline{\tau(X_K, \partial X_K, \alpha)} \\ &= \overline{\tau(X_K, \alpha)} \cdot \overline{\tau(\partial X_K, \alpha)} = \overline{\tau_K(t)} = \tau_K(t^{-1}). \end{aligned}$$

□

Corollary 4.4. *Let $K \subset S^3$ be an oriented knot. Then there exists a representative $\Delta_K(t)$ of the Alexander polynomial with*

$$\Delta_K(t) = \Delta_K(t^{-1}).$$

Proof. It follows from Theorem 4.3 and the definitions that

$$(1 - t^{-1})\Delta_K(t^{-1})^{-1} = (1 - t)\Delta_K(t)^{-1} \in \mathbb{Q}(t)/\{\pm t^i | i \in \mathbb{Z}\}.$$

Since $1 - t = 1 - t^{-1} \in \mathbb{Q}(t)/\{\pm t^i | i \in \mathbb{Z}\}$ we see that also

$$\Delta_K(t) = \Delta_K(t^{-1}) \in \mathbb{Q}(t)/\{\pm t^i | i \in \mathbb{Z}\}.$$

But since $\Delta_K(1) \neq 1$ and since $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$ one can easily show that there exists in fact a representative $\Delta_K(t)$ of the Alexander polynomial with

$$\Delta_K(t) = \Delta_K(t^{-1}).$$

□

4.2. **Fox calculus.**

Given a free group F on generators $x_1, \dots, x_k$ and given $i \in \{1, \dots, k\}$ one can show that there exists a unique $\mathbb{Z}$-linear homomorphism, called Fox derivative

$$\frac{\partial}{\partial x_i}\colon \mathbb{Z}[F] \to \mathbb{Z}[F]$$

with the following properties:

$$\begin{aligned}
\frac{\partial}{\partial x_i} 1 &= 0, \\
\frac{\partial}{\partial x_j} x_j &= \delta_{ij} \\
\frac{\partial}{\partial x_j} uv &= \frac{\partial}{\partial x_j} u + u \frac{\partial}{\partial x_j} v \quad \text{for all } u, v \in \mathbb{Z}[F].
\end{aligned}$$

In the calculations it is useful to note that it follows from these properties, that

$$\frac{\partial}{\partial x_i} x_i^{-1} = -x_i^{-1}.$$

Furthermore, given a presentation $P = \langle x_1, \dots, x_k | r_1, \dots, r_l \rangle$ we denote by $Y(P)$ the corresponding 2-complex with one 0-cell, k 1-cells and l 2-cells. By [Fo53, CF77] the Fox derivatives have the following topological interpretation.

Lemma 4.5. *Let $P = \langle x_1, \dots, x_k | r_1, \dots, r_l \rangle$ be a presentation. Then there exists a lift of the cells of $Y(P)$ to the universal cover $\widetilde{Y(P)}$ such that with respect to the bases the based chain complex $C_*(\widetilde{Y(P)})$ is isomorphic to the chain complex*

$$0 \to \mathbb{Z}[\pi]^l \xrightarrow{\begin{pmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_k} \\ \vdots & & \vdots \\ \frac{\partial r_l}{\partial x_1} & \cdots & \frac{\partial r_l}{\partial x_k} \end{pmatrix}} \mathbb{Z}[\pi]^k \xrightarrow{\begin{pmatrix} 1-x \\ \vdots \\ 1-x_k \end{pmatrix}} \mathbb{Z}[\pi] \to 0.$$

4.3. **Calculating Milnor torsion using Fox calculus.** In this section we will see how one can use Fox derivatives to determine the Milnor torsion of a given knot K.

First of all, for knot groups there is an easy algorithm for reading off a diagram a presentation of deficiency one, i.e. where the number of relators is one less than the number of generators. For example for the trefoil K we have

$$\pi_K = \langle x_1, x_2, x_3 \,|\, x_2^{-1} x_3 x_1 x_3^{-1}, x_3^{-1} x_1 x_2 x_1^{-1} \rangle.$$

The following theorem allows us to determine $\tau_K(t)$ using Fox derivatives.

Theorem 4.6. *Let K be a knot and let $\langle x_1, \dots, x_k | r_1, \dots, r_{k-1} \rangle$ be a presentation of π_K of deficiency one. Let $s \in \{1, \dots, k\}$ such that $\phi_K(x_s)$ is non-trivial. Then*

$$\tau_K(t) = (1 - \phi_K(x_s)) \det \left(\phi_K \Big(\frac{\partial r_i}{\partial x_j} \Big) \right)_{j \neq s}^{-1}.$$

Example. (1) For the trivial knot K we have $\pi_K = \langle x | \rangle = \langle x, y | y \rangle$. It follows that

$$\tau_K(t) = (1 - \phi(y)) \cdot \phi_K \Big(\frac{\partial y}{\partial y} \Big) = 1 - t.$$

Put differently, $\Delta_K(t) = 1$.

(2) For the trefoil K we use the above presentation. In this case $\phi(x_i) = t$ for $i = 1, 2, 3$. It follows that

$$\begin{aligned}
\tau_K(t) &= (1-\phi(x_3))\det\begin{pmatrix} \phi\left(\frac{\partial x_2^{-1}x_3x_1x_3^{-1}}{\partial x_1}\right) & \phi\left(\frac{\partial x_2^{-1}x_3x_1x_3^{-1}}{\partial x_2}\right) \\ \phi\left(\frac{\partial x_3^{-1}x_1x_2x_1^{-1}}{\partial x_1}\right) & \phi\left(\frac{\partial x_3^{-1}x_1x_2x_1^{-1}}{\partial x_2}\right)\end{pmatrix}^{-1} \\
&= (1-\phi(x_3))\det\begin{pmatrix} \phi(x_2^{-1}x_3) & \phi(-x_2^{-1}) \\ \phi(x_3^{-1} - x_3^{-1}x_1x_2x_1^{-1}) & \phi(x_3^{-1}x_1)\end{pmatrix}^{-1} \\
&= (1-t)\det\begin{pmatrix} 1 & -t^{-1} \\ t^{-1}-1 & 1\end{pmatrix} = (1-t)(1-t^{-1}-t^{-2})^{-1}.
\end{aligned}$$

Put differently, $\Delta_K(t) = t^{-1} - 1 + t$.

(3) A similar calculation shows that for the Figure-8 knot we have $\tau_K(t) = (1 - t)(1-3t+t^2)^{-1}$. Put differently, $\Delta_K(t) = t^{-1} - 3t + t$.

(4) More generally, if T_{pq} is the (p, q)-torus knot, then its fundamental group admits the presentation $\langle x, y \mid y^q x^{-p}\rangle$ where $\phi(x) = t^q$ and $\phi(y) = t^p$. It follows that

$$\begin{aligned}
\tau_K(t) &= \phi(1-x^p)\phi\left(\frac{\partial y^q x^{-p}}{\partial y}\right)^{-1} \\
&= \phi(1-x)\phi(1+y+\cdots+y^{q-1})^{-1} \\
&= (1-t^q)(1+t^p+t^{2p}+\cdots+t^{p(q-1)})^{-1}.
\end{aligned}$$

(5) The knot 5_2, i.e. the one knot with five crossings that is not a torus knot, has Alexander polynomial $2t - 5 + 2t^{-1}$.

(6) There exist non-trivial knots with trivial Alexander polynomial, i.e. the Alexander polynomial does not detect the unknot

In the proof of Theorem 4.6 we will need the following proposition.

Proposition 4.7. *Let K be a knot and let $P = \langle x_1, \dots, x_k | r_1, \dots, r_{k-1}\rangle$ be a deficiency one presentation. Then X_K and $Y(P)$ are simple homotopy equivalent.*

Proof. By Theorem 1.23 the Whitehead group of π_K is trivial. It thus suffices to show that $X := X_K$ and $Y := Y(P)$ are homotopy equivalent. We first consider the homology groups. As pointed out above, $H_*(X;\mathbb{Z}) \cong H_*(S^1;\mathbb{Z})$. The following claim shows that the same holds for Y.

Claim.

$$H_*(Y;\mathbb{Z}) \cong H_*(S^1;\mathbb{Z}).$$

It is clear that $H_0(Y;\mathbb{Z}) \cong H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$. Since Y is a 2-complex we know that $H_2(Y;\mathbb{Z})$ is torsion-free. Furthermore, since we are dealing with a deficiency one presentation we have $\chi(Y) = 0$. Since $H_0(Y;\mathbb{Q}) \cong H_1(Y;\mathbb{Q}) \cong \mathbb{Q}$ it follows that $H_2(Y;\mathbb{Q})$ is zero, but this implies that in fact $H_2(Y;\mathbb{Z}) = 0$. All higher homology groups are evidently zero. This concludes the proof of the claim.

It is a consequence of the Sphere Theorem in 3-manifold topology that knot exteriors are aspherical, see e.g. [AFW15, (A.1)] for details. The obvious isomorphism $\pi_1(Y) \to \pi_1(X)$ is thus realized by a map $f\colon Y \to X$. We write $\pi = \pi_1(X)$ and we identify $\pi_1(Y)$ with π via f_*.

Since X is aspherical it suffices to show that Y is also aspherical. By the Hurewicz theorem this means that we have $H_i(X;\mathbb{Z}[\pi]) = 0$ for $i > 1$ and we need to show that $H_i(Y;\mathbb{Z}[\pi]) = 0$ for all $i > 1$.

Since $H_2(Y;\mathbb{Z}) = 0$ we know that the map of free $\mathbb{Z}[\pi]$–modules $C_2(Y;\mathbb{Z}[\pi]) \to C_1(Y;\mathbb{Z}[\pi])$ becomes injective after tensoring over $\mathbb{Z}[\pi]$ with $\mathbb{Z}$. On the other hand, by [Ho82, Corollary 6.2] the group π is 'locally indicable'. The precise definition of 'locally indicable' is of no concern to us. What is important is, that according to [HS83, Theorem 1], this implies that $C_2(Y;\mathbb{Z}[\pi]) \to C_1(Y;\mathbb{Z}[\pi])$ is also injective. This implies that $H_2(Y;\mathbb{Z}[\pi])$ is trivial. $\square$

We can now prove Theorem 4.6.

Proof of Theorem 4.6. Let K be a knot and let $P = \langle x_1,\dots,x_{k+1} | r_1,\dots,r_k\rangle$ be a presentation of π_K of deficiency one. We write $Y = Y(P)$ and as always we denote by $\widetilde{Y}$ the universal cover of Y. Let $s \in \{1,\dots,k\}$ such that $\phi(x_s)$ is non-trivial. Without loss of generality $s = k+1$.

By Corollary 1.18 and Proposition 4.7 we have

$$\tau_K(t) = \tau\Big(Y(P), \pi_K \xrightarrow{\phi_K} \langle t\rangle \to \mathrm{GL}(1,\mathbb{Q}(t))\Big).$$

We will use the description of Lemma 4.5 for $C_*(\widetilde{Y})$. We consider the following vertical short exact sequence of horizontal chain complexes.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \oplus \mathbb{Z}[\pi] \arrow[r, "\begin{pmatrix}1 - x_{k+1}\end{pmatrix}"] \arrow[d] & \mathbb{Z}[\pi] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}[\pi]^k \arrow[r, "\begin{pmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_k} & \frac{\partial r_1}{\partial x_{k+1}} \\ \vdots & & \vdots & \vdots \\ \frac{\partial r_k}{\partial x_1} & \cdots & \frac{\partial r_k}{\partial x_k} & \frac{\partial r_k}{\partial x_{k+1}} \end{pmatrix}"] \arrow[d] & \mathbb{Z}[\pi]^k \oplus \mathbb{Z}[\pi] \arrow[r, "\begin{pmatrix} 1 - x_1 \\ \vdots \\ 1 - x_{k+1} \end{pmatrix}"] \arrow[d] & \mathbb{Z}[\pi] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}[\pi]^k \arrow[r, "\begin{pmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_k} \\ \vdots & & \vdots \\ \frac{\partial r_k}{\partial x_1} & \cdots & \frac{\partial r_k}{\partial x_k} \end{pmatrix}"] & \mathbb{Z}[\pi]^k \oplus 0 \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

The desired equality now follows from tensoring these chain complexes with $\mathbb{Q}(t) \otimes_{\mathbb{Z}[\pi]} -$ and from applying Theorem 1.8 and Lemma 1.6. $\square$

4.4. **Genus and fiberedness of a knot.** A knot K is called *fibered* if there exists a map $X_K \to S^1$ which is a surface bundle over S^1. For example the unknot U is fibered, since $X_U = S^1 \times D^2$ is a surface bundle over S^1. It is somewhat more difficult to see that the trefoil, in fact all torus knots are fibered. See [Rol90] for details.

Theorem 4.8. *If K is a fibered knot with fiber F, then $\Delta_K(t)$ is monic (i.e. its top and bottom coefficients are ± 1) and the degree of $\Delta_K(t)$ equals $2\operatorname{genus}(F)$.*

The Alexander polynomial of the knot 5_2 equals $2t-5+2t^{-1}$. We thus see that this is not fibered. In most, but not all cases, this theorem is strong enough to determine whether or not a given knot is fibered.

Proof. We write $g = \operatorname{genus}(F)$. We denote by $\varphi\colon F \to F$ the monodromy of the fiber bundle. Since F has one boundary component we have $\chi(F) \neq 0$, i.e. φ has a fix point. We use this fix point as the base point for $\pi_1(F)$. By a slight abuse of notation we denote the induced map $\pi_1(F) \to \pi_1(F)$ by φ as well. It follows from the Seifert–van Kampen theorem for splittings along a connected, non-separating subset that

$$\begin{aligned} \pi_K &\cong \langle \pi_1(F), t \,|\, t\pi_1(F)t^{-1} = \varphi(\pi_1(F))\rangle \\ &= \langle x_1, \dots, x_{2g}, t \,|\, tx_1t^{-1}\varphi(x_1)^{-1}, \dots, tx_{2g}t^{-1}\varphi(x_{2g})^{-1}\rangle \\ &= \langle x_1, \dots, x_{2g}, t \,|\, \varphi(x_1)tx_1^{-1}t^{-1}, \dots, \varphi(x_{2g})tx_{2g}^{-1}t^{-1}\rangle. \end{aligned}$$

Note that $\phi(t) = t$ and $\phi(x_1) = \dots = \phi(x_{2g}) = 1$. It follows from Theorem 4.6 and from the definitions that

$$\begin{aligned} \Delta_K(t) &= \det \begin{pmatrix} \phi\Big(\frac{\partial}{\partial x_1}\varphi(x_1)tx_1^{-1}t^{-1}\Big) & \dots & \phi\Big(\frac{\partial}{\partial x_{2g}}\varphi(x_1)tx_1^{-1}t^{-1}\Big) \\ \vdots & & \vdots \\ \phi\Big(\frac{\partial}{\partial x_1}\varphi(x_{2g})tx_{2g}^{-1}t^{-1}\Big) & \dots & \phi\Big(\frac{\partial}{\partial x_{2g}}\varphi(x_{2g})tx_{2g}^{-1}t^{-1}\Big) \end{pmatrix} \\ &= \det\left(B - t\,\mathrm{id}_{2g}\right) \end{aligned}$$

where B denotes the matrix of the action of φ on $H_1(F;\mathbb{Z})$ with respect to the basis given by $x_1, \dots, x_{2g}$. Since $\det(B) = \pm 1$ it follows that $\det\left(B - t\,\mathrm{id}_{2g}\right)$ is a monic polynomial of degree $2g$. □

Remark. It is arguably more natural to prove the theorem by using a CW-structure for X which respects the fiber bundle structure. This approach is used in [Fri14].

Any knot K is the boundary of a Seifert surface, i.e. of a connected orientable properly embedded surface. The minimal genus of a Seifert surface is called the *genus* $g(K)$ of K.

In the following, given a Laurent polynomial $p(t) = \sum_{i=r}^{s} a_i t^i \in \mathbb{Z}[t^{\pm 1}]$ with $a_r \neq 0$ and $a_s \neq 0$ we refer to $s - r$ as the degree of $p(t)$. Furthermore, for a non-zero rational function $f(t) = p(t)q(t)^{-1}$ with $p(t), q(t) \in \mathbb{Z}[t^{\pm 1}]$ we define $\deg(f(t)) = \deg(p(t)) - \deg(q(t))$.

Theorem 4.9. *For any knot K we have*

$$\deg(\Delta_K(t)) \leq 2g(K).$$

Proof. Let F be a Seifert surface for K. We pick a tubular neighborhood $F \times [-1,1]$ of F. We write π_K as an HNN-extension of $\pi_1(X_K \setminus F \times (-1,1))$. If we pick a corresponding presentation and if we do the calculation as in Theorem 4.8 we get the desired result. $\square$

The combination of Theorems 4.8 and 4.9 gives us the following corollary.

Corollary 4.10. *If K is a fibered knot with fiber F, then $2g(K) = 2genus(F) = \deg(\Delta_K(t))$.*

For example, the (p,q)-torus knot is fibered. From the above calculation we see that

$$\begin{array}{rcl} 2g(K) & = & \deg(\Delta_K(t)) \\ & = & -\deg(\tau_K(t)) + 1 \\ & = & -(-p(q-1)+q)+1 = (p-1)(q-1). \end{array}$$

4.5. **Slice knots.** A knot $K \subset S^3$ is called *slice* if K bounds a smoothly embedded disk in D^4. The following theorem was first proved by Fox–Milnor [FM66].

Theorem 4.11. *If K is slice, then there exists a $p(t) \in \mathbb{Z}[t^{\pm 1}]$ such that*

$$\Delta_K(t) = p(t) \cdot p(t^{-1}) \in \mathbb{Z}[t^{\pm 1}]/\{\pm t^i | i \in \mathbb{Z}\}.$$

It is straightforward to verify that the Alexander polynomials of the trefoil and the Figure-8 knot do not have this factorization property. It follows that these two knots are not slice.

The proof of the theorem will require the remainder of this section. Given a knot K we denote in the following the 0-framed surgery by N_K. Recall that N_K is defined as

$$N_K = X_K \cup S^1 \times D^2$$

where we glue a meridian of K to $S^1 \times \{*\}$ and we glue a longitude of K to $\{*\} \times D^2$. It is straightforward to verify that the inclusion $X_K \to N_K$ induces an isomorphism of first homology.

Lemma 4.12.

$$\tau(N_K, \mathbb{Q}(t)) = \tau(X_K, \mathbb{Q}(t)) \cdot (1-t) \in \mathbb{Q}(t)/\{\pm t^i | i \in \mathbb{Z}\}.$$

Proof. It follows from Theorem 2.4 and Corollary 2.2 that

$$\tau(N_K, \mathbb{Q}(t)) = \tau(X_K, \mathbb{Q}(t)) \cdot \tau(S^1 \times D^2) = \tau(X_K, \mathbb{Q}(t)) \cdot (1-t) \in \mathbb{Q}(t)/\{\pm t^i | i \in \mathbb{Z}\}.$$

$\square$

We are now in position to prove Theorem 4.11.

Proof. Let K be a knot which bounds a smoothly embedded disk D in D^4. We denote by Y the result of removing from D^4 an open tubular neighborhood of D.

It is straightforward to verify that $\partial Y = N_K$ and that the inclusion $\mu_K \to Y$ induces an isomorphism on homology. Exactly the same argument as in the proof of Theorem 4.2 shows that $\tau(Y;\mathbb{Q}(t))$ is non-zero.

Furthermore it follows from Theorem 2.7, Theorem 1.8 and Lemma 4.12 and the definition of $\Delta_K(t)$ that

$$\begin{aligned} \overline{\tau(Y;\mathbb{Q}(t))}^{-1} &= \tau(Y,N_K;\mathbb{Q}(t)) \\ &= \tau(Y;\mathbb{Q}(t))\cdot\tau(N_K,\mathbb{Q}(t)) \\ &= \tau(Y;\mathbb{Q}(t))\cdot(1-t)(1-t^{-1})\Delta_K(t)^{-1} \in \mathbb{Q}(t)/\{\pm t^i | i\in\mathbb{Z}\}. \end{aligned}$$

Summarizing, we just showed that there exists an $f(t)\in\mathbb{Q}(t)$, namely $f(t) = (1-t)\tau(N_K,\mathbb{Q}(t))^{-1}$, with

$$\Delta_K(t) = f(t)\cdot f(t^{-1}) \in \mathbb{Q}(t)/\{\pm t^i | i\in\mathbb{Z}\}.$$

But since $\mathbb{Z}[t^{\pm 1}]$ is a unique factorization domain there exists in fact a $p(t)\in\mathbb{Z}[t^{\pm 1}]$ with

$$\Delta_K(t) = p(t)\cdot p(t^{-1}) \in \mathbb{Q}(t)/\{\pm t^i | i\in\mathbb{Z}\}.$$

$\square$

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